Algebra and Discrete Mathematics
Number 2. **(2009).** pp. 45 – 59

RESEARCH ARTICLE

Finding proper factorizations in finite groups

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Communicated by V. I. Sushchansky

Abstract. One focus in group theory has been to establish the properties of a finite group that can be written as the product of two proper subgroups whose properties are known. The investigation here will proceed in the other direction by establishing conditions for when a finite group can be written as a product of two proper subgroups and for when a specific proper subgroup is part of a product of proper subgroups that equals the group. A byproduct of this investigation is a classification of those finite groups which cannot be written as the product of any two proper subgroups.

Introduction

All groups are finite. A group G admits a *proper factorization*, or is *factorizable*, if there are proper subgroups A and B of G such that $G = AB$. Any factorization of $G = AB$ into a product of proper subgroups A and B results in a *maximal factorization* of $G = M_1M_2$, where $A \leq M_1$, $B \leq M_2$ and both M_1 and M_2 are maximal subgroups of G. In [10], Ore showed for solvable groups G, that if H and K are any two non-conjugate maximal subgroups of G, then $G = HK$, and conversely that any maximal factorization of G occurs this way.

Group factorizations have yielded significant results. For example, Itô [5] showed that if $G = AB$ with A and B abelian, then G is metabelian, and Kegel [8] proved that if $G = AB$ with A and B nilpotent, then G is solvable. More recently, this approach has turned to investigating groups which can be written as the product of two simple groups (a good

2000 Mathematics Subject Classification: *20D40; 20E07.*
Key words and phrases: *factorization, supplement, maximal subgroup.*

beginning reference is Walls [14]). An excellent reference on products of groups is the book by Amberg, Franciosi, and de Giovanni [1].

The avenue of research described in the previous paragraphs motivates the following two questions.

1. Given a group G, when can G be written as the product AB of two proper subgroups A and B?

2. Given a specific proper subgroup A of G, when does there exist a proper subgroup B of G such that $G = AB$?

The goal of this article is to answer these two questions. It should be noted that question (1) reduces to the following: Which noncyclic groups have the property that no pair of maximal subgroups are permutable? Scott, in 13.1.5 of [12], showed that a finite solvable group has a proper factorization if and only if it is not a cyclic p-group. A generalization of this result is given in Section 2.

Preliminary results needed to address both questions will be presented in Section 1. An answer to the first question will be given in Section 2 as a corollary to the classification of nonfactorizable groups. A nontrivial group G is *nonfactorizable* if for all proper subgroups A of G there does not exist a proper subgroup B of G such that $G = AB$. To begin to answer the second question, an approach involving aS-groups will be used. A group G is an *aS-group* if it has order 1 or if for every nontrivial subgroup H of G there is a proper subgroup K of G such that $G = HK$. In Section 3, it will be shown that the collection of aS-groups forms a formation. The residual of this formation will be used to determine proper factorizations. The results established through Section 3 will be applied in Section 4, where conditions for determining whether a particular subgroup of a group is in a proper factorization for that group will be examined.

The notation used is standard. The Frattini subgroup of a group G is denoted by $\Phi(G)$, its commutator subgroup by G', and its center by $Z(G)$. If H is a subgroup of G, the centralizer of H in G is denoted by $C_G(H)$ and the index of H in G is denoted by $[G{:}H]$. In the situations where the notation for writing a group G as a product of two of its subgroups A and B causes confusion, writing $G = AB$ will be replaced by $G = A{\cdot}B$. If A is a proper (normal) subgroup of G, it will be denoted by $A < G$ ($A \lhd G$). Finally, if two positive integers s and t are relatively prime, it will be denoted by $(s,t) = 1$.

1. Preliminary results

Consider a group G, and suppose that for a subgroup A of G there exists a subgroup B of G such that $G = AB$. It is somewhat cumbersome to only be able to refer to subgroup B using the factorization of $G = AB$. To help with the terminology, the following definition is made.

Definition 1. *A subgroup A of a group G has a* ***supplement*** *in G if there exists a subgroup B of G such $G = AB$.*

When a group G has a proper factorization $G = AB$ for proper subgroups A and B, the subgroup B is referred to as a proper supplement of A in G.

Lemma 1. *Let N and A be proper subgroups of group G with $N \triangleleft G$ and $N < A$. If A/N has a proper supplement in G/N, then A has a proper supplement in G.*

Proof. Since there exist a proper subgroup B/N of G/N such that $G/N = A/N{\cdot}B/N$, it follows that $G = AB$. □

Lemma 2. *Let A and H be subgroups of G such that A is proper in G and $A < H$. If A has a proper supplement in G, then A has a proper supplement in H.*

Proof. The result follows if $H = G$. Assume H is a proper subgroup of G. Since A has a proper supplement B in G, $G = AB$. By the Modular Identity, $H = G \cap H = AB \cap H = A(B \cap H)$. If $B \cap H = \{1\}$, then $A = H$, a contradiction. If $B \cap H = H$, then $A < H \leq B$, another contradiction. Thus A has a proper supplement in H. □

The converse of Lemma 1 is not true as the group $G = \langle a, b \mid a^5 = b^4 = 1, ab = ba^2\rangle$ indicates. Let $N = \langle a\rangle$ and $A = \langle a, b^2\rangle$. The subgroup A has a proper supplement $\langle b\rangle$ in G, yet A/N has no proper supplement in G/N as it is cyclic of order 4. Likewise, the converse of Lemma 2 is also not true as the group $G = \langle a, b \mid a^9 = b^9 = 1, ab = ba\rangle$ indicates. Consider the subgroups $H = \Phi(G) = \langle a^3, b^3\rangle$ and $A = \langle a^3\rangle$. The subgroup A has a proper supplement in H as $H = A \times \langle b^3\rangle$, yet A has no proper supplement in G as A is contained in the Frattini subgroup of G.

