Algebra and Discrete Mathematics
Volume **22** (2016). Number 1, pp. 1–10

RESEARCH ARTICLE

On colouring integers avoiding t-AP distance-sets

Tanbir Ahmed

Communicated by V. A. Artamonov

Abstract. A t-AP is a sequence of the form $a, a+d, \ldots, a+(t-1)d$, where $a, d \in \mathbb{Z}$. Given a finite set X and positive integers d, t, $a_1, a_2, \ldots, a_{t-1}$, define $\nu(X, d) = |\{(x, y) : x, y \in X, y > x, y - x = d\}|$, $(a_1, a_2, \ldots, a_{t-1}; d) =$ a collection X s.t. $\nu(X, d \cdot i) \geqslant a_i$ for $1 \leqslant i \leqslant t-1$.

In this paper, we investigate the structure of sets with bounded number of pairs with certain gaps. Let $(t-1, t-2, \ldots, 1; d)$ be called a *t-AP distance-set* of size at least t. A k-colouring of integers $1, 2, \ldots, n$ is a mapping $\{1, 2, \ldots, n\} \to \{0, 1, \ldots, k-1\}$ where $0, 1, \ldots, k-1$ are colours. Let $ww(k, t)$ denote the smallest positive integer n such that every k-colouring of $1, 2, \ldots, n$ contains a monochromatic t-AP distance-set for some $d > 0$. We conjecture that $ww(2, t) \geqslant t^2$ and prove the lower bound for most cases. We also generalize the notion of $ww(k, t)$ and prove several lower bounds.

1. Introduction

A t-AP is a sequence of the form $a, a+d, \ldots, a+(t-1)d$, where $a, d \in \mathbb{Z}$. For example, $3, 7, 11, 15$ is a 4-AP with $a = 3$ and $d = 4$.

Given a finite set X and positive integers d, t, $a_1, a_2, \ldots, a_{t-1}$, define

$$\nu(X, d) = |\{(x, y) : x, y \in X, y > x, y - x = d\}|,$$
$$(a_1, a_2, \ldots, a_{t-1}; d) = \text{a collection } X \text{ s.t. } \nu(X, d \cdot i) \geqslant a_i \text{ for } 1 \leqslant i \leqslant t-1.$$

2010 MSC: Primary 05D10.
Key words and phrases: distance sets, colouring integers, sets and sequences.

The t-AP $\{x, x+d, \ldots, x+(t-1)d\}$ (say T) has $\nu(T, d \cdot i) = t - i$ for $1 \leqslant i \leqslant t-1$. On the other hand, a set $(t-1, t-2, \ldots, 1; d)$ (say Y) has $\nu(Y, d \cdot i) \geqslant t - i$ for $1 \leqslant i \leqslant t-1$, but not necessarily contains a t-AP.

A k-colouring of integers $1, 2, \ldots, n$ is a mapping $\{1, 2, \ldots, n\} \to \{0, 1, \ldots, k-1\}$ where $0, 1, \ldots, k-1$ are colours. Let $ww(k,t)$ denote the smallest integer n such that every k-colouring of $1, 2, \ldots, n$ contains a monochromatic set $(t-1, t-2, \ldots, 1; d)$ for some $d > 0$. Here, $(t-1, t-2, \ldots, 1; d)$ is a *t-AP distance-set* of size at least t. The existence of $ww(k,t)$ is guaranteed by van der Waerden's theorem [1]. Given positive integers k, t, and n, a *good k-colouring* of $1, 2, \ldots, n$ contains no monochromatic t-AP distance set. We call such a good k-colouring, a *certificate* of the lower bound $ww(k,t) > n$. We write a certificate as a sequence of n colours each in $\{0, 1, \ldots, k-1\}$, where the i-th ($i \in \{1, 2, \ldots, n\}$) colour corresponds to the colour of the integer i.

