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# Codes of groupoids with one-sided quasigroup conditions

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Abstract. We propose a method of description of quasigroup conditions defining one-sided quasigroup classes and one-sided quasigroup varieties by number codes. This method can be used for all one-sided quasigroups. In order to construct the code words we use the technique of Steinitz numbers.

## 1. Introduction

We propose a technique of describing quasigroup conditions defining one-sided quasigroup classes and one-sided quasigroup varieties by number codes. To obtain these descriptions we apply some combinatorial methods involving, among other things, the concept of the cycle type of a permutation. We use a technique which arises from an idea of E. Steinitz (see [10, p. 250]). Today this construction is known as Steinitz numbers ([3, 9]) (or supernatural or surnatural numbers ([8])). These numbers have found applications in various parts of algebra, especially in group theory, field theory and some related areas. Examples of such applications can be found in [5, 6, 7]. Construction of the Steinitz numbers — slightly reformulated — is recalled briefly to make the paper self-contained.

Our method has common roots with one introduced in [4], but is slightly more general and applies to all one-sided quasigroups. In the main theorem (Theorem 9.2) we show that the quasigroup conditions defining one-sided quasigroup classes and one-sided quasigroup varieties

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as well as conditions establishing relations between these classes can be precisely expressed by sequences of extended natural numbers interpreted as Steinitz numbers. This result was announced without proof at the AAA 78 – 78th Conference on General Algebra (Bern 2009). Some examples and applications are also given.

## 2. Basic notation

To avoid misunderstandings, we recall some basic notation and terminology:

- $\mathbb{N}$ denotes the set of natural numbers (nonnegative integers).
- $\mathbb{N}_1 := \mathbb{N} - \{0\}$ denotes the set of positive integers.
- $n \mid m$ where $n, m \in \mathbb{N}_1$ means that $n$ divides $m$.
- $\mathbb{P} = \{p_1, p_2, \ldots\}$ where $p_1 < p_2 < \cdots$ denotes the set of prime numbers.
- $\bar{\mathbb{N}} := \mathbb{N} \cup \{\aleph_0\}$ denotes the extension of $\mathbb{N}$ by adding a new largest element $\aleph_0$.
- $\bar{\mathbb{N}}_1 := \bar{\mathbb{N}} - \{0\}$.
- $\bar{\mathfrak{N}} = (\bar{\mathbb{N}}, \leq)$ and $\bar{\mathfrak{N}}_1 = (\bar{\mathbb{N}}_1, \leq)$ are both bounded chains.
- $|A|$ denotes the cardinality of the set $A$.
- If $A \subseteq \mathbf{Cn}$ is a set of cardinal numbers, then $\mathrm{lcm}(A)$ denotes the least common multiple of $A$.
- Let $A$ and $B$ be sets. Then $A^B := \{f \mid f \colon B \longrightarrow A\}$ denotes the set of all maps from $B$ to $A$, often identified with their graphs, so viewed as subsets of $B \times A$.

## 3. Steinitz numbers

### 3.1. Construction

The *Steinitz numbers* ([10]) are the relational system $\mathfrak{S} = (\mathbb{S}, \leq)$ where $\mathbb{S} := \bar{\mathbb{N}}^{\mathbb{P}}$ is the set of sequences $(s_p)_{p\in\mathbb{P}}$, where each $s_p$ is a natural number or $\aleph_0$, and $\leq$ is defined coordinatewise by the relation from $\bar{\mathfrak{N}} = (\bar{\mathbb{N}}, \leq)$. More precisely, if $s_1 = (s_{1,p})_{p\in\mathbb{P}}, s_2 = (s_{2,p})_{p\in\mathbb{P}} \in \mathbb{S}$ then

$$s_1 \leq s_2 \;:\Leftrightarrow\; \forall\, p \in \mathbb{P} \;\; s_{1,p} \leq s_{2,p}.$$

For reasons that will become clear shortly, we sometimes write $s_1 \mid s_2$ instead of $s_1 \leq s_2$ and call this relation 'divisibility'.

### 3.2. Closed Steinitz numbers

In the next sections we show that one can assign Steinitz numbers to quasigroups. To enable this assignment for every right (and left) quasigroup we need to extend the Steinitz numbers, just as $\mathbb{N}$ was extended to $\bar{\mathbb{N}}$.