The last lemma of the section is obvious and presented without proof.

Lemma 3. *If G is a group such that $G = HK$ for subgroups H and K, then for any normal subgroup N of G, $G/N = NH/N{\cdot}NK/N$.*

It should be noted that even if H and K are proper nontrivial subgroups of G, the same cannot be said for NH/N and NK/N in G/N.

2. Nonfactorizable groups

To determine when a group G admits a proper factorization, a study of nonfactorizable groups is essential.

Definition 2. *A group G is* ***nonfactorizable*** *if $|G| \neq 1$ and for all proper subgroups A of G, there does not exist a proper subgroup B of G such that $G = AB$.*

The following proposition follows directly from Lemmas 1 and 3.

Proposition 1. *Let G be a group and N be a proper normal subgroup of G.*

(i) If G is nonfactorizable, then G/N is nonfactorizable.

(ii) If G/N is nonfactorizable for $N \leq \Phi(G)$, then G is nonfactorizable.

The maximal factorizations of all of the finite simple groups have been determined by Liebeck, Praeger, and Saxland in [9]. One of the byproducts of this investigation is that the nonfactorizable simple groups are also known. The abelian simple groups of order greater than 1 are nonfactorizable, and the alternating groups A_n, for $n \geq 5$, are all factorizable. The nonfactorizable simple groups of Lie type and the nonfactorizable Sporadic simple groups are listed in Table 4.1. The number q is a power of a prime.

Theorem 1. *Let G be a nonfactorizable group. Then G is one of the following three types of groups:*

(i) a cyclic p-group;

(ii) a nonabelian nonfactorizable simple group (as listed in Table 4.1);

(iii) a perfect group with $\Phi(G) \neq \{1\}$ and $G/\Phi(G)$ a nonabelian nonfactorizable simple group.

Proof. Consider the commutator subgroup G' of G. If G' is a proper subgroup of G, then $G' \leq \Phi(G)$ implies G is nilpotent. Consequently, G is cyclic of prime power order.

Now consider the case where $G' = G$ or G is perfect. If G contains no proper normal subgroups other than the trivial one, then G is a nonabelian simple group. The nonfactorizable nonabelian simple groups are known (as determined by Liebeck, Praeger, and Saxl [9]) and are listed in Table 4.1.

If G contains a proper nontrivial normal subgroup, then it must each be contained in $\Phi(G)$. Thus $\Phi(G)$ is the unique maximal normal subgroup of G. By Proposition 1, $G/\Phi(G)$ is also nonfactorizable and must be a type of group listed in Table 4.1. □

Groups of Lie Type	
Simple Group	Conditions
$A_n(q)$	$n = 2$ and $q \neq 7, 11, 19, 23, 25, 59$ or $q \equiv 1 \pmod 4$
	$n \geq 3$ when n prime and $n \mid q - 1$
$U_n(q)$	n odd, except for $U_3(3), U_3(5), U_9(2)$
$B_n(q)$	$n \geq 3$ and q even
$G_2(q)$	$q \neq 3^t$ or $q \neq 4$
$F_4(q)$	$q \neq 2^t$
$E_6(q)$	
$E_7(q)$	
$E_8(q)$	
${}^2B_2(q)$	
${}^2D_n(q)$	$n \geq 4$, n even
${}^3D_4(q)$	
${}^2G_2(q)$	
${}^2F_4(q)$	
${}^2E_6(q)$	
Sporadic Groups	
$M_{22}, Mc, CO_3, CO_2, Fi_{23}, Fi'_{24}, HN, Th$	
$B, M, J_1, O'N, J_3, Ly, J_4$	

Table 4.1: Nonfactorizable nonabelian simple groups

An answer to the question concerning when a group admits a proper factorization is an immediate consequence of this result.

Corollary 1. *For a noncyclic group G, if $G' \neq G$, then G admits a proper factorization.*

Given a group G with $G' = G$, Scott, in 13.1.8 of [12], showed that if G is a finite nonsolvable group such that every non-abelian composition factor group of G has a proper factorization, then G has a proper factorization. Using Lemma 2, another corollary to Theorem 1 is obtained.

Corollary 2. *Let H and K be subgroups of a group G such that $H < K$ and K is a cyclic p-group. Then H has no proper supplement in G.*

The groups of the type mentioned in (i) and (ii) of Theorem 1 are well-known. At this point, the groups mentioned in part (iii) of Theorem 1 will be briefly examined and will be referred to as Type III groups for the remainder of this section. Type III groups are perfect and for more information on perfect groups, the book by Holt and Plesken [4] is an excellent reference. As a reminder, a group G is *quasisimple* if G is perfect and $G/Z(G)$ is simple.

