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Frattini theory for N-Lie algebras

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Abstract. We develop a Frattini Theory for n-Lie algebras by extending theorems of Barnes' to the n-Lie algebra setting. Specifically, we show some sufficient conditions for the Frattini subalgebra to be an ideal and find an example where the Frattini subalgebra fails to be an ideal.

1. Introduction

Frattini theory has been studied extensively and has a rich history both in group theory and later in Lie algebras. The Frattini subalgebra is the intersection of all proper maximal subalgebras and we shall denote it $\phi(A)$. In 1967, Barnes proved the following theorem for the Frattini subalgebra: if $B, C \lhd A$ where $C \subset \phi(A) \bigcap B$ and B/C is nilpotent, then B is nilpotent [4 p.348 Theorem 5]. A corollary to this is the following: if $\phi(A) \lhd A$, then $\phi(A)$ is nilpotent. This theorem raises an obvious question: when is $\phi(A) \lhd A$? For groups it is known that $\phi(A) \lhd A$ always holds because all automorphisms permute maximal subgroups. Barnes and Chao [5 p.233 Theorem 3] proved that A is a nilpotent Lie algebra if and only if $\phi(A) = A^2$. If A is nilpotent, clearly $\phi(A) = A^2 \lhd A$. In 1968, Barnes strengthened this statement and proved that if A is a solvable Lie algebra, then $\phi(A) \lhd A$ [2 p.348 Lemma 3.4]. One might believe $\phi(A) \lhd A$ is true in general for Lie algebras, but over $\mathbb{F}_2 = \{0, 1\}$ if A is the cross product Lie algebra, then $\phi(A) \not\lhd A$.

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The purpose of this paper is to develop a Frattini theory for n-Lie algebras. Where an n-Lie algebra, as introduced by Filipov [6], is an algebra equiped with an n-linear, skew symmetric n-ary bracket that satisfies the following Jacobi-like identity:

$$[[x_1,\dots,x_n],y_2,\dots,y_n]=\sum_{i=1}^{n}[x_1,\dots,x_{i-1},[x_i,y_2,\dots y_n],x_{i+1},\dots,x_n]$$

We also recall the following definitions given by Filipov for A an n-Lie algebra:

Derivations

If δ is a linear transformation such that

$$([a_1,\dots,a_n])\delta=\sum_{i=1}^{n}[a_1,\dots,(a_i)\delta,\dots,a_n]$$

for all $a_j \in A$ then δ is a *derivation* of A.

Right Multiplications

$R_{(y)} = [\_\_, y_2, \dots, y_n]$ where (y) will always denote the set $y_2, \dots, y_n$ of $n-1$ vectors right justified in the n-bracket. Example: $xR_{(y)} = [x, y_2, \dots, y_n]$.

R(A)

$R(A)$ is the vector space generated by all right multiplications of A.

In [7 Theorem 2.2] we proved for A an n-Lie algebra, A is nilpotent if and only if $\phi(A) = A^2$ which coresponds to Barnes' and Chao's work. In this paper we will prove the n-Lie algebra version of Barnes' Frattini theorem and continue to examine when $\phi(A) \lhd A$ for n-Lie algebras. We will do so by proving the following three theorems:

Theorem 1. *Let A be an n-Lie algebra. If $B, C \lhd A$ where $C \subset \phi(A) \bigcap B$ and B/C is nilpotent, then B is nilpotent.*

Theorem 2. *If A is a solvable n-Lie algebra, then $\phi(A) \lhd A$.*

Theorem 3. *If A is the n-Lie cross product over $\mathbb{F}_2$, then $\phi(A) \ntriangleleft A$.*

2. Proof of Theorem 1

We will prove the theorem by contradiction. Assume that the hypothesis holds but B is not nilpotent. By Engel's theorem there exists an $R \in R(B)$ such that $BR^s \neq 0$ for all s. We apply Fitting's lemma. Let I be the final image and K be the Fitting null component. Recall that $A = I \bigoplus K$ and K is a subalgebra of A. Since $B \lhd A$, we observe that $I \subset B$. Furthermore, since B/C is nilpotent, $I \subset C \subset \phi(A)$. Now since $A = I \bigoplus K$ we see that $A = \phi(A) + K$. Since K is a subalgebra it is contained in some maximal subalgebra M and $A = \phi(A) + M$. Since M is a maximal subalgebra, $\phi(A) \subset M$ and $M = A$ but this is a contradiction as M is a maximal subalgebra. This proves Theorem 1. □

Corollary 1. *If $D \lhd A$ and $D \subset \phi(A)$, then D is nilpotent.*

Corollary 2. *If $\phi(A) \lhd A$, then $A/\phi(A)$ is nilpotent if and only if A is nilpotent.*

The proof of these corollaries follow from standard Lie algebra arguments using Theorem 1.