Let $\bar{\mathbb{P}} = \mathbb{P} \cup \{\aleph_0\}$ and $\bar{\mathbb{S}} := \bar{\mathbb{N}}^{\bar{\mathbb{P}}}$. Members of $\bar{\mathbb{S}}$ are called *closed Steinitz numbers*. Just as in $\mathbb{S}$, the relation $\leq$ on $\bar{\mathbb{S}}$ is defined coordinatewise, and sometimes called 'divisibility'. We write $\bar{\mathfrak{S}} = (\bar{\mathbb{S}}, \leq) = (\bar{\mathbb{S}}, |)$.

### 3.3. Selected properties

We simplifiy the notations as follows:

- $(n_1, \ldots, n_k, \overset{\infty}{\ldots})$ denotes the Steinitz number $(s_p)_{p \in \mathbb{P}}$ such that: $s_{p_1} = n_1, \ldots, s_{p_k} = n_k$ and $s_{p_i} = n_k$ for $i \in \mathbb{N}$ and $i > k$.
- $(n_1, \ldots, n_k, \overset{\infty}{\ldots})(n_{\aleph_0})$ denotes the closed Steinitz number $(s_p)_{p \in \bar{\mathbb{P}}}$ such that:
  $s_{p_1} = n_1, \ldots, s_{p_k} = n_k$, $s_{p_i} = n_k$ for $i \in \mathbb{N}$, $i > k$ and $s_{\aleph_0} = n_{\aleph_0}$.

The easy proof of the following fact is omitted.

**Fact 3.1.**

(i) *The relational systems $\mathfrak{S}$ and $\bar{\mathfrak{S}}$ are both complete distributive bounded lattices. $(\aleph_0, \overset{\infty}{\ldots})$ and $(0, \overset{\infty}{\ldots})$ are the unit and null respectively in $\mathfrak{S}$, whereas $(\aleph_0, \overset{\infty}{\ldots})(\aleph_0)$ and $(0, \overset{\infty}{\ldots})(0)$ are the unit and null respectively in $\bar{\mathfrak{S}}$.*

(ii) *There are two natural monomorphic embeddings of relational systems:*

$$(\ ) \colon \mathfrak{N}_1 \xrightarrow{\text{mon}} \mathfrak{S} \quad (n \mapsto (n)), \tag{1}$$

$$\langle\ \rangle \colon \bar{\mathfrak{N}}_1 \xrightarrow{\text{mon}} \bar{\mathfrak{S}} \quad (n \mapsto \langle n \rangle), \tag{2}$$

*defined as follows:*

- $(1) = (0, \overset{\infty}{\ldots}), \quad \langle 1 \rangle = (0, \overset{\infty}{\ldots})(0)$.
- *If $n \in \mathbb{N}$, $n > 1$ and $n = 2^{n_1} 3^{n_2} \ldots p^{n_k}$ is the prime factorization of $n$, then*

$$(n) = (n_1, \ldots, n_k, \overset{\infty}{\ldots}), \quad \langle n \rangle = (n_1, \ldots, n_k, \overset{\infty}{\ldots})(0).$$

- $\langle \aleph_0 \rangle = (0, .\overset{\infty}{.}.)(1)$.

(iii) *$\mathfrak{S}$ is a proper sublattice of $\bar{\mathfrak{S}}$ but because these lattices have different units, $\mathfrak{S}$ is not a bounded sublattice of $\bar{\mathfrak{S}}$.*

It is now clear that if $n, m \in \mathbb{N}$ and $n, m \geq 1$, then $(n) \leq (m)$ means exactly that $n \mid m$, which explains the name 'divisibility' for the order relation in $\mathfrak{N}$.

### 3.4. Examples

(i) Consider the number 360 in the decimal system. Then:
$(360) = (3, 2, 1, 0, .\overset{\infty}{.}.)$ or less formally but more intuitively
$(360) = 2^3 3^2 5^1 7^0 \dots p_i^0 \dots$;
$\langle 360 \rangle = (3, 2, 1, 0, .\overset{\infty}{.}.)(0)$ or less formally but more intuitively
$\langle 360 \rangle = 2^3 3^2 5^1 7^0 \dots p_i^0 \dots \aleph_0^0$.