Proposition 2. *Let p be a prime. If G is a Type III group with $|\Phi(G)| = p$, then $\Phi(G) = Z(G)$ and G is quasisimple.*

Proof. Consider $C_G(\Phi(G))$, the centralizer of $\Phi(G)$ in G. Since $\Phi(G)$ is abelian and $C_G(\Phi(G)) \trianglelefteq G$, it must be that $C_G(\Phi(G)) = \Phi(G)$ or G.

Given that $p \big| |G/\Phi(G)|$, there is a Sylow p-subgroup S_p of G such that $\Phi(G) < S_p$. Since $\Phi(G) \cap Z(S_p) \neq \{1\}$, $\Phi(G) \leq Z(S_p)$. As a result, there is an element $x \in S_p$ where $x \notin \Phi(G)$ and $x \in C_G(\Phi(G))$. Consequently, $C_G(\Phi(G)) = G$ and $\Phi(G) = Z(G)$. Since $G/Z(G)$ is simple, G is quasisimple. $\square$

One collection of Type III groups that satisfy the conditions of Proposition 2 are the groups $SL(2, q)$, for $q \neq 7, 11, 19, 23, 25, 59$ or $q \equiv 1 \pmod 4$. In this case, $|\Phi(SL(2, q))| = 2$ and $SL(2, q)/\Phi(SL(2, q)) = PSL(2, q)$, a nonfactorizable simple group.

Let G be a Type III group. Using the fact that G is perfect and $G/\Phi(G)$ is simple, it is easy to show that if $\Phi(G)$ is minimal normal in G with $|\Phi(G)| = p^\alpha$ and $\alpha \geq 2$, then $C_G(\Phi(G)) = \Phi(G)$, and more generally that there exists a normal subgroup N of G such that $N < \Phi(G)$, $\Phi(G)/N$ is abelian, and $C_{G/N}(\Phi(G)/N) = \Phi(G)/N$ or G/N. However, at this point, it is open as to whether Type III groups exist that satisfy these conditions.

3. The $\mathfrak{aS}$-residual

Recall Lemma 1, which states that for a group G, if $A/N < G/N$ has a proper supplement in G/N, then A has a proper supplement in G. Consequently, if every subgroup of G/N had a proper supplement in G/N, then every subgroup H of G, such that $N < H < G$, would have a proper supplement in G. This motivates the following definition.

Definition 3. *A group G is an* **aS*-group*** *if it has order 1 or if every nontrivial subgroup has a proper supplement.*

These types of groups, which were studied and classified in by Kappe and Kirtland in [6], are an extension of K-groups. A *K-group* G satisfies the property that for each subgroup A of G there is a subgroup B of G such that $A \cap B = \{1\}$ and $G = \langle A, B \rangle$. K-groups were first introduced and studied by Suzuki in [13]. One main result from [6], concerning the classification of aS-groups, is presented here.

Theorem 2. *A group G is an aS-group if and only if G is supersolvable with elementary abelian Sylow subgroups.*

In this section it will be shown that the collection of aS-groups forms a formation. The search for a proper factorization of an arbitrary group G is assisted by the existence of a proper normal subgroup N of G such that G/N is an aS-group. This motivates the examination of the residual of the collection of aS-groups, which will also be done in this section.

Theorem 3. *The class of aS-groups, denoted by $\mathfrak{a}\mathfrak{S}$, forms a formation.*

Proof. Let G be an aS-group, and let $N \trianglelefteq G$. If $N = G$, then G/N is trivially an aS-group. Assume that $N \neq G$, and let P/N be a Sylow p-subgroup of G/N. Thus there is a Sylow p-subgroup R of G such that R is not contained in N. By Theorem 2, R is elementary abelian. Since $RN/N \cong R/(R \cap N)$, the Sylow p-subgroup RN/N of G/N is also elementary abelian. Given that all Sylow p-subgroups are conjugate, P/N is also elementary abelian.

Consider a chief series for G/N. Trivially, it can be extended to a chief series for G. Since, by Theorem 2, all chief factors for G are cyclic, all chief factors for G/N are cyclic. Thus by Theorem 2, G/N is an aS-group.

Now let N_1 and N_2 be normal subgroups of G, with $N_1 \cap N_2 = \{1\}$, such that G/N_1 and G/N_2 are both are aS-groups. Since G/N_1 and G/N_2 are supersolvable, $G/(N_1 \cap N_2) \cong G$ is also supersolvable.

Let P be a Sylow p-subgroup of G. Then PN_1/N_1 and PN_2/N_2 are Sylow p-subgroups of G/N_1 and G/N_2 respectively. Since G/N_1 and G/N_2 are aS-groups, PN_1/N_1 and PN_2/N_2 are elementary abelian by Theorem 2. Thus $\Phi(P) \subseteq N_1$ and $\Phi(P) \subseteq N_2$. As a result, $\Phi(P) \subseteq N_1 \cap N_2 = \{1\}$ and P is elementary abelian. By Theorem 2, G is an aS-group. $\square$

Essentially, the formation $\mathfrak{a}\mathfrak{S} = \mathfrak{U} \cap \mathfrak{E}$, where $\mathfrak{U}$ is the formation of supersolvable groups and $\mathfrak{E}$ is the formation of groups whose Sylow subgroups are elementary abelian. The formation $\mathfrak{a}\mathfrak{S}$ is not saturated as indicated by the cyclic group of order 4.