3. Proof of Theorem 2

We begin the proof of the Theorem 2 with the following lemmas:

Lemma 1. *If I is a minimal ideal of A a solvable n-Lie algebra, then*

1) $[I, I, A \ldots, A] = 0$

2) *$I \bigcap M = 0$ or $I \bigcap M = I$ for all M maximal subalgebras of A.*

Proof. First we show 1. We observe

$$\begin{aligned}
&[[I, I, A, \ldots, A], A, \ldots, A] = \\
&\qquad = [[I, A, A, \ldots, A], I, \ldots, A] + [I, [I, A, \ldots, A]A, \ldots, A] + \\
&\qquad\qquad + [I, I, [A, \ldots, A], A, \ldots, A] \subset [[I, I, A, \ldots, A]A, \ldots, A]
\end{aligned}$$

and $[I, I, A, \ldots, A] \lhd A$. Since A is solvable, $[I, I, A, \ldots, A]$ is properly contained in I. Indeed, if $I = [I, I, A, \ldots, A]$, we can prove that $I = [I, I, A, \ldots, A] \subset A^{(n)}$ for all n. We do so inductively. Obviously $[I, I, A, \ldots, A] \subset A^{(2)}$ and taking the inductive step, we assume that $I = [I, I, A, \ldots, A] \subset A^{(n)}$. Then $I = [I, I, A, \ldots, A] \subset [A^{(n)}, A^{(n)}, A, \ldots, A] = A^{(n+1)}$ and as a result $I = [I, I, A, \ldots, A] \subset A^{(n)}$ for all n. Since A is solvable, I must be 0 contradicting the minimality condition of I. Hence,

$[I, I, A, \ldots, A]$ is properly contained in I. But the only way this can happen is if $[I, I, A, \ldots, A] = 0$ otherwise we will again contradict the minimality condition of I. This proves 1.

Now we show 2. Assume that $I \bigcap M$ is properly contained in I. Since I is not contained in M we observe M is properly contained in $I+M$ and $I+M$ is a subalgebra. The only way this can happen is if $I+M = A$. Now using 1 we observe that

$$[I \bigcap M, A, \ldots, A] = [I \bigcap M, M+I, \ldots, M+I] = \\ = [I \bigcap M, M, \ldots, M] + 0 + \ldots, +0 \in I \bigcap M$$

hence $I \bigcap M \lhd A$. As a result $I \bigcap M = 0$, otherwise we contradict the minimality of I. This proves the lemma. □

Lemma 2. *Let D be a nilpotent derivation of A, an n-Lie algebra over a field $\mathbb{F}$ and $D^{m+1} = 0$. Then $\exp(D) = \sum_{i=0}^{m} \frac{D^i}{i!}$ is an automorphism of A under the following field considerations: either char$(\mathbb{F}) = 0$ or, if char$(\mathbb{F}) = p \neq 0$ and $D^k = 0$ for some minimal k, then $k < \frac{p-1}{2}$.*

Proof. By Leibnitz's rule for n-Lie algebras we see that

$$(*) \left[x_1, x_2, \ldots, x_n\right] \frac{D^k}{k!} = \\ = \frac{1}{k!} \sum_{i_1+\ldots+i_n=k} \binom{k}{i_1, i_2, \ldots, i_n} [x_1 D^{i_1}, x_2 D^{i_2}, \ldots, x_n D^{i_n}] = \\ = \sum_{i_1+\ldots+i_n=k} \left[\frac{x_1 D^{i_1}}{i_1!}, \frac{x_2 D^{i_2}}{i_2!}, \ldots, \frac{x_n D^{i_n}}{i_n!}\right].$$