A certificate of lower bound $ww(k,t) > n$ that avoids a monochromatic arithmetic progression, may still be invalid, since it may contain a monochromatic distance set. For example, while looking for a certificate of lower bound of $ww(2,4)$, if the set $X = \{1, 2, 3, 5, 9, 10\}$ (which does not contain a 4-AP) is monochromatic, then the colouring is "bad" as $\nu(X,1) = 3$, $\nu(X,2) = 2$, and $\nu(X,3) = 1$.

In this paper, we perform computer experiements to observe the patterns of certificates of $ww(k,t) > n$. We conjecture that $ww(2,t) \geqslant t^2$ and prove the lower bound for most cases. We also generalize the notion of $ww(k,t)$ and provide several lower bounds.

2. Some values and bounds

With a primitive computer search algorithm, we have computed the following values and bounds of $ww(k,t)$. Theorem 1 gives a lower bound for $ww(k,t)$. A computed lower bound is presented only if it improves over the bound given by Theorem 1.

Theorem 1. *Given $k \geqslant 2, t \geqslant 3$, if $t \leqslant 2k+1$, then*

$$ww(k,t) > k(t-1)(t-2).$$

Proof. Let $n = k(t-1)(t-2)$ and consider the colouring

$$f : \{1, 2, \ldots, n\} \to \{0, 1, \ldots, k-1\}.$$

Let $X_i = \{x \in X : f(x) = i\}$. Take the certificate

$$(0^{t-1}1^{t-1} \cdots (k-1)^{t-1})^{t-2}.$$

Table 1. $ww(k,t)$

k/t	3	4	5	6	7	8
2	9	19	33	43	64	85
3	17	39	> 56	> 67	> 97	> 121
4	33	$\geqslant 70$	> 85	> 102	> 134	
5	$\geqslant 44$	> 86	> 135	> 141	> 181	> 242
6	> 56	> 106	> 175	> 221	> 254	> 287
7	> 73	> 142	> 214	> 278	> 298	> 380
8	> 91	> 168	> 246	> 338	> 390	> 484
9	> 115	> 198	> 302	> 398	> 478	> 567
10	> 127	> 233	> 365	> 464	> 558	> 691
11	> 146	> 275	> 417	> 581	> 672	> 806
12	> 157	> 315	> 474	> 649	> 769	> 927
13	> 174	> 337	> 550	> 760	> 840	> 1085
14	> 198	> 405	> 594	> 828	> 949	> 1220
15	> 229	> 434	> 666	> 904	> 1087	> 1334
16	> 230	> 493	> 784	> 1015	> 1236	> 1517
17	> 270	> 525	> 849	> 1082	> 1375	> 1676
18	> 298	> 589	> 932	> 1211	> 1509	> 1841
19	> 337	> 629	> 988	> 1338	> 1635	> 2027
20	> 348	> 689	> 1098	> 1445	> 1850	> 2249
21	> 364	> 756	> 1179	> 1561	> 2014	> 2487
22	> 401	> 824	> 1288	> 1701	> 2153	> 2632
23	> 422	> 890	> 1354	> 1868	> 2249	> 2820
24	> 476	> 948	> 1459	> 1952	> 2563	> 3107
25	> 500	> 1033	> 1592	> 2125	> 2746	> 3284

We show that for each d, there exists j with $1 \leqslant j \leqslant t-1$ such that $\nu(X_i, d \cdot j) < t - j$ for each $i \in \{0, 1, \ldots, k-1\}$. The largest difference between two monochromatic numbers in the certificate is

$$n - (k-1)(t-1) - 1 = (t-1)(k(t-2) - k - 1) - 1 < k(t-1)(t-3).$$

Since the existence of a monochromatic set $(t-1, t-2, \ldots, 1; d)$ in X requires $\nu(X_i, d \cdot (t-1)) \geqslant 1$, we have $d < k(t-3)$. We have the following cases:

(a) $1 \leqslant d \leqslant k-1$: Take $x, y \in \{1, 2, \ldots, n\}$ such that $y = x + d(t-1)$. But by our choice of d, we have $f(y) = (f(x) + d) \mod k \neq f(x)$, that is, x and y cannot be monochromatic. So, $\nu(X_i, (t-1) \cdot d) = 0 < 1$ for each $i \in \{0, 1, \ldots, k-1\}$.

