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On the quasi-primary decomposition of HK-torsion theories

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Abstract. The paper is devoted to the study of quasi-primary decompositions of torsion theories in the rings which derivatives. It is shown that every *HK*-torsion theory of the differential noetherian completely bounded ring it is an intersection of finite number of quasi-primary *HK*-torsion theories.

Introduction

Primary decomposition is a presentation of the ideal (or submodule) as an intersection of primary ideals (submodules). Recently differentially prime and primary differential ideals are investigated. In particular, Khadjiev and Çallıalp [8] developed a theory of differentially prime ideals in associative and non-associative differential rings by generalizing a number of results known for associative rings without derivations.

On the other hand, starting with 1970th torsion theory intensively develops over an ordinary rings. Different substitutes of prime ideals appeared within torsion theory. The most famous is the concept, which belongs to Lambeck and Michler, it is constructed by critical modules. This theory, in particular, allows to solve the problem about generalizing the theory of primary decomposition on broader classes of noncommutative rings. Recall a work of Storrer, in which primary decompositions of modules are obtained as a result of application of the technique of atomic and rationally complete modules, which in a way simplifies torsion-theoretic approach.

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In this paper a differentially prime radical is defined as the intersection of all differentially prime differential ideals. The notion of quasi-primary torsion theory in a category of differential modules is introduced; the restriction of such torsions to its full subcategory of differentially uniform modules are investigated. The culminating point is generalization of the theory of quasi-primary decomposition on differentially noetherian torsions in a category of differentially uniform modules [10]. For technical reasons, some properties of #-operator for differential modules are established. Basing on this operator #-filters are studied; they prove to be useful when investigating the differential HK-filters.

All the rings considered in this paper are assumed to be associative with nonzero identity, and all the modules are unitary left modules, unless otherwise specified. The word "ideal" will be used to mean a two-sided ideal. $R-$Mod and $R-$DMod denote the categories of left R-modules and module homomorphisms and left differential R-modules and differential homomorphisms respectively.

Let R be a differential ring with the set of n pairwise commutative derivations $\Delta = \{\delta_1, \delta_2, \ldots, \delta_n\}$ and let M be a left differential module over the differential ring R. The differential structure on the module M is defined by the set $D = \{d_1, d_2, \ldots, d_n\}$ of pairwise commutative module derivations, consistent with the corresponding ring derivations. Assume that at least one of the derivations from the sets Δ and D is nontrivial.

If I is a left ideal of the ring R and $S \subseteq R$ is an arbitrary subset, then the set $(I : S) = \{r \in R | rS \subseteq I\}$ is a left ideal of R. In particular, when $S = \{a\}$, where $a \in R$, $(I : a)$ denotes the left ideal of R given by $\{r \in R | ra \in I\}$. If I is a differential ideal of the differential ring R, then $(I : S)$ and $(I : a)$ are differential ideals.

For $a \in R$, $m \in M$ we use the following notations:

$$a^{(i_1,\ldots,i_n)} = (\delta_1^{i_1} \circ \ldots \circ \delta_n^{i_n})(a), \qquad m^{(i_1,\ldots,i_n)} = (d_1^{i_1} \circ \ldots \circ d_n^{i_n})(a),$$

$$a^{(\infty)} = \{a^{(i_1,\ldots,i_n)} | i_1, i_2, \ldots, i_n \in \mathbb{N} \cup \{0\}\},$$

$$m^{(\infty)} = \{m^{(i_1,\ldots,i_n)} | i_1, i_2, \ldots, i_n \in \mathbb{N} \cup \{0\}\}.$$

For any left differential ideal I and any element $a \in R$ the left ideal $(I : a^{(\infty)})$ is differential and the equality $((I : a^{(\infty)}) : b^{(\infty)}) = (I : (ab)^{(\infty)})$ holds for any $a, b \in R$.

In the paper a standard ring-theoretic terminology will be used, following [2], [9].