Definition 4. *Let G be a group. The $\mathfrak{a}\mathfrak{S}$-**residual**, denoted by $G^{\mathfrak{a}\mathfrak{S}}$ or more simply by $G^{\mathcal{S}}$, is the intersection of all of the normal subgroups N of G such that $G/N \in \mathfrak{a}\mathfrak{S}$.*

The $\mathfrak{a}\mathfrak{S}$-residual $G^{\mathcal{S}}$ for a group G is an important subgroup with regards to proper factorizations for G. Before this aspect is explored, a few properties of $G^{\mathcal{S}}$ are presented.

Lemma 4. *For a group G and the $\mathfrak{a}\mathfrak{S}$-residual $G^{\mathcal{S}}$ of G, the following properties hold:*

(i) $\Phi(G) \leq G^{\mathcal{S}}$;

(ii) G' *is a proper subgroup of* G *if and only if* $G^{\mathcal{S}}$ *is a proper subgroup of* G;

(iii) If $N \trianglelefteq G$, *then* $N^{\mathcal{S}} \leq G^{\mathcal{S}}$ *and* $G^{\mathcal{S}}N/N = (G/N)^{\mathcal{S}}$;

(iv) For a group L *with* $\mathfrak{a}\mathfrak{S}$*-residual* $L^{\mathcal{S}}$, $(G \times L)^{\mathcal{S}} = G^{\mathcal{S}} \times L^{\mathcal{S}}$;

(v) $G^{\mathcal{S}} \cap Z(G) \leq \Phi(G)$;

(vi) For any subgroup H *of* G, $\Phi(H) \leq G^{\mathcal{S}}$.

Proof. Statement (i) follows from the fact that $\Phi(G)G^{\mathcal{S}}/G^{\mathcal{S}} \leq \Phi(G/G^{\mathcal{S}}) = 1_{G/G^{\mathcal{S}}}$. The forward direction of (ii) follows from the fact that G/G' is abelian. The converse follows from the fact that $G/G^{\mathcal{S}}$ is supersolvable.

To prove the first part of (iii) note that the result follows if $N \leq G^{\mathcal{S}}$. If N is not contained in $G^{\mathcal{S}}$, then $G^{\mathcal{S}}N/G^{\mathcal{S}}$ is a nontrivial subgroup of the aS-group $G/G^{\mathcal{S}}$. Since $G^{\mathcal{S}}N/G^{\mathcal{S}}$ is also an aS-group (3.5 in [6]) and $G^{\mathcal{S}}N/G^{\mathcal{S}} \cong N/(G^{\mathcal{S}} \cap N)$, $N^{\mathcal{S}} \leq G^{\mathcal{S}}$.

To prove the second part of (iii), let $(G/N)^{\mathcal{S}} = S/N$. Given that $(G/N)/(S/N) \cong G/S$, which is an aS-group, $G^{\mathcal{S}} \leq S$. Thus $G^{\mathcal{S}}N/N \leq S/N = (G/N)^{\mathcal{S}}$. If $G^{\mathcal{S}}N = G$, then $G/N = G^{\mathcal{S}}N/N \leq (G/N)^{\mathcal{S}}$ and $G^{\mathcal{S}}N/N = (G/N)^{\mathcal{S}}$. Now suppose that $G^{\mathcal{S}}N < G$. Then it follows that $(G/G^{\mathcal{S}})/(G^{\mathcal{S}}N/G^{\mathcal{S}}) \cong G/G^{\mathcal{S}}N$. Since $G^{\mathcal{S}} \leq G^{\mathcal{S}}N < G$ and $G/G^{\mathcal{S}}$ is an aS-group, then $G/G^{\mathcal{S}}N$ is an aS-group (see [6]). Thus $(G/N)^{\mathcal{S}} \leq G^{\mathcal{S}}N/N$ and $G^{\mathcal{S}}N/N = (G/N)^{\mathcal{S}}$.

To start the proof of (iv) note that since $(G \times L)/(G^{\mathcal{S}} \times L^{\mathcal{S}}) \cong G/G^{\mathcal{S}} \times L/L^{\mathcal{S}}$, which is an aS-group, $(G \times L)^{\mathcal{S}} \leq G^{\mathcal{S}} \times L^{\mathcal{S}}$. By (iii), both $G^{\mathcal{S}}$ and $L^{\mathcal{S}}$ are contained in $(G \times L)^{\mathcal{S}}$. Thus $G^{\mathcal{S}} \times L^{\mathcal{S}} \leq (G \times L)^{\mathcal{S}}$ and $G^{\mathcal{S}} \times L^{\mathcal{S}} = (G \times L)^{\mathcal{S}}$.

For (v), assume that $G^{\mathcal{S}} \cap Z(G) = K \neq \{1\}$ and that K is not contained in $\Phi(G)$. Thus there is a maximal subgroup M of G such that $G = KM$. Let $g \in G$. Then $g = km$, where $k \in K$ and $m \in M$, and $M^g = M^{km}$. Since $k \in Z(G)$, $M^g = M^{km} = M^m = M$ and $M \lhd G$. Since M is maximal in G, $[G{:}M] = p$ for some prime p. Given that G/M is an aS-group and $M \lhd G$, $G^{\mathcal{S}} \leq M$, which implies that $K \leq M$, a contradiction. Thus $K \leq \Phi(G)$.