(b) $k \leqslant d \leqslant t-3$: Take $x, y \in \{1, 2, \ldots, n\}$ such that $y = x + d(t-a)$ where a is such that $(t-3)(t-a) \leqslant (t-1)(k-1)$ and $k(t-a) \geqslant k$, which gives us a bound

$$t - \frac{(t-1)(k-1)}{t-3} \leqslant a \leqslant t - k.$$

Such an a exists since

$$\frac{(t-1)(k-1)}{t-3} = k - 1 + \frac{2(k-1)}{t-3} \geqslant k - 1 + \frac{2(k-1)}{(2k+1)-3} \geqslant k.$$

(c) $d = t-2$: In each block of $t-1$ colours, there is one pair of integers at distance $t-2$, and there are $t-2$ such blocks for each colour. So, $\nu(X_i, 1 \cdot d) = \nu(X, t-2) = t-2 < t-1$ for each $i \in \{0, 1, \ldots, k-1\}$.

(d) $(t-1) \leqslant d \leqslant (k-1)(t-1)$: Take $x, y \in \{1, 2, \ldots, n\}$ such that $y = x + d = x + q(t-1) + r$ where $1 \leqslant q \leqslant k-1$ and $0 \leqslant r \leqslant t-2$. Suppose $x = q_x(t-1) + r_x$ with $0 \leqslant r_x \leqslant t-2$. Then

$$\begin{aligned} f(x) &= q_x \mod k; \\ f(y) &= (f(x) + q + \lfloor (r + r_x)/(t-1) \rfloor) \mod k. \end{aligned}$$

If $r > 0$, then $q \leqslant k-2$, which implies $q + \lfloor (r+r_x)/(t-1) \rfloor \leqslant (k-2)+1 = k-1$. If $r = 0$, then $q \leqslant k-1$, which implies $q + \lfloor (0 + r_x)/(t-1) \rfloor \leqslant (k-1) + 0 = k-1$. Therefore, $f(y) \neq f(x)$; and x and y cannot be monochromatic. So, $\nu(X_i, d \cdot 1) = 0 < t-1$ for each $i \in \{0, 1, \ldots, k-1\}$.

Since $t \leqslant 2k+1$, we have

$$\begin{aligned} d < k(t-3) = kt - 3k &= (kt - k - 2k) \\ &\leqslant (kt - k - (t-1)) = (k-1)(t-1). \end{aligned}$$

Hence, we are done and there is no monochromatic t-AP distance set in X. □

Conjecture 1. For $t \geqslant 3$, $ww(2,t) \geqslant t^2$.

Lemma 1. *For $t \geqslant 3$ and $t \neq 2^u$ with $u \geqslant 2$, $ww(2,t) \geqslant t^2$.*

Proof. Let $t = 2^u + v$ with $1 \leqslant v \leqslant 2^u - 1$. Let $n = t^2 - 1$ and $X = \{1, 2, \ldots, n\}$, and consider the colouring $f : X \to \{0, 1\}$. Let $m = n - 1 - (t-1) = q \cdot 2^u + r$ with $0 \leqslant r \leqslant 2^u - 1$.

Now, take the certificate

$$\begin{cases} 01^{t-1}(0^{2^u}1^{2^u})^{q/2}0^r, & \text{if } q \equiv 0 \pmod 2; \\ 01^{t-1}(0^{2^u}1^{2^u})^{\lfloor q/2 \rfloor}0^{2^u}1^r, & \text{if } q \equiv 1 \pmod 2. \end{cases}$$

We need to show that for each d, there exists j with $1 \leqslant j \leqslant t-1$ such that $\nu(X, d \cdot j) < t-j$. Since the existence of a monochromatic set $(t-1, t-2, \ldots, 1; d)$ in X requires $\nu(X, d \cdot (t-1)) \geqslant 1$, we have $d(t-1) < t^2-1$, that is, $1 \leqslant d \leqslant t$. Let $X_i = \{x \in X : f(x) = i\}$.