(ii) The sequence $(3, 2, 1, 0, .\overset{\infty}{.}.)(1)$ is a 'proper' closed Steinitz number, i.e.

$$\forall\, n \in \mathbb{N}_1 \quad \langle n \rangle \neq (3, 2, 1, 0, .\overset{\infty}{.}.)(1).$$

(iii) Evidently $\langle 360 \rangle = (3, 2, 1, 0, .\overset{\infty}{.}.)(0) \mid (3, 2, 1, 0, .\overset{\infty}{.}.)(1)$.

## 4. Groupoids with right (left) quasigroup properties

### 4.1. Right (left) quasigroups and quasigroups

#### Definitions

By a groupoid we mean a pair $\mathfrak{G} = (G, \cdot\,)$ with universe $G$ and binary operation

$$\cdot : G \times G \longrightarrow G \quad ((x, y) \mapsto xy).$$

In the following, the Gothic letters (without or with subscripts), like $\mathfrak{G}$, $\mathfrak{H}$, $\mathfrak{G}_i$ and $\mathfrak{H}_j$ stand for groupoids only.

By a *right* (resp. *left*) *quasigroup* we mean a groupoid $\mathfrak{G}$ such that for all $a, b \in G$ the equation $xa = b$ (resp. $ax = b$) has a unique solution. By a *quasigroup* we mean a groupoid which is a right and left quasigroup simultaneously.

#### Duality

For a groupoid $\mathfrak{G} = (G, \cdot\,)$ we have the *dual groupoid* $\mathfrak{G}^{\leftarrow} = (G, \circ\,)$ where $x \circ y := yx$. Clearly $(\mathfrak{G}^{\leftarrow})^{\leftarrow} = \mathfrak{G}$. Let $t$ be a term over a language appropriate for groupoid theory. Let $\mathfrak{G}$ be a groupoid. Then the interpretation $t^{\mathfrak{G}^{\leftarrow}}$ is named the *dual sentence* to the interpretation $t^{\mathfrak{G}}$. If a groupoid $\mathfrak{G}$

is a right quasigroup then its dual groupoid is a left quasigroup and vice versa. This duality establishes a symmetrical correspondence between 'right' and 'left' versions of statements. For conciseness we formulate almost all statements below in one (right) version only.

**Combinatorial approach**

In combinatorial terminology, a groupoid $\mathfrak{G}$ is a right quasigroup if and only if for every $a \in G$ the right translation

$$r_a^{\mathfrak{G}} \colon G \longrightarrow G \; (x \mapsto xa)$$

is a bijection. Thus, we have the function

$$r^{\mathfrak{G}} \colon G \longrightarrow S_G \; (x \mapsto r_x^{\mathfrak{G}})$$

where $S_G$ denotes the set of bijections of $G$. Denote by $\mathcal{QG}^*$ the class of right quasigroups and by $\mathcal{QG}^*[G]$ the set of right quasigroups with universe $G$. In this way we obtain the bijection

$$\varrho_G^* \colon \mathcal{QG}^*[G] \longrightarrow (S_G)^G \; (x \mapsto r^x). \tag{3}$$

Informally but intuitively, a right quasigroup structure on $G$ can be seen as a bundle of bijections $(r_a^{\mathfrak{G}})_{a \in G}$. More formally, one can define a map $\varphi : G \times G \to G$ by

$$\varphi(u, v) = a :\Leftrightarrow ua = v$$

Then for each $a \in G$ the fibre $\varphi^{-1}(a) = \{(u, v) \mid \varphi(u, v) = a\} = \{(u, ua) \mid a \in G\}$, so it is precisely (the graph of) $r_a^{\mathfrak{G}}$ (cf. [1]).

This interpretation is illustrated in Figure 1. The fibres are presented as vertical lines. Horizontal coordinates determine the values of the right translations. The universe of the quasigroup is $G = \{a_\iota \mid \iota \in I\}$.