1. Operator # and its properties

Recall from [11] that a *differential* of the subset X of the D-module M is a set

$$X_\# = \{x \in M \mid x^{(i_1,i_2,\dots,i_n)} \in X \text{ for all } i_1, i_2, \dots, i_n \in \mathbb{N} \cup \{0\}\}.$$

The operator $(\quad)_\#$ preserves some algebraic structures on subsets of the D-module.

Proposition 1. *Let M and N be D-modules over Δ-ring R and let $f : M \to N$ be a differential module homomorphism. The operator $(\quad)_\#$ on subsets of D-module has the following properties.*

1. *If X is a subset of the D-module M, then $X_\# \subseteq X$ and $(X_\#)_\# = X_\#$.*
2. *If X is a subset of the D-module M, then $X_\# = X$ if and only if the set X is differentially closed in M.*
3. *If X, Y are subsets of the D-module M and $X \subseteq Y$, then $X_\# \subseteq Y_\#$.*
4. *If $\{X_i\}_{i \in I}$ is a family of subsets of M, then*
$$\left(\bigcap_{i\in I} X_i\right)_\# = \bigcup_{i\in I} (X_i)_\# \quad \text{and} \quad \left(\bigcup_{i\in I} X_i\right)_\# = \bigcap_{i\in I} (X_i)_\# .$$
5. *If X, Y are subsets of the D-module M over the δ-ring R, A is a subset of R, then*
$$X_\# + Y_\# = (X+Y)_\# \quad \text{and} \qquad (AX)_\# = A_\# X_\#.$$
6. *If X is an arbitrary subset of the D-module N and $f : M \to N$ is a differential module epimorphism, then $f^{-1}(X_\#) = (f^{-1}(X))_\#$.*
7. *If X is an arbitrary subset of the D-module M and $f : M \to N$ is a differential module homomorphism, then $f(X_\#) \subseteq (f(X))_\#$.*

   If $f : M \to N$ is injective, then the equality $f(X_\#) = (f(X))_\#$ holds.

Proof. See [11]. □

Proposition 2. *If X is an arbitrary subset of the D-module M over the Δ-ring R and $a \in R$ then*

$$\left(X : a^{(\infty)}\right)_\# = \left(X_\# : a^{(\infty)}\right).$$

Proof. Without loss of generality, we may consider ordinary differential rings and modules. Remind that

$$\left(X : a^{(\infty)}\right) = \left\{x \in M \,\middle|\, ax \in X, a'x \in X, a''x \in X, ..., a^{(n)}x \in X\right\}.$$

Denote $a' = \delta(a)$, $x' = d(x)$.

Suppose $x \in \left(X_\# : a^{(\infty)}\right)$. Then $ax \in X_\#$, $a'x \in X_\#$, $a''x \in X_\#$, $\ldots$, $a^{(n)}x \in X_\#$ etc. It follows $\left(a^{(i)}x\right)^{(j)} \in X$, for any $i, j \in \mathbb{N} \cup \{0\}$, in particular $(ax)' = a'x + ax' \in X$. But $a'x \in X$, so $ax' \in X$. From $(a'x)' = a''x + a'x' \in X$ we have $a'x' \in X$, so $(ax)'' \in X$ implies $ax'' \in X$. By analogy it may be established that $ax^{(n)} \in X$ for any $n \in \mathbb{N} \cup \{0\}$. In the same way we may prove that $a'x^{(i)} \in X$ for any $i \in \mathbb{N} \cup \{0\}$. Applying induction, it is easy to ascertain that $a^{(j)}x^{(i)} \in X$ for every $i, j \in \mathbb{N} \cup \{0\}$. It follows that $x^i \in \left(X : a^{(\infty)}\right)$, and so $x \in \left(X : a^{(\infty)}\right)_\#$. This proves the inclusion $\left(X_\# : a^{(\infty)}\right) \subseteq \left(X : a^{(\infty)}\right)_\#$.