Hence

$$[\exp D(x_1), \exp D(x_2), \ldots, \exp D(x_n)] = \\ = \left[\sum_{i_1=1}^{m} \frac{x_1 D^{i_1}}{i_1!}, \sum_{i_2=1}^{m} \frac{x_2 D^{i_2}}{i_2!}, \ldots, \sum_{i_n=1}^{m} \frac{x_n D^{i_n}}{i_n!}\right] = \\ = \sum_{i_1,\ldots,i_n=0}^{m} \left[\frac{x_1 D^{i_1}}{i_1!}, \frac{x_2 D^{i_2}}{i_2!}, \ldots, \frac{x_n D^{i_n}}{i_n!}\right] = \\ = \sum_{k=1}^{nm} \sum_{i_1+\ldots+i_n=k} \left[\frac{x_1 D^{i_1}}{i_1!}, \frac{x_2 D^{i_2}}{i_2!}, \ldots, \frac{x_n D^{i_n}}{i_n!}\right] = \\ = \sum_{k=1}^{nm} \frac{[x_1, x_2, \ldots, x_n] D^k}{k!} = \sum_{k=1}^{m} \frac{[x_1, x_2, \ldots, x_n] D^k}{k!} =$$

$$= \exp D([x_1, x_2, \ldots, x_n])$$

Hence $\exp(D)$ is an n-Lie algebra homomorphism. Furthermore

$$(\exp(D))(\exp(-D)) = (\exp(D))\left(\sum_{i=0}^{m} \frac{(-D)^i}{i!}\right) =$$
$$= \sum_{i,j=0}^{m} \frac{D^j(-D)^i}{j!i!} = \sum_{k=0}^{2m} \frac{1}{k!} \sum_{i=0}^{k} \binom{k}{i} D^i(-D)^{k-i} =$$
$$= \sum_{k=0}^{2m} \frac{1}{k!}(D-D)^k = I$$

Similarly, $(\exp(-D))(\exp(D) = I$. Hence $\exp(-D) = (\exp(D))^{-1}$ and $\exp(D)$ is an automorphism. This proves Lemma 2. □

Proof of Theorem 2

The proof follows closely that of Barnes' [2 p.348 Lemma 3.4] analogous proof for Lie Algebras. We induct on $dim(A)$. Let I be a minimal ideal of A and let ϕ_I denote the intersection of all maximal subalgebras M such that $I \subset M$.

Then

$$\phi_I/I = (\bigcap_{M|I\subset M} M)/I = \bigcap_{M|I\subset M} M/I = \phi(A/I).$$

By the induction hypothesis we have that $\phi_I/I \lhd A/I$ and hence $\phi_I \lhd A$. If $I \subset \phi(A)$, then $\phi(A) = \phi_I \lhd A$ and we're done.
From here the proof can be broken up and conducted in two cases:

1) I is not a subset of $\phi(A)$ and $C_A(I) \neq I$.

2) I is not a subset of $\phi(A)$ and $C_A(I) = I$.

We define $C_A(I) = \{x \in A | [x, I, A, \ldots, A] = 0\}$ and call it *the centralizer of I in A*.

Note that since $I \lhd A$, then $C_A(I) \lhd A$. Indeed, for $x \in C_I(A)$ we see by using the Jacobian property of n-Lie algebras that

$$[[x, A, \ldots, A], I, \ldots, A] = [[x, I, A \ldots, A], A, \ldots, A] +$$
$$+ [x, [A, I, \ldots, A], A \ldots, A] = 0 + [x, I, A \ldots, A] = 0 + 0.$$

Hence $C_A(I) \lhd A$. We now resume the proof.
Case 1
Suppose I is not a subset of $\phi(A)$ and $C_A(I) \neq I$. Due to this assumption

there exists M, a maximal subalgebra of A such that $I \bigcap M$ is a proper subset of I. By our Lemma 1 $I \bigcap M = 0$. Recall that $C_A(I) = \{x \in A | [x, I, A, \ldots, A] = 0\}$. Recall also that if $I \lhd A$, then $C_A(I) \lhd A$. Suppose $I \subset C_A(I)$, then $C_A(I) \bigcap M \neq 0$ and

$$[C_A(I) \bigcap M, A, \ldots, A] = [C_A(I) \bigcap M, M + I, \ldots, M + I] =$$
$$= [C_A(I) \bigcap M, M, \ldots, M] + 0 + \ldots, +0 \subset C_A(I) \bigcap M$$

because $C_A(I) \lhd A$ and M is a subalgebra. Hence $C_A(I) \bigcap M \lhd A$. As a result for each M a maximal subalgebra there exists $J \subset M$ where J is a minimal ideal of A.