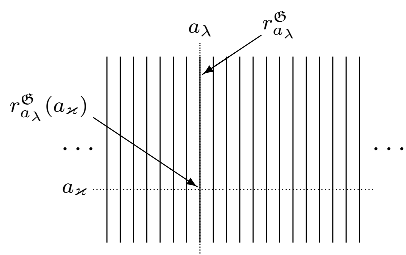


Figure 1: Combinatorial interpretation of a right quasigroup $\mathfrak{G}$

### Cycle types

For any set $G$, every permutation of $G$ can be (uniquely) decomposed into disjoint cycles. This yields the map

$$\tau\colon S_G \longrightarrow \mathbf{Cn}^{\bar{\mathbb{N}}_1} \ (x \mapsto \tau(x) = \vec{x} = (\vec{x}_i)_{i\in\bar{\mathbb{N}}_1}) \tag{4}$$

where $f_i \in \mathbf{Cn}$ denotes the number of cycles of length $i$ appearing in the disjoint cycle decomposition of $f$ ($f_{\aleph_0}$ denotes the number of infinite cycles). This map is called the *cycle type description* of the permutation $f$ (see [2]).

If a right quasigroup $\mathfrak{G}$ is fixed then we write simply $r_a$ instead of $r_a^{\mathfrak{G}}$ and $r$ instead of $r^{\mathfrak{G}}$. The map giving the cycle type description of all the permutations $r_a$ for $a \in G$,

$$\vec{r} := \tau r\colon G \longrightarrow \mathbf{Cn}^{\bar{\mathbb{N}}_1} \ (x \mapsto \tau(r_x) = \vec{r_x} = (\vec{r_{xi}})_{i\in\bar{\mathbb{N}}_1})$$

is called the *cycle type* of $\mathfrak{G}$. Recall that $\varrho_G^*(\mathfrak{G}) = r^{\mathfrak{G}} = r \in (S_G)^G$. Thus we have the following map (cf. (3)):

$$\tau^G\varrho_G^*\colon \mathcal{QG}^*[G] \longrightarrow \left(\mathbf{Cn}^{\bar{\mathbb{N}}_1}\right)^G \ (x \mapsto r^{\vec{x}}) \tag{5}$$

where $\tau^G$ is the product map defined on $(S_G)^G$ as follows:

$$\tau^G((f_a)_{a\in G}) := (\tau(f_a))_{a\in G}.$$

## 5. Cycle codes. Code numbers

### 5.1. Definitions and main concepts

### Full cycle codes

Let $\vec{r}$ be the cycle type of a right quasigroup $\mathfrak{G}$. Let $i \in \bar{\mathbb{N}}_1$. The *i-th support* of $\mathfrak{G}$ is the set

$$\mathrm{supp}_i\vec{r} := \{a \mid \vec{r_{ai}} \neq 0\}. \tag{6}$$

By the *full cycle code* of $\mathfrak{G}$ we mean the sequence

$$\varsigma(\vec{r}) = (\varsigma_i)_{i\in\bar{\mathbb{N}}_1} := \sum_{a\in G}\vec{r_a} = \sum_{a\in G}(\vec{r_{ai}})_{i\in\bar{\mathbb{N}}_1} = (\sum_{a\in G}\vec{r_{ai}})_{i\in\bar{\mathbb{N}}_1} \tag{7}$$

where

$$\varsigma_i = \sum_{a\in G} r_{a,i} := \begin{cases} \sum\limits_{a\,\in\,\mathrm{supp}_i\vec{r}} \vec{r_{ai}} & \text{if} \quad \begin{cases} |\,\mathrm{supp}_i\vec{r}\,| < \aleph_0 \\ \text{and} \\ \forall\, a \in \mathrm{supp}_i\vec{r} \ \ \vec{r_{ai}} < \aleph_0 \end{cases} ; \\ |\,\mathrm{supp}_i\vec{r}\,|(\bigcup\{\,\vec{r_{ai}} \mid a \in \mathrm{supp}_i\vec{r}\,\}) & \text{otherwise.} \end{cases}$$

The full cycle code has a connection with the cardinality of the groupoid. This connection can be easily seen for finite groupoids (cf. (10) below).

### Cycle codes

By the *cycle code* of $\mathfrak{G}$ we mean the sequence of closed Steinitz numbers

$$\vec{\mathfrak{G}} := (\varepsilon_i)_{i\in\bar{\mathbb{N}}_1} \tag{8}$$

associated with the full cycle code $\varsigma(\vec{r})$ (cf. (7)) as follows:

$$\varepsilon_i := \begin{cases} \langle i\rangle & \text{if } \varsigma_i \neq 0, \\ \langle 1\rangle = (0,\overset{\infty}{\ldots})(0) & \text{otherwise.} \end{cases}$$

The cycle code carries information about the 'torsion' structure of the groupoid (cf. $(*^t)$ below). From the technical point of view $\vec{\mathfrak{G}} \in \bar{\mathbb{S}}^{\bar{\mathbb{N}}_1}$ and can be interpreted as an infinite matrix.

