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RESEARCH ARTICLE

A new characterization of groups with central chief factors

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Abstract. In [1] it is proved that a locally nilpotent group is an (X)-group arising the question whether the converse holds. In this paper we derive some interesting properties and give a complete characterization of (X)-groups. As a consequence we obtain a new characterization of groups whose chief factors are central and it follows also that there exists an (X)-group which is not locally nilpotent, thus answering the question raised in [1]. We also prove a result which extends one on finitely generated nilpotent groups due to Gruenberg.

Dedicated to Professor Miguel Ferrero
on occasion of his 70-th anniversary

Introduction

Most families of known groups are defined in terms of certain identities or series. Clearly this is the most obvious way to define a family of groups. But for the sake of applications it might be interesting to define a family of groups by means of a non-trivial set theoretical property and or non-trivial equivalent relation. In [1] the following definition is given:

A group G is said to satisfy condition (X) if for every pair of elements $g, h \in G$ we have that either $g[g,G] = h[h,G]$ or $g[g,G] \bigcap h[h,G] = \emptyset$.

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A group G which satisfies condition (X) will be called an (X)*-group*. Note that if G is an (X)-group then the sets $g[g, G]$ induce a partition on G.

In [1] it is proved that for finite groups condition (X) is equivalent to nilpotency and that locally nilpotent groups are (X)-groups. The question is raised whether the converse holds, i.e., if an (X)-group is locally nilpotent.

Here we give a description of (X)-groups by means of well known properties.

Theorem 1. *In a group G the following conditions are equivalent.*

1. *G is an (X)-group.*
2. *Every chief factor of G is a central factor.*
3. *Every homomorphic image of G is a Z-group.*
4. *Every homomorphic image of G is residually central.*

We recall that a Z-group is a group which has a generalized central series and that G is a residually central group if for each $1 \neq x \in G$ we have $x \notin [x, G]$. See [4] and [6] for an account of these properties (see also [2]). The equivalence between items 2 and 3 and the implication from 3 to 4 can be found in [6] pages 5 and 6. For the sake of completeness a proof will be presented here.

In [6], page 5, Robinson quotes examples due to P. Hall and Merzljakov showing that a group satisfying item 3) of Theorem 1 may have non-cyclic free subgroups. So, it is clear from the theorem above that an (X)-group is not necessarily locally nilpotent. See also [7] where it is proved that for each prime p there is a non locally nilpotent residually finite p-group satisfying item 3) and which is also an $\tilde{N}$-group.

Since a locally nilpotent group is a Z-group (see [4] - 12.4.2) it also follows from the theorem above that a locally nilpotent group is an (X)-group. This result is Theorem A of [1].

A result due to Gruenberg states that a finitely generated torsion free nilpotent group is a residually finite p-group for every prime p. See [4] - 5.2.21. We use property (X) to prove an extension of this result giving at the same time a short proof.

Theorem 2. *Let G be a torsion free nilpotent group, p any prime and $1 \neq x \in G$. Then there is $N \lhd G$ such that $x \notin N$ and $Z(G/N)$ is locally cyclic p-group. In particular, if G is finitely generated then G/N is a finite p-group.*

In other words, this theorem implies that any torsion free nilpotent group is residually a locally finite p-group with cyclic center for any prime p.

The fact that we obtain shorter proofs of some known results leads us to pose the following problem:

Which classes of groups can be defined by a non-trivial partition?

For example can soluble groups be defined by a non-trivial partition? If this were the case, could it be used to characterize groups of odd order? For more results on groups which are the union of certain subsets see [3] and its list of references.

In section two we prove some properties of (X)-groups. In the last section we prove our main results. Notation is standard largely following that of [4]. In particular, the term generalized central series follows the definition given in [4].

1. Some properties of (X)-groups

In this section we shall prove some properties of (X)-groups. It is easy to see that condition (X) is inherited by homomorphic images (see [1]). We state this as a lemma leaving the proof to the reader.