Suppose $q \equiv 0 \pmod 2$. Then we have the following two cases:

(a) $d \equiv 1 \pmod 2$: Take $x, y \in X$ such that $y = x + d \cdot 2^u$.

(a1) $v+1 \leqslant x < y \leqslant n-r$: Since $f(x) \in \{0,1\}$ and $f(x+1 \cdot 2^u) = (f(x)+1) \mod 2 \neq f(x)$, we have $f(y) = f(x+d \cdot 2^u) \neq f(x)$. So, two monochromatic integers cannot both be in $\{v+1, v+2, \ldots, n-r\}$.

(a2) $2 \leqslant x \leqslant v$ and $y \leqslant n-r$: Since $f(x) = 1$ and $f(x+1 \cdot 2^u) = 1 = f(x)$, we have $f(y) = (x + d \cdot 2^u) = f(x)$. So, there are exactly $v-1$ pairs of integers with colour 1 at distance $d \cdot 2^u$.

(a3) $x = 1$ and $y \leqslant n-r$: Since $f(x) = 0$, $f(x+1 \cdot 2^u) = 1 \neq f(x)$, and $1 + 2^u > v$, using case (i) we have 0 pair of integers with colour 0 at distance $d \cdot 2^u$.

(a4) $x \geqslant 1$ and $n-r+1 \leqslant y \leqslant n$: Since $f(y) = 0$ and $r < 2^u$, we have $f(y - 1 \cdot 2^u) = 1$, which implies $f(y - d \cdot 2^u) = 1 \neq f(y)$. That is, adding r trailing zeros does not change the number of monochromatic pairs at distance $d \cdot 2^u$.

Therefore, for each $i \in \{0,1\}$, we have $\nu(X_i, d \cdot 2^u) \leqslant v-1 < v = t-2^u$.

(b) $d \equiv 0 \pmod 2$: Let $d = 2^w \cdot d_o$, with d_o being an odd number and $w \geqslant 1$. Then $\nu(X_i, d \cdot 2^{u-w}) = \nu(X_i, d_o \cdot 2^u) \leqslant v-1 < v = t - 2^u$ (by case (i)) for each $i \in \{0,1\}$.

The case $q \equiv 1 \pmod 2$ is similar. □

Lemma 2. *Suppose $t = 2^u$ for some $u \geqslant 2$ and $t-1$ is prime. Then for each $r \in \{1, 2, \ldots, u\}$ and for each $d_o \in \{1, 3, \ldots, 2^{u-r}-1\}$, there exists $s \in \{1, 2, 3, \ldots, d_o - 1\}$ such that d_o divides $s(t-1) - 2^{u-r}$.*

Proof. Since $t-1$ is prime, we have $gcd(t-1, d_o) = 1$, and hence the linear congruence $(t-1)s \equiv 2^{u-r} \pmod{d_o}$ has a solution. Extended Euclid Algorithm yields $x, y \in \mathbb{Z}$ such that $(t-1) \cdot x + d_o \cdot y = 1$. Then $s = (x \cdot 2^{u-r}) \mod d_o$. Since $d_o \not| x$ and $d_o \not| 2^{u-r}$, we have $s \neq 0$. Hence $s \in \{1, 2, \ldots, d_o - 1\}$. □

Lemma 3. *If $t = 2^u$ for $u \geqslant 2$ with $t-1$ prime, then $ww(2,t) \geqslant t^2$.*

Proof. Take the certificate $0(1^{t-1}0^{t-1})^{t/2}1^{t-2}$. We have the following two cases:

(a) $d \equiv 1 \pmod 2$: Take $x, y \in X$ such that $y = x + d \cdot (t-1)$.

(a1) $2 \leqslant x \leqslant t^2 - t + 1 = n - (t-2)$: Since $f(x) \in \{0,1\}$ and $f(x + 1 \cdot (t-1)) = (f(x)+1) \mod 2 \neq f(x)$, we have $f(y) = f(x + d \cdot (t-1)) \neq f(x)$. So, two monochromatic integers cannot both be in $\{2, 3, \ldots, n-(t-2)\}$.