To prove $(v$i), consider the case that H is not contained in $G^{\mathcal{S}}$. Then $HG^{\mathcal{S}}/G^{\mathcal{S}}$ is a nontrivial subgroup of $G/G^{\mathcal{S}}$. Consequently, by Theorem 2, $H/(H \cap G^{\mathcal{S}})$ has elementary abelian Sylow p-subgroups. Thus, by Theorem 2.3 in [2], $\Phi(H) \leq G^{\mathcal{S}}$. □

There do exist groups G where $\Phi(G) < G^{\mathcal{S}}$ as indicted by the alternating group A_4. For this group, $\Phi(A_4) = \{1\}$, but $\mathfrak{a}\mathfrak{S}$-residual $G^{\mathcal{S}}$ is the Sylow 2-subgroup of A_4. However, the following result can be proven.

Theorem 4. *If G is nilpotent group, then $\Phi(G) = G^{\mathcal{S}}$.*

Proof. By Lemma 4, all that needs to be shown is that $G^{\mathcal{S}} \leq \Phi(G)$. Since G is nilpotent, $G = S_1 \times \cdots \times S_t$, where for each i, $1 \leq i \leq t$, S_i is a Sylow p_i-subgroup of G. Given that $\Phi(G) = \Phi(S_1) \times \cdots \times \Phi(S_t)$, it follows that $G/\Phi(G) \cong S_1/\Phi(S_1) \times \cdots \times S_t/\Phi(S_t)$. Since for each i, $1 \leq i \leq t$, $S_i/\Phi(S_i)$ is elementary abelian, $G/\Phi(G)$ is an aS-group. As a result, $G^{\mathcal{S}} \leq \Phi(G)$. □

There do exist nonnilpotent groups G where $\Phi(G) = G^{\mathcal{S}}$. Consider the group $G = \langle a, b, c \mid a^3 = b^3 = c^2 = 1, ab = ba, ac = cb, bc = ca\rangle$. Here, $\Phi(G) = G^{\mathcal{S}} = \{1\}$, yet G is not nilpotent. For a moment, consider the case when $\Phi(G) = G^{\mathcal{S}}$ for some group G. By Lemma 4, this would imply that $\Phi(P) \leq \Phi(G)$ for all Sylow p-subgroups P of G. Both Bechtell in [2] and Doerk in [3] independently studied and established numerous results concerning groups with this Sylow p-subgroup containment property. One additional result is mentioned here.

Lemma 5. *If $\Phi(G) = G^{\mathcal{S}}$, then for each Sylow p-subgroup P of G, $P \cap \Phi(G) = \Phi(P)$.*

Proof. Since $\Phi(G) = G^{\mathcal{S}}$, G is supersolvable. By Lemma 4, $\Phi(P) \leq G^{\mathcal{S}} = \Phi(G)$, and thus $\Phi(P) \leq P \cap \Phi(G)$. Furthermore, the fact that $P' \leq \Phi(P) \leq \Phi(G)$ results in $P' \leq P \cap \Phi(G)$. Consequently, by Corollary 8 in [11], $P \cap \Phi(G) \leq \Phi(P)$. □

Determining when $G^{\mathcal{S}}$ has a proper supplement motivates results established in Section 5. Clearly, if $G^{\mathcal{S}}$ is proper and nontrivial in a group G, then $G^{\mathcal{S}}$ will have a proper supplement when $G^{\mathcal{S}} \neq \Phi(G)$. The only case that remains to examine is when $G^{\mathcal{S}}$ is nilpotent.

Theorem 5. *Let $G^{\mathcal{S}}$ be a proper, nontrivial, and nilpotent subgroup of a non-nilpotent group G. Then $G^{\mathcal{S}}$ has a proper supplement in G if and only if there exists a prime p where $p \big| |G^{\mathcal{S}}|$ such that p does not divide $[G{:}G^{\mathcal{S}}]$ or $G^{\mathcal{S}} \cap P$ has a proper supplement in P for each Sylow p-subgroup P of G.*

Proof. Given that $G^{\mathcal{S}}$ is nilpotent, G is solvable. Furthermore, since G is non-nilpotent, there exist at least two distinct primes that divide the order of G.

First, suppose $G^{\mathcal{S}}$ has a proper supplement in G and that for all primes p_i, $1 \leq i \leq t$, where $p_i \big| |G^{\mathcal{S}}|$, that $p_i \big| [G{:}G^{\mathcal{S}}]$. Since $G^{\mathcal{S}}$ is nilpotent, $G^{\mathcal{S}} = P_1^* \times \cdots \times P_t^*$ where for each i, P_i^* is the Sylow p_i-subgroup of $G^{\mathcal{S}}$. If each $P_i^* \leq \Phi(G)$, then $\Phi(G) = G^{\mathcal{S}}$, a contradiction. Thus, without loss of generality, P_1^* is not contained in $\Phi(G)$. Consequently, since $P_1^* \lhd G$, there is a proper subgroup H of G such that $G = P_1^* H$. Let P_1 be a Sylow p_1-subgroup of G. Since $G^{\mathcal{S}} \cap P_1 = P_1^*$, $G^{\mathcal{S}} \cap P_1$ has a proper supplement in P_1 by Lemma 2.