Lemma 1. *If a group G satisfies property (X) then so does every homomorphic image of G.*

Lemma 2. *Let G be an (X)-group. For every $x \in G$ the following holds:*

1. *$x \notin [x, G]$ and hence G is residually central.*
2. *There exists $N \lhd G$ with $x \notin N$ but $x^p \in N$ for some rational prime number p. Moreover the intersection of all normal subgroups of G/N is $\langle xN \rangle$, and the center $Z(G/N)$ is locally cyclic.*

Proof. If $x \in [x, G]$ then $1 \in x[x, G]$ and thus $x = 1$. Using item one and Zorn's Lemma we can find a maximal normal subgroup N of G subject to the condition that $x \notin N$ and $[x, G] \subset N$. For every $yN \in Z(G/N)$ with $y \notin N$ it is clear that $x \in \langle y \rangle N$; in particular, for every non zero integer k with $x^k \notin N$ we have $x \in \langle x^k \rangle N$. Hence, the order of xN in G/N is finite and prime. That $Z(G/N)$ is locally cyclic is a consequence of the fact that it is locally a finite abelian group with an element xN belonging to every proper subgroup. □

The normal subgroup N of item 2 need not be maximal. In fact if G is nilpotent then its Frattini subgroup is nontrivial. Hence, taking

$x \in Frat(G)$ the normal subgroup N can not be maximal because x belongs to all maximal subgroups of G.

By Lemma 2, first item, a minimal normal subgroup of an (X)-group must be central and hence, by Lemma 1 and [4] -12.4.1, every (X)-group has a generalized central series and all chief factors are central. Note that if G is finite this reasoning implies that G is nilpotent (see also [2]). A very different proof ot this last result is given in [1]. Our next result generalizes this. It is weaker than the theorem presented in [6], page 7 (the residually central groups satisfying min-n are the hypercentral Chernikov groups). On the other hand its proof has the advantage of being very simple directly applying condition (X).

Theorem 3. *Let G be an (X)-group and suppose that G satisfies min-n. Then G is hypercentral, hence locally nilpotent and by consequence it is the product of finitely many Chernikov p-groups for various primes p.*

Proof. Let $x \in G$. If $y \in [x, G]$ then $[y, G]$ is a proper subgroup of $[x, G]$. For if not then $y \in [x, G] = [y, G]$ which contradicts Lemma 2. Hence we can construct a strictly descending chain $([x_n, G])_{n \in \mathbf{N}}$. Using the hypothesis of minimality this chain shall reach the identity and thus there exists $n \in \mathbf{N}$ such that x_n is a central element. This proves that G has nontrivial center. Since homomorphic images of G also satisfies the minimal condition on normal subgroups it follows from Lemma 1 that all homomorphic images of G have non-trivial center, which is a necessary and suficient condition for G to be hypercentral (see [5], page 50). By [[4], 12.2.4] G is locally nilpotent and thus, by [[4], 12.1.8], G is the direct product of finitely many Chernikov p-groups for various primes p. □

Theorem 4. *Let G be a finitely generated (X)-group. G is nilpotent if and only if G is soluble.*

Proof. Suppose that G is a finitely generated soluble (X)-group. If G were not nilpotent then, by a result of Robinson and Wehrfritz [[4], 15.5.3], G would have a finite non-nilpotent image which contradicts the fact that a finite (X)-group must be nilpotent. □

Theorem 5. *Let G be an (X)-group. A maximal subgroup M of G has either infinite or prime index. Moreover M is normal if and only if $[G : M]$ is finite.*

Proof. Let M be a maximal subgroup of finite index and let $N = M_G$ be the core of M in G. Then N is normal subgroup and of finite index in G (see [[4], 1.6.9]). By Theorem 3 G/N is nilpotent and hence, by [[4], 5.4.7], M is normal and of prime index in G. Now let $M \lhd G$ be a normal maximal subgroup. Suppose that G/M is not of prime order. Then, by Lemma 2, G/M is non-simple. But this contradicts the maximality of M and thus G/M has prime index. □

Corollary 1. *Let G be a locally nilpotent group and M a maximal subgroup of G. Then M is normal and has prime index in G.*

Proof. By Theorem A of [1] G is an (X)-group and, by [[4], 12.1.5], all maximal subgroups of G are normal. The result now follows from Theorem 5. □

2. Proofs of the Theorems

We start this section stating a result whose proof is straightforward.