(a2) $x = 1$ and $y \leqslant n-(t-2)$: Since $f(x) = 0$, $f(x+1\cdot(t-1)) = 1 \neq f(x)$, using case (i) we have 0 pair of integers with colour 0 at distance $d \cdot (t-1)$.

(a3) $x \geqslant 1$ and $n-(t-2)+1 \leqslant y \leqslant n$: Since $f(y) = 0$, we have $f(y - 1 \cdot (t-1)) = 1$, which implies $f(y - d \cdot (t-1)) = 1 \neq f(y)$. That is, adding $t-2$ trailing ones does not change the number of monochromatic pairs at distance $d \cdot (t-1)$.

Therefore, for each $i \in \{0,1\}$, we have $\nu(X_i, d \cdot (t-1)) = 0 < 1$.

(b) $d \equiv 0 \pmod 2$: For $d \in \{2, 4, \ldots, t\}$ and $j \in \{1, 2, \ldots, t-1\}$,

$$2 \leqslant d \cdot j \leqslant t(t-1) = t^2 - t.$$

For a given $d \in \{2, 4, \ldots, t\}$, we show that there exists (j, w) with $j \in \{1, 2, \ldots, t-1\}$ and $w \in \{1, 3, 5, \ldots, t-1\}$ such that $d \cdot j = w(t-1)-1$. In that case, since $d \cdot j \leqslant t(t-1)$, we have $w \leqslant t$; and also since $d \cdot j$ is even, $w(t-1)$ is odd, which implies w is odd, that is, $w \in \{1, 3, 5, \ldots, t-1\}$.

Let $d = 2^r d_o$ with $1 \leqslant r \leqslant u$ and odd number $d_o \in \{1, 3, \ldots, 2^{u-r}-1\}$ (since $d \leqslant t = 2^u$). For a w to exist and satisfy $d \cdot j = w(t-1)-1$, we need

$$2^r d_o \cdot j = w(t-1) - 1 = wt - (w+1),$$

that is, 2^r divides $w+1$ (since 2^r divides $wt = w2^u$). Let $w = s \cdot 2^r - 1$ with $s \in \{1, 2, \ldots, d_o - 1\}$. The chosen s requires to satisfy that d_o divides $(w(t-1)-1)/2^r = s(t-1) - 2^{u-r}$. By Lemma 2, such an s exists.

It can be observed that for a given $d \in \{2, 4, \ldots, t\}$, if $d \cdot j_1 = w_1 \cdot (t-1) - 1$ for some $j_1 \in \{1, 2, 3, \ldots, t-1\}$ and $w_1 \in \{1, 3, 5, \ldots, t-1\}$, then

$$d \cdot j_2 = w_2 \cdot (t-1) + 1,$$

with $j_1 + j_2 = t-1$ and $w_1 + w_2 = d$. We claim that $\nu(X_1, d \cdot t_i) < t - j_i$ for at least one $i \in \{1,2\}$. If $\nu(X_1, d \cdot j_1) < t - j_1$, then we are done. Suppose

$$\nu(X_1, d \cdot j_1) = \nu(X_1, w_1(t-1)-1) = \frac{t}{2} - \frac{w_1 - 1}{2} \geqslant t - j_1,$$

which implies $t/2 + (w_1-1)/2 \leqslant j_1$. Now,

$$\begin{aligned}\nu(X_1, d \cdot j_2) &= \nu(X_1, w_2(t-1)+1) = \frac{t}{2} - \frac{w_2 - 1}{2} = \frac{t}{2} - \frac{d - w_1 - 1}{2} \\ &= \frac{t}{2} + \frac{w_1 - 1}{2} + 1 - \frac{d}{2} \leqslant j_1 + 1 - \frac{d}{2} = t - 1 - j_2 + 1 - \frac{d}{2} < t - j_2.\end{aligned}$$

Similarly, we can show that $\nu(X_0, d\cdot j) < t - j$ for some $j \in \{1, 2, \ldots, t-1\}$. □

3. Generalized distance-sets

Here we consider variants of $ww(k,t)$ with different variations of parameters in a distance set.