To prove the converse inclusion, we let $x \in \left(X : a^{(\infty)}\right)_\#$. Then $a^{(j)}x^{(i)} \in X$ for any $i, j \in \mathbb{N} \cup \{0\}$, in particular $ax' \in X$, $a'x' \in X$, $a''x' \in X$, $\ldots$, $a^{(n)}x' \in X, \ldots$, which means that $x' \in \left(X : a^{(\infty)}\right)$. In the same way $ax^{(i)} \in X$, $a'x^{(i)} \in X$, $a''x^{(i)} \in X \ldots$, $a^{(n)}x^{(i)} \in X$ etc., i. e. $x^{(i)} \in \left(X : a^{(\infty)}\right)$ for any $i \in \mathbb{N} \cup \{0\}$. Taking into account the above reasons and the fact that $\left(a^{(j)}x^{(i)}\right)^{(k)}$ is a sum of possible products of derivatives from the elements $a \in R$ and $x \in M$, we have that $\left(a^{(j)}x^{(i)}\right)^{(k)} \in X$ for every $i, j, k \in \mathbb{N} \cup \{0\}$. It easily follows $x \in \left(X_\# : a^{(\infty)}\right)$. It proves the inclusion $\left(X : a^{(\infty)}\right)_\# \subseteq \left(X_\# : a^{(\infty)}\right)$. □

Note that the previous proposition naturally follows that if N is a submodule of M, then $N_\#$ is a differential submodule of M, and if N is a differential submodule of the D-module M, then $N_\# = N$.

2. Differential kernel functors and #-filters

Remind that a functor $\sigma : R-\mathrm{DMod} \longrightarrow R-\mathrm{DMod}$ is called a *differential kernel functor* in the category $R - \mathrm{DMod}$ [10], if the following conditions hold:

1. $\sigma(M)$ is a differential submodule of M for each $M \in R - \mathrm{DMod}$;
2. If $f \in \mathrm{DHom}_R(M, N)$, then $f(\sigma(M)) \subseteq \sigma(N)$;
3. $\sigma(N) = N \bigcap \sigma(M)$ for every differential submodule N of the differential module M.

Kernel functors were investigated in [10], [15], [16], [17].

Let $\sigma : R - \text{Mod} \to R - \text{Mod}$ be a kernel functor. Define a functor $\sigma_{\#} : R - \text{DMod} \to R - \text{DMod}$ in such a way: $\sigma_{\#}(M) \stackrel{df}{=} (\sigma(M))_{\#}$ for each $M \in R - \text{Mod}$.

Proposition 3. *The functor* $\sigma_{\#} : R - \text{DMod} \to R - \text{DMod}$ *is a differential kernel functor.*

Proof. $\sigma_{\#}(M) = (\sigma(M))_{\#}$ is obviously a differential submodule of M for each $M \in R - \text{Mod}$.

If $f : M \to N$ is a differential homomorphism, then

$$f(\sigma_{\#}(M)) = f\left((\sigma(M))_{\#}\right) \subseteq (f(\sigma(M)))_{\#},$$

by Proposition 1, and $f(\sigma(M))_{\#} \subseteq (\sigma(N))_{\#} = \sigma_{\#}(N)$. Hence $\sigma_{\#}$ is a differential preradical in $R - \text{DMod}$.

Let N be a differential submodule of M. Then $\sigma_{\#}(N) = (\sigma(N))_{\#} = (N \bigcap \sigma(M))_{\#}$, and so by Proposition 1,

$$\left(N \bigcap \sigma(M)\right)_{\#} = N_{\#} \bigcap (\sigma(M))_{\#} = N \bigcap \sigma_{\#}(M).$$

Hence $\sigma_{\#}(N) = N \bigcap \sigma_{\#}(M)$. □

Another example of differential kernel functor provide the functor of differential socle, i. e. the functor, which puts in correspondence to each differential module the sum of its differentially simple submodules.

The differential kernel functor $\sigma_{\#}$ defines a differential torsion theory $\sigma_{\#} = \left(\mathcal{T}_{\sigma_{\#}}, \mathcal{F}_{\sigma_{\#}}\right)$, where $\mathcal{T}_{\sigma_{\#}} = \{M \in R - \text{Mod} \,|\, \sigma_{\#}(M) = M\}$ is a $\sigma_{\#}$-torsion class, and $\mathcal{F}_{\sigma_{\#}} = \{M \in R - \text{Mod} \,|\, \sigma_{\#}(M) = 0\}$ is a $\sigma_{\#}$-torsion-free class.