### Code numbers

By the *code number* of $\mathfrak{G}$ we mean the closed Steinitz number

$$\langle\mathfrak{G}\rangle := \max\{\varepsilon_i \mid i \in \bar{\mathbb{N}}_1\}. \tag{9}$$

The code number carries summary information about the 'torsion' and 'equational' structure of the groupoid (see Theorem 9.2 below).

## 5.2. Examples

Let $\mathfrak{G}_1$ be the right quasigroup with universe $\mathbb{N}_1$ determined by the family of permutations $(f_n)_{n\in\mathbb{N}_1}$ defined by

$$f_n = (1\ 2) \cup \ldots \cup (2n-1\ \ 2n) \cup (2n+1\ \ \ldots\ \ 2n+p_n) \cup \mathrm{id} \quad (n \in \mathbb{N}_1).$$

Thus

$$\begin{aligned} &f_1 = (1\ 2) \cup (3\ 4) \cup \mathrm{id}, \\ &f_2 = (1\ 2) \cup (3\ 4) \cup (5\ 6\ 7) \cup \mathrm{id}, \\ &f_3 = (1\ 2) \cup (3\ 4) \cup (5\ 6) \cup (7\ 8\ 9\ 10\ 11) \cup \mathrm{id}, \\ &\ldots \end{aligned}$$

$\mathfrak{G}_1$ has the infinite Cayley table:

| | $f_1$ | $f_2$ | $f_3$ | $\dots$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\mathfrak{G}_1$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | $\dots$ |
| 1 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | $\dots$ |
| 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | $\dots$ |
| 3 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | $\dots$ |
| 4 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | $\dots$ |
| 5 | 5 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | $\dots$ |
| 6 | 6 | 7 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | $\dots$ |
| 7 | 7 | 5 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | $\dots$ |
| 8 | 8 | 8 | 9 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | $\dots$ |
| 9 | 9 | 9 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | $\dots$ |
| 10 | 10 | 10 | 11 | 11 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | $\dots$ |
| 11 | 11 | 11 | 7 | 12 | 12 | 12 | 12 | 12 | 12 | 12 | 12 | $\dots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ . |

Thus:

- $\vec{r} = ((\vec{r_j}_i)_{i\in\bar{\mathbb{N}}_1})_{j\in\mathbb{N}_1}$ where

$$\vec{r_j}_i = \begin{cases} \aleph_0 & \text{if} \quad j \in \mathbb{N}_1, \quad i = 1; \\ 2 & \text{if} \quad j = 1, \quad i = 2; \\ j & \text{if} \quad j \geq 2, \quad i = 2; \\ 1 & \text{if} \quad j \geq 2, \quad i = p_j; \\ 0 & \text{otherwise.} \end{cases}$$

- $\varsigma(\vec{r}) = (\varsigma_i)_{i\in\bar{\mathbb{N}}_1}$ where

$$\varsigma_i = \begin{cases} \aleph_0 & \text{if} \quad i \in \{1,2\}; \\ 1 & \text{if} \quad i \in \mathbb{P},\ i \neq 2; \\ 0 & \text{otherwise.} \end{cases}$$

- $\vec{\mathfrak{G}_1} = (\varepsilon_i)_{i\in\bar{\mathbb{N}}_1}$ where

$$\varepsilon_i = \begin{cases} \langle p_i \rangle & \text{if} \quad i \in \mathbb{N}_1; \\ \langle 1 \rangle & \text{if} \quad i = \aleph_0. \end{cases}$$

- $\langle \mathfrak{G}_1 \rangle = (1, \overset{\infty}{\ldots})(0)$.