Lemma 3. *The following conditions on a group G are equivalent.*

1. *G is an (X)-group.*
2. *for all x and y in G, $x \in y[y, G]$ implies $[x, G] = [y, G]$.*
3. *for all x and y in G, $x \in [y, G]$ implies $[yx, G] = [y, G]$.*

In any group G, $x \in y[y, G]$ implies $[x, G] \subset [y, G]$. Therefore, in order to prove that G satisfies (X) it suffices to show that $x \in y[y, G]$ implies $[x, G] \supset [y, G]$ for any x and y in G.

Proof of Theorem 1. **(1) implies (2).** This follows by Lemma 1 and item 1 of Lemma 2.
(2) implies (3). By Lemma 1, it is enough to show that G is a Z-group. This is an obvious consequence of the fact that every group has a generalized normal series and that every normal series can be refined to a composition series (see [4] or [6]).
(3) implies (4). See [6], page 6.
(4) implies (1). Suppose G satisfies (4) and consider $x, y \in G$ such that $x \in y[y, G]$. It is enough to show that $[y, G] \subset [x, G]$.

The bar denotes the image in $\overline{G} = G/[x, G]$. The fact that $\overline{x} \in Z(\overline{G})$ implies that $[\overline{y^{-1}x}, \overline{G}] = [\overline{y^{-1}}, \overline{G}] = [\overline{y}, \overline{G}]$. Since $x \in y[y, G]$, we have that

$$\overline{y^{-1}x} \in [\overline{y}, \overline{G}] = [\overline{y^{-1}x}, \overline{G}].$$

But $\overline{G}$ is residually central and so $\overline{y^{-1}x} = \overline{1}$, that is, $y \in x[x, G]$. Therefore we have $[y, G] \subset [x, G]$ and this concludes the proof. □

To finish this section we proceed toward the proof of the extension of Gruenberg's result. We first make some observations which we shall use during the proof.

Let G be a group and denote by $\gamma_n(G)$ the terms of the lower central series of G. Let $x \in \gamma_n(G)$. If $\langle x \rangle \bigcap [x, G] = \langle x^{k_1} \rangle \neq 1$ then $[x^{k_1}, G] \subset \gamma_{n+1}(G)$. So, if G is nilpotent then there exists k such that $\langle x^k \rangle \bigcap [x^k, G] = 1$. Note that the same holds if we substitute nilpotency by min-n. In particular this is the case when G is a Chernikov group.

Proof of Theorem 2. Let G be a torsion free nilpotent group, $x \in G$ and p a prime. We will prove that there is $N \lhd G$ such that $x \notin N$ and G/N is a locally finite p-group.

By the remark above, there exists $k \in \mathbf{N}$ such that $\langle x^k \rangle \bigcap [x^k, G] = 1$. Consider the family $\mathcal{F} = \{N \lhd G : x^k \notin N, \ \langle [x^k, G], x^{pk} \rangle \subset N\}$ and the subgroup $N_0 = \langle [x^k, G], x^{pk} \rangle$. We claim that $N_0 \in \mathcal{F}$. In fact working with the quotient $G/[x^k, G]$ it is clear that N_0 is a normal subgroup of G and since G is torsion free and $\langle x^k \rangle \bigcap [x^k, G] = 1$ it follows easily that $x^k \notin N_0$.

Clearly $\mathcal{F}$ is inductive and thus Zorn's Lemma guarantees the existence of a maximal element N. Set $\overline{G} = G/N$. As in the proof of Lemma 2 it follows that the center of $\overline{G}$ is locally a cyclic p-group. In particular if G is finitely generated then $\overline{G}$ is finite and we obtain Gruenberg's result (see [4] - 5.2.22 or [5] - Theorem 2.24)). □

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