To the differential kernel functor corresponds a differential preradical filter

$$\mathfrak{F}_{\sigma_{\#}} = \left\{I\text{— left differential ideal of } R \,\middle|\, R/I \in \mathcal{T}_{\sigma_{\#}}\right\}.$$

The order relation on the class of all differential kernel functors may be defined by the rule: $\sigma \leq \tau$ if and only if $\sigma(M) \subseteq \tau(M)$ for all $M \in R - \text{DMod}$.

The class of all differential kernel functors forms a complete lattice. For every pair of differential kernel functors σ and τ there exists its meet $\sigma \wedge \tau$ and join $\sigma \vee \tau$ defined by the rule:

$$(\sigma \wedge \tau)(M) = \sigma(M) \cap \tau(M),$$

$$(\sigma \vee \tau)(M) = \sigma(M) + \tau(M)$$

for all $M \in R - \text{Mod}$.

Then we have define $\#$-filter as follows: $(\mathfrak{F}_{\sigma})_{\#} \stackrel{df}{=} \mathfrak{F}_{\sigma_{\#}}$.

Proposition 4. *Let M be a left differential R-module, $\sigma, \tau : R-\mathrm{Mod} \longrightarrow R-\mathrm{Mod}$ be kernel functors in $R-\mathrm{Mod}$. The operator $(\quad)_{\#}$ for kernel functors has the following properties:*

1. $\sigma_{\#} \leq \sigma$;
2. $(\sigma_{\#})_{\#} = \sigma_{\#}$;
3. $\sigma_{\#} = \sigma$ *if and only if the set* $\sigma(M)$ *is a differential submodule of* M;
4. *If* $\sigma \leq \tau$, *then* $\sigma_{\#} \leq \sigma_{\#}$;
5. $(\wedge_{i\in I}\sigma_i)_{\#} = \wedge_{i\in I}(\sigma_i)_{\#}$;
6. $\sigma_{\#} \vee \tau_{\#} \leq (\sigma \vee \tau)_{\#}$.

Proof. 1. It is obvious that $\sigma_{\#}(M) = (\sigma(M))_{\#} \subseteq \sigma(M)$.

2. $(\sigma_{\#})_{\#}(M) = (\sigma_{\#}(M))_{\#} = \Big((\sigma(M))_{\#}\Big)_{\#}$, by Proposition 1, it equals to $(\sigma(M))_{\#} = \sigma_{\#}(M)$.
3. The equality $\sigma_{\#}(M) = \sigma(M)$ holds if and only if $(\sigma(M))_{\#} = \sigma(M)$, but it is possible if and only if $\sigma(M)$ is a differential submodule of M.
4. If $\sigma \leq \tau$, then $\sigma_{\#}(M) = (\sigma(M))_{\#} \subseteq (\tau(M))_{\#} = \tau_{\#}(M)$.
5. We have $(\wedge_{i\in I}\sigma_i)_{\#}(M) = \left(\left(\bigcap_{i\in I}\sigma_i\right)(M)\right)_{\#} = (\bigcap_{i\in I}(\sigma_i(M)))_{\#}$, and by Propositiion 1, $\left(\bigcap_{i\in I}(\sigma_i(M))\right)_{\#} = \bigcap_{i\in I}(\sigma_i(M))_{\#} =$ $= \bigcap_{i\in I}\Big((\sigma_i)_{\#}(M)\Big) = \wedge_{i\in I}(\sigma_i)_{\#}(M)$.
6. $(\sigma_{\#} \vee \tau_{\#})(M) = \sigma_{\#}(M) + \tau_{\#}(M) = (\sigma(M))_{\#} + (\tau(M))_{\#}$, and so $(\sigma(M))_{\#} +$ $+ (\tau(M))_{\#} \subseteq (\sigma(M) + \tau(M))_{\#} = (\sigma \vee \tau)_{\#}(M)$.