Conversely, suppose there is a prime p such that $p \big| |G^{\mathcal{S}}|$, but p does not divide $[G{:}G^{\mathcal{S}}]$. In this case, $\Phi(G) \neq G^{\mathcal{S}}$ and $G^{\mathcal{S}}$ has a proper supplement in G. Now suppose there is a prime p such that $p \big| |G^{\mathcal{S}}|$, $p \big| [G{:}G^{\mathcal{S}}]$, and for each Sylow p-subgroup P of G that $P = (G^{\mathcal{S}} \cap P)H_p$ for a proper subgroup H_p of P. By Lemma 4, $\Phi(P) \leq G^{\mathcal{S}}$. As a result, $\Phi(P) \leq G^{\mathcal{S}} \cap P$.

Suppose that $G^{\mathcal{S}} \cap P \leq \Phi(G)$. Given that $P' \leq \Phi(P)$, $P' \leq P \cap \Phi(G)$. By Corollary 8 in [11], this imples that $P \cap \Phi(G) \leq \Phi(P)$. As a result, $G^{\mathcal{S}} \cap P \leq P \cap \Phi(G) \leq \Phi(P)$, a contradiction. Consequently, $G^{\mathcal{S}} \cap P$ is not contained in $\Phi(G)$. Thus, since $G^{\mathcal{S}} \cap P \lhd G$, there exists a proper subgroup H of G such that $G = (G^{\mathcal{S}} \cap P)H$. As a result, $G = G^{\mathcal{S}} H$. □

4. Determining proper factorizations

The main purpose of studying the $\mathfrak{a}\mathfrak{S}$-residual $G^{\mathcal{S}}$ of a group G is to obtain results concerning proper factorizations.

Theorem 6. *Let $\{1\} < H < G$. If $G^{\mathcal{S}} \cap H \neq H$, then H has a proper supplement in G.*

Proof. Since $G^{\mathcal{S}} \cap H \neq H$, it must be that $G^{\mathcal{S}} \neq G$. If $G = HG^{\mathcal{S}}$, then H has a proper supplement.

Suppose that $G \neq HG^{\mathcal{S}}$. Then $HG^{\mathcal{S}}/G^{\mathcal{S}}$ is a nontrivial and proper subgroup of $G/G^{\mathcal{S}}$, which is an aS-group. Thus there is a proper subgroup $K/G^{\mathcal{S}}$ of $G/G^{\mathcal{S}}$ such that $G/G_{\mathcal{S}} = HG^{\mathcal{S}}/G^{\mathcal{S}} \cdot K/G^{\mathcal{S}}$. Consequently, $G = HK$ where K is proper in G. □

A number of results are an immediate consequence of Theorem 6.

Corollary 3. *Let G be a group whose order is divisible by two or more primes. Then for each prime p, where $p \big| |G/G^{\mathcal{S}}|$, each Sylow p-subgroup P has a proper supplement in G.*

Corollary 4. *Let G be a noncyclic group with $G^{\mathcal{S}} \neq G$.*

(i) Every maximal subgroup M of G has a proper supplement in G.

(ii) The group G admits a proper factorization where one of the proper subgroups is cyclic.

Proof. To prove (i), consider the case that $G \neq G^{\mathcal{S}} M$. Then $M = G^{\mathcal{S}}$ and $M \lhd G$. If $M = \Phi(G)$, then G is a cyclic group. Thus $M \neq \Phi(G)$ and it has a proper supplement in G.

For (ii), note that $G^{\mathcal{S}} \neq G$ implies that there exists an element $g \in G \backslash G^{\mathcal{S}}$ such that $g \neq 1$. Since G is noncyclic, $\langle g \rangle \neq G$ and $\langle g \rangle \cap G^{\mathcal{S}} \neq \langle g \rangle$. Thus by Theorem 6, $\langle g \rangle$ has a proper supplement in G. □

Corollary 5. *Let G be a group such that $G^{\mathcal{S}} \neq G$. Then G is a cyclic p-group or G admits a proper factorization $G = NH$ where $N \lhd G$, H is supersolvable, and $N \cap H = G^{\mathcal{S}}$.*

Proof. Suppose G is not a p-group and $G^{\mathcal{S}} < G$. The result follows from the fact that $G/G^{\mathcal{S}}$ is supersolvable and in the formation $\mathfrak{a}\mathfrak{S}$. □

Determining proper factorizations involving normal subgroups of a group is more easily determined as normal subgroups are permutable. Recall that a subgroup A of a group G is *permutable* if $AB = BA$ for all subgroups B of G. Essentially, if a normal subgroup is not contained in the Frattini subgroup (or is non-nilpotent), it will have a proper supplement. Some more detailed critrea for when a normal subgroup has a proper supplement are given in [7]. The only difficult case is determining when a nilpotent normal subgroup has a proper supplement. This, along with Theorem 5, motivates the following result.