We see that

$$\phi(A) = \bigcap M = \bigcap_{J \subset M} \phi_J \lhd A.$$

Case 2
Suppose $C_A(I) = I$ and I is not contained in $\phi(A)$. We will show that $\phi(A) = 0$. As in Case 1, due to this assumption and Lemma 1 there exists M, a maximal subalgebra of A such that $I \bigcap M = 0$. For all $m \in M$ we prove that $m \notin \phi(A)$. Since $m \notin I = C_A(I)$ there exists $i \in I$ and $a_i's \in A$ such that $[m, i, a_3, a_4, \ldots, a_n] = mR_a \neq 0$. Note that R_a is a derivation on I. Since $[I, I, A \ldots, A] = 0$ we see that $R_a^2 = 0$ and by Lemma 2, $\exp R_a = 1 + R_a$ is an automorphism of A . Set $N = \exp R_a(M)$ a maximal subalgebra. If $m \in N$ then $m = n(1 + R_a) = n + nR_a$ for some $n \in M$. Since $n, m \in M$ we see that $m - n = nR_a \in M$. Hence $nR_a \in I \bigcap M = 0$ and $m = n$. But this means that $mR_a = nR_a = 0$ which contradicts the fact that $mR_a \neq 0$. This implies that $M \bigcap N = 0$ and in turn that $\phi(A) = 0 \lhd A$. This proves the theorem. □

4. Proof of Theorem 3

As A is simple, it is enough to show that $\phi(A) \neq 0$ and $\phi(A) \neq A$. This is due to the following fact: a subspace $S \subset A$ of codimension 1 is a subalgebra if and only if

$$v = \sum_{i=1}^{n+1} x_i \in S.$$

Let's prove this fact. Note that S has a basis of the form

$$\{x_i + \lambda_i x_{n+1} | \lambda_i \in \mathbb{F}_2\}_{i=1}^{n}$$

and $x_1, \dots, x_n$ is the standard basis. This can be easily shown by induction on n.

Indeed if $n = 2$ and $\{v_1, v_2\} = \{v_1, x_1 + x_2 + x_3\}$ is a basis for S, then $\{v_1, v_1 + x_1 + x_2 + x_3\}$ is as well and since $0 \neq v_1 \neq x_1 + x_2 + x_3$, we see that $v_1 + v_2 = \sum_{i=1}^{3} \lambda_i x_i$ where at least one λ_i is zero.

Now we induct on n. We consider A_{n-1} the $(n-1)$-Lie algebra defined by

$$[v_{i_1}, \dots, \widehat{v_{i_j}}, \dots, v_{i_{n-1}}]_{n-1} = [v_{i_1}, \dots, \widehat{v_{i_j}}, \dots, v_{i_{n-1}}, v_n] = v_{i_j}$$

where $i_k \neq n$ for all k. By the induction hypothesis, $\{x_i + \lambda_i x_{n+1}\}_{i=1,\neq n}^{n+1}$ is a basis for A_{n-1} and in turn, $\{x_i + \lambda_i x_{n+1}, v_n\}_{i=1,\neq n}^{n+1}$ is a basis for A. We note that

$$v_n = \sum_{i=1,\neq n}^{n+1} t_i x_i + x_n,$$

otherwise we do not have a basis. We observe

$$v_n - \sum_{i=1,\neq n}^{n+1} t_i(x_i + \lambda_i x_{n+1}) = x_n + \sum_{i=1,\neq n}^{n+1} t_i \lambda_i x_{n+1}.$$

Replacing v_n with the right hand side above, we obtain $\{x_i + \lambda_i x_{n+1}\}_{i=1}^{n}$ as a basis for A where $\lambda_n = \sum_{i=1,\neq n}^{n+1} t_i \lambda_i$. Using this basis, we note the only non-zero product is

$$w = [x_1 + \lambda_1 x_{n+1}, \dots, x_n + \lambda_n x_{n+1}] = x_{n+1} + \sum_{i=1}^{n} \lambda_i x_i.$$

But $w \in S$ if and only if $w = \sum_{i=1}^{n} t_i(x_i + \lambda_i x_{n+1})$ if and only if $\lambda_i = t_i$, for all i and $\sum_{i=1}^{n} t_i \lambda_i = 1$. As a result

$$\sum_{i=1}^{n} t_i \lambda_i = \sum_{i=1}^{n} \lambda_i^2 = \sum \lambda_i = 1.$$

Hence $w \in S$ if and only if $\sum_i \lambda_i = 1$.

On the other hand, $v = \sum_{i=1}^{n+1} x_i \in S$ if and only if $v = \sum_{i=1}^{n}(x_i + \lambda_i x_{n+1})$ which is equivalent to the fact that $\sum_i \lambda_i = 1$. This completes the proof and the paper. □

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