□

Remind that a nonempty collection $\mathcal{F}$ of left differential ideals of the differential ring R is said to be a *differential preradical HK-filter* of R (see [3]) if the following conditions hold:

HK1. If $I \in \mathcal{F}$ and $I \subseteq J$, where J is a left differential ideal of R, then $J \in \mathcal{F}$;

HK2. If $I \in \mathcal{F}$ and $I \in \mathcal{F}$, then $I \bigcap J \in \mathcal{F}$;

HK3. If $I \in \mathcal{F}$, then $(I : a^{(\infty)}) \in \mathcal{F}$ for each $a \in R$.

If a differential preradical filter $\mathcal{F}$ satisfies an extra condition

HK4. If $I \subseteq J$ with $J \in \mathcal{F}$ and $(I : a^{(\infty)}) \in \mathcal{F}$ for all $a \in J$, then $I \in \mathcal{F}$,

then the filter $\mathcal{F}$ is called a *differential radical HK-filter.*

Proposition 5. *Let $\mathcal{F}$ be a preradical filter of the left ideals of the differential ring R and $\left(I : a^{(\infty)}\right) \in \mathcal{F}$ for every $I \in \mathcal{F}$ and every $a \in R$. Then $\mathcal{F}_{\#}$ is a preradical HK-filter of the ring R. If, in addition, $\mathcal{F}$ is a radical filter, then $\mathcal{F}_{\#}$ is a radical HK-filter of the noetherian ring R.*

Proof. Let $I \in \mathcal{F}_{\#}$ and $I \subseteq J$, where I, J are left differential ideals of R. Then there exists a left ideal $K \in \mathcal{F}$ such that $K_{\#} = I$. Consider the left ideal $K + J$ of the Δ-ring R. Since $K \subseteq K + J$ and $\mathcal{F}$ is a preradical filter, then $K + J \in \mathcal{F}$. Since J is differential, it holds $(K + J)_{\#} = K_{\#} + J_{\#} = K_{\#} + J = I + J = J$. Hence $J \in \mathcal{F}_{\#}$. Thus the condition HK1 holds.

Let $I, J \in \mathcal{F}_{\#}$, and K, L be left ideals of $\mathcal{F}$, such that $K_{\#} = I$, $L_{\#} = J$. Then $K_{\#} \bigcap L_{\#} = (K \bigcap L)_{\#}$, by Proposition 1. Since $K \bigcap L \in \mathcal{F}$, $I \bigcap J \in \mathcal{F}_{\#}$, and HK3 is proved.

For HK2, suppose $I = K_{\#}$ for some $K \in \mathcal{F}$ and let $a \in R$. Then

$$\left(I : a^{(\infty)}\right) = \left(K_{\#} : a^{(\infty)}\right) = \left(K : a^{(\infty)}\right)_{\#}.$$

Since by assumption $\left(K : a^{(\infty)}\right) \in \mathcal{F}$, we see that $\mathcal{F}_{\#}$ satisfies the condition HK2. Hence $\mathcal{F}_{\#}$ is a preradical filter.

Assume now that $\mathcal{F}$ is a radical filter of left ideals, which satisfies the condition pointed in the statement. Let $J \subseteq I$ be left differential ideals, where $I \in \mathcal{F}_{\#}$ and for every $a \in J$ $\left(I : a^{(\infty)}\right) \in \mathcal{F}_{\#}$. Then there exist left ideals $K, K_a \in \mathcal{F}$, for which $I = K_{\#}$ and $\left(I : a^{(\infty)}\right) = (K_a)_{\#}$. Since the underlying ring is noetherian, then $I = Rb_1 + Rb_2 + ... + Rb_s$ for some elements $b_1, b_2, ..., b_s \in K$. Now $T = \left(I : b_1^{(\infty)}\right) \cap \cdots \cap \left(I : b_s^{(\infty)}\right) = (K_{b_1})_{\#} \cap \cdots \cap (K_{b_s})_{\#} = (K_{b_1} \cap \cdots \cap K_{b_s})_{\#} \in \mathcal{F}_{\#}$. But $TI \supseteq ((K_{b_1} \cap \cdots \cap K_{b_s})\, K)_{\#} \in \mathcal{F}_{\#}$. Then $TI \subseteq J$ follows $J \in \mathcal{F}_{\#}$. □