Theorem 7. *Let N be a proper and nontrivial nilpotent subgroup of a non-nilpotent group G such that $N \lhd G$ and N is of square-free order. Then N has a proper supplement in G if and only if there exists a prime p where $p \big| |N|$ such that p does not divide $[G{:}N]$ or $N \cap P$ has a proper supplement in P for each Sylow p-subgroup P of G.*

Proof. First, suppose that $G = NS$ for some proper subgroup S of G and that for all primes p, where $p \big| |N|$, that $p \big| [G{:}N]$. Since N is nilpotent and normal in G, $N = N_{p_1} \times \cdots \times N_{p_t}$, where each N_{p_i}, for $1 \leq i \leq t$, is a Sylow p_i-subgroup of N and $N_{p_i} \lhd G$. If for each i, $N_{p_i} \leq \Phi(G)$, then $N \leq \Phi(G)$, a contradiction. Thus, without loss of generality, N_{p_1} is not contain in $\Phi(G)$. Thus $G = N_{p_1}H$, where H is a proper subgroup of G. Let P_1 be a Sylow p_1-subgroup of G. By Lemma 2, $N \cap P_1 = N_{p_1}$ will have a proper supplement in P_1.

Conversely, suppose there is a prime p such that $p \big| |N|$ and p does not divide $[G{:}N]$. In this case, $N \neq \Phi(G)$ and N has a proper supplement in G. Now consider the case where all primes that divide $|N|$ also divide $[G{:}N]$. Then for some prime p that divides $|N|$, $N_p = N \cap P$ has a proper supplement in each Sylow p-subgroup P of G.

Suppose that $N_p \leq \Phi(G)$. Given that $|N_p| = p$, $N_p \leq Z(P)$. Since $N_p \lhd G$ and $N_p \leq \Phi(G) \cap Z(P)$, Theorem 5 from [11] implies that $N_p \leq \Phi(P)$. This is a contradiction. As a result, N_p is not contained in $\Phi(G)$

and $G = N_pH$ for some proper subgroup H of G. Consequently, $G = NH$. $\square$

For both Theorems 5 and 7, the fact that, for some prime p, $P \cap \Phi(G) \leq \Phi(P)$ for all Sylow p-subgroups P of G was the main catalist. This motivates the following theorem. The proof of this theroem, which is similar to the proof of Theorem 7, is not given.

Theorem 8. *Let N be a proper and nontrivial nilpotent subgroup of a non-nilpotent group G such that $N \lhd G$ and $P \cap \Phi(G) \leq \Phi(P)$ for each Sylow p-subgroup of G. Then N has a proper supplement in G if and only if there exists a prime p where $p \big| |N|$ such that p does not divide $[G{:}N]$ or $N \cap P$ has a proper supplement in P for each Sylow p-subgroup of G.*

Attention will now be given to arbitray subgroups. Recall that for a proper normal subgroup N of a group G, N will have a proper supplement or be part of a proper factorization for G if and only if N is not contained in the Frattini subgroup of G. While this condition can be extended to arbitrary subgroups of nilpotent groups, it cannot be extended to arbitrary subgroups of non-nilpotent groups. As an example, consider the group $G = \langle a, b \mid a^5 = b^4 = 1, ab = ba^2\rangle$. For this group, $\Phi(G) = \{1\}$, yet $\langle b^2\rangle$ does not have a proper supplement. Another condition that cannot be generalized is the one established in the Schur-Zassenhaus Theorem.

Theorem 9 (The Schur-Zassenhaus Theorem)**.** *If N is a normal subgroup of a group G such that $(|N|, [G{:}N]) = 1$, then N has a complement in G and all complements are conjugate.*

At first glance, it does seem that the following statement could be true: if H is a proper subgroup of a group G such that $|H|$ is relatively prime to its index $[G{:}H]$, then H will have a proper supplement in G. However, this is not the case. Consider the sporadic simple group M_{22}. It has a maximal subgroup H of order $5760 = 2^7 3^2 5$, which is relatively prime to its index $[M_{22}{:}H] = 77$. However, H has no proper supplement in M_{22} as M_{22} is nonfactorizable.

However, using the $\mathfrak{a}\mathfrak{S}$-residual $G^{\mathcal{S}}$, one generalization can be made.

Theorem 10. *Let G be a group with $G^{\mathcal{S}} \neq G$, and let H be a nontrivial and proper subgroup of G such that $(|H|, [G{:}H]) = 1$.*

(i) If H is not contained in G', then H has a proper supplement in G.

(ii) If H is abelian and $H \cap Z(G) \neq \{1\}$, then H has a proper supplement in G.

Proof. Given that $G^{\mathcal{S}} < G$, Lemma 4 implies that $G' < G$. To prove (i), first note that if $G'H = G$, then H has a proper supplement in G.

Assume that $G'H \neq G$. If $G' = H$, then $H \lhd G$ and H has a proper supplement in G by Theorem 9.

Suppose $G' \neq H$. Since H is not contained in G', $G'H/G'$ is a proper nontrivial normal subgroup of G/G'. Let $|G'H/G'| = |H/(H \cap G')| = d$. By the Third Isomorphism Theorem, $|(G/G')/(G'H/G')| = |G/G'H| = \frac{|G|}{|G'H|}$. However, given that $|G'H| = \frac{|G'||H|}{|H \cap G'|} > |H|$, $\frac{|G|}{|G'H|} = l$, where $(l, |H|) = 1$. Since $d \big| |H|$, $(l, d) = 1$ and $G'H/G'$ has a supplement K/G' by Theorem 9. Consequently, $G = HK$ and H has a proper supplement in G.