Let $\mathfrak{G}_2$ be the right quasigroup with universe $\mathbb{N}_1$ determined by the family of permutations $(f_n)_{n\in\mathbb{N}_1}$ defined by

$$f_n := \begin{cases} g & \text{if } n = 1, \\ (1 \ldots n-1) \cup \mathrm{id}_{\mathbb{N}_1 - \{1,\ldots,n-1\}} & \text{if } n > 1, \end{cases}$$

where $g$ is the permutation of $\mathbb{N}_1$ defined as follows:

$$g(k) = \begin{cases} k-2 & \text{if } k \text{ is odd, } k \geq 3, \\ k+2 & \text{if } k \text{ is even, } k \geq 2, \\ 2 & \text{if } k = 1. \end{cases}$$

Thus:

- $\vec{r} = ((\vec{r_j}_i)_{i\in\bar{\mathbb{N}}_1})_{j\in\mathbb{N}_1}$ where

$$\vec{r_j}_i = \begin{cases} 0 & \text{if} \quad j = 1, \quad i \neq \aleph_0; \\ 1 & \text{if} \quad j = 1, \quad i = \aleph_0; \\ \aleph_0 & \text{if} \quad j > 1, \quad i = 1; \\ 1 & \text{if} \quad j > 1, \quad i = j; \\ 0 & \text{otherwise.} \end{cases}$$

- $\varsigma(\vec{r}) = (\varsigma_i)_{i\in\bar{\mathbb{N}}_1}$ where

$$\varsigma_i = \begin{cases} \aleph_0 & \text{if} \quad i = 1; \\ 1 & \text{otherwise.} \end{cases}$$

- $\vec{\mathfrak{G}_2} = (\langle i \rangle)_{i\in\bar{\mathbb{N}}_1}$.
- $\langle \mathfrak{G}_2 \rangle = (\aleph_0, \overset{\infty}{\ldots})(1)$.

There is no positive integer $n$ such that $\langle n \rangle = \langle \mathfrak{G}_2 \rangle$.

## 6. Cycle codes and code numbers of finite groupoids

For all finite and some other one-sided quasigroups one can construct so called code tables. These tables — presented as matrices — allow us to clearly present the meaning of the notions introduced above.

### 6.1. Code tables

Let $\mathfrak{G} = (G, \cdot)$ where $G = \{g_1, g_2, \ldots, g_n\}$ is a finite right quasigroup with Cayley table

| $\mathfrak{G}$ | $g_1$ | $g_2$ | $\cdots$ | $g_n$ |
|---|---|---|---|---|
| $g_1$ | $g_{11}$ | $g_{12}$ | $\cdots$ | $g_{1n}$ |
| $g_2$ | $g_{21}$ | $g_{22}$ | $\cdots$ | $g_{2n}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $g_n$ | $g_{n1}$ | $g_{n2}$ | $\cdots$ | $g_{nn}$ |

For finite groupoids the code parameters defined in (7), (8) and (9) can be clearly presented in the following *code table* (the formally infinite sequences are written in shorter forms, with null elements omitted):

| $(r_g)_{g\in G}$ | $\vec{r_{g_1}}$ | $\vec{r_{g_2}}$ | $\cdots$ | $\vec{r_{g_n}}$ | $\varsigma(\vec{r})$ | $\vec{\mathfrak{G}}$ |
|---|---|---|---|---|---|---|
| 1 | $r_{g_1,1}$ | $r_{g_2,1}$ | $\cdots$ | $r_{g_n,1}$ | $\varsigma_1$ | $\varepsilon_1$ |
| 2 | $r_{g_1,2}$ | $r_{g_2,2}$ | $\cdots$ | $r_{g_n,2}$ | $\varsigma_2$ | $\varepsilon_2$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $n$ | $r_{g_1,n}$ | $r_{g_2,n}$ | $\cdots$ | $r_{g_n,n}$ | $\varsigma_n$ | $\varepsilon_n$ |
| | | | | | | $\langle\mathfrak{G}\rangle$ |

On can check that the following formula holds:

$$\sum_{i\in\mathbb{N}_1} i\varsigma_i = |G|. \tag{10}$$

### 6.2. Example

Let $\mathfrak{G}_3$ be the right quasigroup with Cayley table

| $\mathfrak{G}_3$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 1 |
| 1 | 1 | 1 | 1 | 2 | 2 |
| 2 | 2 | 2 | 3 | 3 | 3 |
| 3 | 3 | 4 | 4 | 4 | 4 |
| 4 | 4 | 3 | 2 | 1 | 0 |