Let $gww(k, t; a_1, a_2, \ldots, a_{t-1})$ (with $a_i \geqslant 1$) denote the smallest positive integer n such that any k-colouring of $1, 2, \ldots, n$ contains monochromatic set $(a_1, a_2, \ldots, a_{t-1}; d)$ for some $d > 0$. In this definition,

$$gww(k, t; t-1, t-2, \ldots, 1) = ww(k, t).$$

Observation 1. Let us write $gww(2, t; a_1, a_2, \ldots, a_{t-1})$ as $gww(2, t, r)$, where $a_i = r$ for $1 \leqslant i \leqslant t-1$. It is trivial that $gww(2, t, t-1) \geqslant ww(2, t)$. Table 2 contains a few computed values of $gww(2, t, r)$.

Table 2. $gww(2, t, r)$

t/r	1	2	3	4	5	6	7	8
3	9	13						
4	13	21	29					
5	33	37	41	49				
6	41	45	57	65	74			
7	49	53	69	85	92	$\geqslant 96$		
8	57	61	85	105	114	> 118	> 123	
9	129	133	137	> 140	> 144	> 148	> 152	> 156
10	145	149	153					
11	161	165	169					
12	177	181	185					
13	193	197	> 200					
14	209	213	> 216					
15	225	229	> 232					
16	241	245	> 248					
17	513	> 516						
33	> 2048	> 2052						
65	> 8192							

Lemma 4. *For $u \geqslant 1$ and $1 \leqslant v \leqslant 2^u$,*

$$gww(2, 2^u + v, 1) \geqslant (2^u + v - 1)2^{u+1} + 1.$$

Proof. Consider $t = 2^u + v$ ($t \geqslant 5$) and let $n = (2^u + v - 1)2^{u+1} = (t-1)2^{u+1}$ and $X = \{1, 2, \ldots, n\}$. Consider the colouring $f : X \to \{0, 1\}$ and take the certificate $(0^{2^u}1^{2^u})^{t-1}$.

Let $X_i = \{x \in X : f(x) = i\}$. We claim that this 2-colouring of X does not contain a monochromatic set $(1, 1, \dots, 1; d)$ for any $d > 0$, that is, for each d with $1 \leqslant d < 2^{u+1}$ and for each $i \in \{0, 1\}$, there exists $j \in \{1, 2, \dots, t-1\}$ such that $\nu(X_i, d \cdot j) = 0$.

(a) $d \equiv 1 \pmod 2$: Take $x, y \in X$ such that $y = x + d \cdot 2^u$. Since d is odd, if $f(x) = 0$, then $f(x + d \cdot 2^u) = 1$ and vice-versa. Hence, $\nu(X_i, d \cdot 2^u) = 0$ for each $i \in \{0, 1\}$

(b) $d \equiv 0 \pmod 2$: Let $d = 2^w \cdot d_o$, with d_o being an odd number and $w \geqslant 1$. Then for each $i \in \{0, 1\}$,

$$\nu(X_i, d \cdot 2^{u-w}) = \nu(X_i, d_o \cdot 2^u) = 0 \text{ (by case } (a)).$$

So, X does not contain a monochromatic set $(1, 1, \dots, 1; d)$ for any $d > 0$. □

Conjecture 2. For $u \geqslant 1$ and $1 \leqslant v \leqslant 2^u$,

$$gww(2, 2^u + v, 1) = (2^u + v - 1)2^{u+1} + 1.$$

Lemma 5. *For $u \geqslant 2$ and $1 \leqslant v \leqslant 2^u$,*

$$gww(2, 2^u + v, 2) \geqslant (2^u + v - 1)2^{u+1} + 5.$$

Proof. Let $t = 2^u + v$ $(t \geqslant 5)$, $n = (2^u + v - 1)2^{u+1} + 4 = (t-1)2^{u+1} + 4$, and $X = \{1, 2, \dots, n\}$. Consider the colouring $f : X \to \{0, 1\}$ and take the certificate $000(10^{2^u-3}1101^{2^u-3}00)^{t-2}(10^{2^u-3}11)(01^{2^u-3})011$. We show that this colouring of X does not contain a monochromatic set $(2, 2, \dots, 2; d)$ for any $d > 0$, that is, for each d with $1 \leqslant d \leqslant 2^{u+1}$ and for each $i \in \{0, 1\}$, there exists $j \in \{1, 2, \dots, t-1\}$ such that $\nu(X_i, d \cdot j) \leqslant 1$.