For (ii), let $h \in H \cap Z(G)$, where $|h| = p$ for a prime p that divides the order of H. Thus there is a Sylow p-subgroup P of G, $P \cap Z(G) \neq \{1\}$. Given that P is a proper abelian subgroup of G, $G' \cap Z(G) \cap P = \{1\}$. Since $P \cap Z(G) \neq \{1\}$, P cannot be contained in G'. Thus H is not contained in G' and H has a supplement in G by part (i). □

The next natural step would be to generalize the result established in Theorem 5 to arbitrary subgroups. However, this is not possible as demonstrated in Example 1.

Example 1. There do exists groups G where a proper factorization for a Sylow p-subgroup P of G does not give rise to a proper factorization of G. Consider the symmetric group S_4, where $S_4 = A_4\langle b \rangle$ with $|b| = 2$. Since the Sylow 2-subgroups of S_4 are isomorphic to the dihedral group D_4, there is an element $a \in A_4$ such that $|a| = 2$ and $a \notin \Phi(P_2)$, where P_2 is a Sylow 2-subgroup of S_4. Consequently, $\langle a \rangle$ will have a proper supplement in P_2. Suppose that $S_4 = \langle a \rangle K$ for some proper subgroup K of S_4. Since $|a| = 2$ and $\langle a \rangle \cap K = \{1\}$, K is normal in S_4 and has order 12. Thus $K = A_4$, which is a contradiction as $a \in A_4$.

In addition, as demonstrated in Example 2, adding the stronger condition that $P \cap \Phi(G) \leq \Phi(P)$ for all Sylow p-subgroups of G, is not enough to make the condition true (the condition that if $H \leq G$, and there exists a prime p such that $p \big| |H|$, $p \big| [G{:}H]$, and $P = (H \cap P)K$ then H has a supplement in G).

Example 2. Let $G = S_4 \times \mathbb{Z}_9$ with $\mathbb{Z}_9 = \langle z \rangle$. By Example 1, there exists an element $a \in S_4$ such that $|a| = 2$ and $\langle a \rangle$ has no proper supplement in S_4. Since $\Phi(G) = \langle z^3 \rangle$, for each Sylow 2- or 3-subgroup P of G, $P \cap \Phi(G) \leq \Phi(P)$. Consider the subgroup $H = \langle a, z^3 \rangle = \langle a \rangle \times \langle z^3 \rangle$ of G and suppose that $G = HK$ for some proper subgroup K of G. Since $\langle z^3 \rangle \lhd G$, $G = \langle a \rangle(\langle z^3 \rangle K)$. Given that $\langle z^3 \rangle \leq \Phi(G)$, $\langle z^3 \rangle K = G$ implies $G = K$, a contradiction. Thus $\langle z^3 \rangle K$ is a proper subgroup of G. Consequently, $S_4 = G \cap S_4 = \langle a \rangle(\langle z^3 \rangle K) \cap S_4 = \langle a \rangle(\langle z^3 \rangle K \cap S_4)$. Since $\langle z^3 \rangle K \cap S_4 \neq S_4$, $\langle a \rangle$ has a supplement in S_4, a contradiction.

A first step at approaching this generalization to arbitray subgroups motivates the following theorem.

Theorem 11. *Let G be a group such that $\Phi(G) = G^{\mathcal{S}}$ and consider a proper nontrivial subgroup H of G. Then $G = HK$ for a proper subgroup K of G if and only if for some prime p such that $p\big||H|$ then p does not divide $[G{:}H]$ or for each Sylow p-subgroup P of G where $H \cap P \neq \{1\}$, $P = (H \cap P)K_p$ where K_p is a proper subgroup of P.*

Proof. Suppose that $G = HK$ for some proper subgroup K of G and that for each prime p such that $p\big||H|$ that p also divides $[G{:}H]$. Now suppose that for all Sylow subgroups P of G where $H \cap P \neq \{1\}$, that $H_p = H \cap P$ has no proper supplement in P. Then $H_p \leq \Phi(P)$, and by Lemma 5, $H_p \leq \Phi(G)$. As a result, each Sylow subgroup of H is contained in $\Phi(G)$. Thus $H \leq \Phi(G)$, a contradiction. Thus for some prime p, where $p\big||H|$ and $H_p = H \cap P \neq \{1\}$, $P = H_pK_p$ for some proper subgroup K_p of P.

Conversely, consider a proper nontrivial subgroup H of G. Suppose there is a prime p such that $p\big||H|$ and p does not divide $[G{:}H]$. If $H \leq \Phi(G)$, then $p\big||\Phi(G)|$ and p does not divide $[G{:}\Phi(G)]$, a contradiction. Thus H is not contained in $\Phi(G)$. Consequently, H is not contained in $G^{\mathcal{S}}$. In this case, H has a proper supplement by Theorem 6.

Now consider the case where for some prime p, where $p\big||H|$ and $p\big|[G{:}H]$, that $P = (H \cap P)K_p$ for some Sylow p-subgroup P of G where $H \cap P \neq \{1\}$ and K_p is proper in P. If $H \cap P \leq \Phi(G)$, then $H \cap P \leq \Phi(P)$ by Lemma 5. This contradiction implies that $H \cap P$ is not contained in $\Phi(G)$. Thus H is not contained in $\Phi(G) = G^{\mathcal{S}}$. Consequently, by Theorem 6, H has a proper supplement in G. $\square$

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Received by the editors: 28.08.2007
and in final form 08.10.2009.