The code table of $\mathfrak{G}_3$ is:

| $(r_a)_{a\in G_3}$ | $\vec{r_0}$ | $\vec{r_1}$ | $\vec{r_2}$ | $\vec{r_3}$ | $\vec{r_4}$ | $\varsigma(\vec{r})$ | $\vec{\mathfrak{G}_3}$ | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 5 | 5 | 2 | 1 | 0 | 13 | $\langle 1\rangle$ | $=$ | $(0,\overset{\infty}{\ldots})(0)$ |
| 2 | 0 | 0 | 0 | 0 | 0 | 0 | $\langle 1\rangle$ | $=$ | $(0,\overset{\infty}{\ldots})(0)$ |
| 3 | 0 | 0 | 1 | 0 | 0 | 1 | $\langle 3\rangle$ | $=$ | $(0,1,0,\overset{\infty}{\ldots})(0)$ |
| 4 | 0 | 0 | 0 | 1 | 0 | 1 | $\langle 4\rangle$ | $=$ | $(2,0,\overset{\infty}{\ldots})(0)$ |
| 5 | 0 | 0 | 0 | 0 | 1 | 1 | $\langle 5\rangle$ | $=$ | $(0,0,1,0,\overset{\infty}{\ldots})(0)$ |
| | | | | | | | $\langle\mathfrak{G}_3\rangle$ | $=$ | $(2,1,1,0,\overset{\infty}{\ldots})(0)$ |

Thus $\langle\mathfrak{G}_3\rangle = (2,1,1,0,\overset{\infty}{\ldots})(0) = \langle 60\rangle$.

## 7. Connections of codes of groupoids with other notions

For convenience of the reader we repeat some definitions from [4] to make our presentation self-contained.

Consider the function

$$\sigma\colon \mathbf{Cn}^{\bar{\mathbb{N}}_1} \longrightarrow \mathcal{P}(\bar{\mathbb{N}}_1)$$

where $\mathcal{P}(\bar{\mathbb{N}}_1)$ denotes the power set of $\bar{\mathbb{N}}_1$ and $\sigma(g) := \{i \in \bar{\mathbb{N}}_1 \mid g(i) \neq 0\}$ for $g \in \mathbf{Cn}^{\bar{\mathbb{N}}_1}$. We call $\sigma(g)$ the *support* of $g$. The composition $\sigma\tau$ (cf. (4)) assigns to every permutation $f \in S_G$ its *cycle support* $\sigma(\vec{f})$. For its product function we have $(\sigma\tau)^G((f_a)_{a\in G}) = \sigma^G(\vec{f}) = (\sigma(\vec{f_a}))_{a\in G}$. Let $\varrho_G^{*\sigma}$ be the function defined on $\mathcal{QG}^*[G]$ as follows:

$$\varrho_G^{*\sigma}(\mathfrak{G}) := (\sigma\tau)^G(\varrho_G^*(\mathfrak{G})).$$

We call $\varrho_G^{*\sigma}(\mathfrak{G})$ the *cycle support* of the right quasigroup $\mathfrak{G}$. Consider $\operatorname{im}\varrho_G^{*\sigma}(\mathfrak{G})$, the image of the function $\varrho_G^{*\sigma}(\mathfrak{G})$. Define

$$\Sigma_{\mathfrak{G}}^* := \bigcup \operatorname{im}\varrho_G^{*\sigma}(\mathfrak{G}).$$

It is clear that the set $\Sigma_{\mathfrak{G}}^*$ is nonempty. Moreover, $n \in \Sigma_{\mathfrak{G}}^*$ if and only if there exists $a \in G$ such that there exists a cycle of length $n \in \bar{\mathbb{N}}_1$ in the disjoint cycle decomposition of $r_a^{\mathfrak{G}}$. By the *cycle weight* of $\mathfrak{G}$ we mean (cf. [4]) the number

$$\omega_{\mathfrak{G}}^* := \operatorname{lcm}(\Sigma_{\mathfrak{G}}^*).$$

It is clear that this is a positive integer and it does not exist for all right quasigroups. As in [4] we say that a right quasigroup $\mathfrak{G}$ has *finite cycle type* if

$$0 < |\Sigma_{\mathfrak{G}}^*| < \aleph_0 \quad \wedge \quad \Sigma_{\mathfrak{G}}^* \subseteq \mathbb{N}_1. \tag{11}$$

A right quasigroup $\mathfrak{G}$ has *countable cycle type* if $\Sigma_{\mathfrak{G}}^* \subseteq \mathbb{N}_1$. Some applications of the above notions apear in Theorems 9.1 and 10.1 of Section 9.