Note that the largest difference between two integers with colour 0 in the colouring is $(n-2) - 1 = n - 3 = (t-1) \cdot 2^{u+1} + 1 = p$ (say); and the largest difference between two integers with colour 1 in the colouring is $n - 4 = (t-1) \cdot 2^{u+1} = p - 1$.

(a) $d = 2^{u+1}$: Note that $d \cdot (t-1) = (t-1)2^{u+1} = n - 4 = p - 1$. The only pair (x, y) with $f(x) = f(y) = 0$ and $y = x + d \cdot (t-1)$ is $(1, n-3)$ and the only pair (x, y) with $f(x) = f(y) = 1$ and $y = x + d \cdot (t-1)$ is $(4, n)$. Hence, we have $\nu(X_i, d \cdot (t-1)) = 1$ for each $i \in \{0, 1\}$.

(b) $d \equiv 1 \pmod 2$: Write the certificate as

$$000A_0A_1 \dots A_{2t-5}, A_{2t-4}C11,$$

where $A_i = 10^{2^u-3}11$ if $i \equiv 0 \pmod 2$, $A_i = 01^{2^u-3}00$ if $i \equiv 1 \pmod 2$, and $C = 01^{2^u-3}0$. Take $x, y \in \{4, 5, \dots, n - 2^u - 2\}$ such that $y = x + d \cdot 2^u$

for some odd $d < 2^{u+1}+1$. Suppose $x = 3 + i \cdot 2^u + j$, that is, $f(x)$ is the j-th ($1 \leqslant j \leqslant 2^u$) bit in A_i. Then $y = 3 + (i+d) \cdot 2^u + j$, that is, $f(y)$ is the j-th bit in A_{i+d}. If $i \equiv 1 \pmod 2$, then $(i+d) \equiv 0 \pmod 2$ (since d is odd), and vice-versa. Therefore, $f(x) \neq f(y)$. So, two monochromatic integers at distance $d \cdot 2^u$ cannot both be in $\{4, 5, \ldots, n - 2^u - 2\}$.

Now, take $x, y \in \{4, 5, \ldots, n-2\}$ such that $y = x + d \cdot 2^u$ and $y \in \{n - 2^u - 1, n - 2^u, \ldots, n-2\}$. Since $|C| < 2^u$, x must be in $\{4, 5, \ldots, n - 2^u - 2\}$. With similar reasoning as above, it can be shown that two monochromatic integers at distance $d \cdot 2^u$ cannot both be in $\{4, 5, \ldots, n-2\}$. Following are the remaining cases:

(b1) If $x = 1$, then $x + d \cdot 2^u = 3 + d \cdot 2^u - 2 = 3 + (d-1) \cdot 2^u + (2^u - 2) = y$ (say). We have $f(x) = 0$. Again $(2^u - 2)$-th bit in A_{d-1} is also zero since d is odd and $A_{d-1} = 10^{2^u-3}11$.

(b2) If $x = 2, 3$ (where $f(x) = 0$), then with similar reasoning as above, $f(x + d \cdot 2^u) = 1$.

(b3) If $y = n - 1$ (where $f(y) = 1$), then

$$\begin{aligned} y - d \cdot 2^u &= (t-1)2^{u+1} + 3 - d \cdot 2^u \\ &= 3 + (2t - 3 - d)2^u + 2^u = x \text{ (say)}. \end{aligned}$$

Since d is odd, $2t - 3 - d$ is even, that is, $A_{2t-3-d} = 10^{2^u-3}11$. The 2^u-th element in A_{2t-3-d} is one, that is $f(x) = 1$.