3. Quasi-prime and quasi-primary torsion theories

Remind that a pretorsion theory (torsion theory) in the category of left R-modules is called 1-*pretorsion theory* (1-*torsion theory*), if the corresponding preradical filter (radical filter) has the basis of principal left ideals. Every 1-pretorsion (torsion) theory defines a set $\Sigma(\mathfrak{F}) =$

$\{a \in R\,|\,Ra \in \mathfrak{F}\}$, which is a left Ore set and is multiplicatively closed. Conversely, each subset of the ring R with the properties give above defines some 1-pretorsion theory.

Let τ be some 1-torsion theory in the category $R-\text{Mod}$. Consider the set

$$\Omega = \{\sigma \in R-\text{Mod}\,|\sigma \text{ is a } HK\text{-torsion theory}, \tau \wedge \sigma = \{R\}\}$$

together with the partial order defined in the usual way. It is easy to prove that this set is inductively ordered and, by Zorn's lemma, there exists maximal elements in Ω. The maximal of the pretorsion theories in Ω is called a *quasi-prime HK-pretorsion theory.*

The existence of quasi-prime torsion theories can also be established by using the method of transfinite induction.

Definition 1. *Quasi-prime HK-torsion theory is a quasi-prime HK-pretorsion theory which is a torsion theory.*

Example 1. Let P be a quasi-prime ideal of R. Then $S = R\backslash P$ is a dm-system. $\{Ra\,|a \in S\}$ is a basis of the radical filter $\mathcal{E}_P$. This filter is a 1-filter. Every maximal filter of the ones which do not meet $\mathcal{E}_P$ is a quasi-prime filter.

Definition 2. *Quasi-prime radical of the HK-torsion theory σ is an intersection $\sqrt{\sigma}$ of all quasi-prime HK-torsion theories τ such that $\sigma \leq \tau$, i. e.*

$$\sqrt{\sigma} = \bigcap_{\tau \geq \sigma} \tau.$$

Definition 3. *Quasi-primary HK-torsion theory σ is a HK-torsion theory such that a quasi-prime radical of which $\sqrt{\sigma}$ is a quasi-prime HK-torsion theory.*

Theorem 1. *Every HK-torsion theory of the differential noetherian completely bounded ring has the quasi-primary decomposition, i. e. it is an intersection of finite number of quasi-primary HK-torsion theories, which in fact is irreducible.*

Proof. The proof follows from the above definition, propositions and some additional reasoning.

Let σ be an arbitrary HK-torsion theory in the category $R-\text{DMod}$ and $\mathcal{F}$ is a corresponding differential HK-filter. Then, by Generalized Gabriel-Maranda theorem $\mathcal{F} = (\mathcal{F}_{\bar{\sigma}})_{\#}$. Following [1] the torsion theory $\bar{\sigma}$ over the noetherian ring is an intersection of finite number of irreducible torsion theories $\sigma_1, \ldots, \sigma_n$. Now use the operator $\#$ to the equality $\bar{\sigma} =$

$\sigma_1 \wedge \cdots \wedge \sigma_n$ and see what happens to the corresponding *HK*-filters, we obtain

$$\mathcal{F} = (\mathcal{F}_{\sigma_1})_{\#} \cap \cdots \cap (\mathcal{F}_{\sigma_n})_{\#}\,.$$

Thus, to prove the theorem, it is enough to show that each of the *HK*-filters $(\mathcal{F}_{\sigma_i})_{\#}$ is quasi-primary. In other words, it needs to prove that the operator # maps irreducible torsion theories into quasi-primary. It is easy to get, considering the fact that over a completely bounded noetherian ring every irreducible torsion theory is prime. Due to this fact, all torsion theories $\sigma_1, \ldots, \sigma_n$ are prime, and their #-images are quasi-prime, so are quasi-primary. □

Note that the theorem shows that a completely bounded noetherian ring is semidefinable in the sense of Golan. It may be used to get the quasi-primary decomposition of periodical with respect to differential torsion theory differential modules over noetherian differential rings.

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