(b4) If $y = n$ (where $f(y) = 1$), then

$$\begin{aligned} y - d \cdot 2^u &= (t-1)2^{u+1} + 4 - d \cdot 2^u \\ &= 3 + (2t - 2 - d)2^u + 1 = x \text{ (say)}. \end{aligned}$$

Since d is odd, $2t - 2 - d$ is odd, that is, $A_{2t-2-d} = 01^{2^u-3}00$. The 1-st element in A_{2t-2-d} is zero, that is $f(x) = 0$.

Hence $\nu(X_i, d \cdot 2^u) \leqslant 1$ for each $i \in \{0, 1\}$.

(c) Otherwise ($d \neq 2^{u+1}$ and $d \equiv 0 \pmod 2$): Let $d = 2^w \cdot d_o$, with d_o being an odd number and $w \geqslant 1$. Then for each $i \in \{0, 1\}$,

$$\nu(X_i, d \cdot 2^{u-w}) = \nu(X_i, d_o \cdot 2^u) \leqslant 1 \text{ (by case } (b)).$$

Therefore, X does not contain a monochromatic set $(2, 2, \ldots, 2; d)$ for any $d > 0$. □

Conjecture 3. For $r \geqslant 2$ and $t \geqslant 2^r + 1$,

$$gww(2, t, r) = (t-1)2^{\lfloor \log_2 (t-1) \rfloor + 1} + 2^r + 1.$$

Observation 2. We observe the following experimental results:
(a) Primitive search gives $gww(2,10,9) > 186$;
(b) Using the certificate

$$\begin{cases} 01^{t+1}(0^{t-1}1^{t-1})^{t/2+1}, & \text{if } t \equiv 0 \pmod 2; \\ 01^{t+1}(0^{t-1}1^{t-1})^{\lfloor t/2 \rfloor+1}0^{t-1}, & \text{if } t \equiv 1 \pmod 2, \end{cases}$$

we obtain the following lower bounds with $12 \leqslant t \leqslant 48$:

$$\begin{array}{ccc} gww(2,12,11) > 168, & gww(2,14,13) > 224, & gww(2,17,16) > 323, \\ gww(2,18,17) > 360, & gww(2,20,19) > 440, & gww(2,24,23) > 624, \\ gww(2.30,29) > 960, & gww(2,32,31) > 1088, & gww(2,33,32) > 1155, \\ gww(2,38,37) > 1520, & gww(2,42,41) > 1848, & gww(2,44,43) > 2024. \end{array}$$

(c) $0^{35}(1^{18}0^{18})^{2}1^{20}0^{17}(1^{19}0^{18})^{5}1^{19}0^{17}1^{20}0^{10}1^{4}$ proves $gww(2,19,18) > 399$.
(d) $0^{41}1^{21}0^{21}(1^{22}0^{21})^{10}1^{15}$ proves $gww(2,22,21) > 528$.

Conjecture 4. For $t \geqslant 4$, $gww(2,t,t-1) \geqslant (t+1)^2$.

Conjecture 5. For $t \geqslant 5$, $gww(2,t,t-1) < 2^{t+1}$.

We do not have enough data to make stronger upper bound conjecture for $gww(2,t,t-1)$, but it may be possible that $gww(2,t,t-1) < t^3$.

Acknowledgements

The author would like to thank Hunter Snevily (deceased) for suggesting the problem, Clement Lam and Srecko Brlek for their support, and Andalib Parvez for carefully reading the manuscript.

References

[1] B. L. van der Waerden, Beweis einer Baudetschen Vermutung, *Nieuw Archief voor Wiskunde*, **15** (1927), 212–216.

Contact information

T. Ahmed Laboratoire de Combinatoire et d'Informatique Mathématique, UQAM, Montréal, Canada
E-Mail(s): `tanbir@gmail.com`

Received by the editors: 05.10.2015
and in final form 15.01.2016.