It is clear that $\mathfrak{G}$ has finite cycle type if and only if $\omega_{\mathfrak{G}}^*$ is 'well defined', i.e. $\omega_{\mathfrak{G}}^* \in \mathbb{N}_1$. Evidently there exist right quasigroups not having finite cycle type (for example $\mathfrak{G}_1$ and $\mathfrak{G}_2$ from Section 5.2). For them, the cycle weight does not exist. The code number introduced in this paper is a more general notion and applies for all right quasigroups. The following theorem gives connections between these notions:

**Theorem 7.1.** *Let $\mathfrak{G}$ be a right quasigroup. Then*

$$\forall\, n \in \mathbb{N}_1 \;\; (\omega_{\mathfrak{G}}^* = n \iff \langle \mathfrak{G} \rangle = \langle n \rangle\,). \tag{12}$$

*Proof.* $\Rightarrow$. Because (11) is satisfied in $\mathfrak{G}$, the following formula holds:

$$\exists\, p \in \mathbb{P}\ \ \forall\, m \in \Sigma^*_{\mathfrak{G}}\ \ \exists\, (\alpha_{m,2}, \dots, \alpha_{m,p}) \in \mathbb{N}^p\ \ m = 2^{\alpha_{m,2}} 3^{\alpha_{m,3}} \dots p^{\alpha_{m,p}}.$$

Thus $\omega^*_{\mathfrak{G}} = 2^{\alpha_2} 3^{\alpha_3} \dots p^{\alpha_p}$ where $\alpha_i = \max\{\alpha_{m,i} \mid m \in \Sigma^*_{\mathfrak{G}}\}$. Hence $\langle \omega^*_{\mathfrak{G}} \rangle = (\alpha_2, \alpha_3, \dots, \alpha_p, 0, \overset{\infty}{\dots})(0)$ where $\alpha_i \neq 0$ if and only if for some $a \in G$ there exists a cycle of length $k \in \bar{\mathbb{N}}_1$ in the disjoint cycle decomposition of $r^{\mathfrak{G}}_a$ and $i \mid k$.

Let us consider the cycle code of $\mathfrak{G}$, i.e. the sequence $\vec{\mathfrak{G}} = (\varepsilon_i)_{i \in \bar{\mathbb{N}}_1}$ (see (8)). Thus there exists $l \in \bar{\mathbb{N}}_1$ such that $\varepsilon_i = \langle 1 \rangle$ for all $i \geq l$. Moreover, for $i < l$ we know that $\varepsilon_i \neq \langle 1 \rangle$ if and only if for some $a \in G$ there exists a cycle of length $i \in \bar{\mathbb{N}}_1$ and $i \neq 1$ in the disjoint cycle decomposition of $r^{\mathfrak{G}}_a$. Thus $\varepsilon_i \neq \langle 1 \rangle$ if and only if $i \in \Sigma^*_{\mathfrak{G}}$ and $i \neq 1$. Hence $\langle \mathfrak{G} \rangle = \max\{\varepsilon_i \mid i \in \Sigma^*_{\mathfrak{G}}\}$. Thus $\langle \mathfrak{G} \rangle = \langle \omega^*_{\mathfrak{G}} \rangle$.

$\Leftarrow$. Let $\langle \mathfrak{G} \rangle = \langle n \rangle$ for $n \in \mathbb{N}_1$. Then $\Sigma^*_{\mathfrak{G}}$ satisfies (11). Thus $\omega^*_{\mathfrak{G}}$ exists and $\omega^*_{\mathfrak{G}} = m$ for some $m \in \mathbb{N}_1$. Applying the part ($\Rightarrow$) proved previously we see that $\langle \mathfrak{G} \rangle = \langle n \rangle = \langle m \rangle$. Thus $n = m$ and $\omega^*_{\mathfrak{G}} = n$. □

## 8. Power conditions. Power classes

In this section we focus our attention on conditions defining quasigroup properties.

### 8.1. Main notions

The family of *'power' terms* $\{xy^n \mid n \in \mathbb{N}_1\}$ is inductively defined as follows:

$$xy^1 := xy, \quad xy^n := (xy^{n-1})y,$$

and similarly for $\{{}^n yx \mid n \in \mathbb{N}_1\}$. We recall some 'power' conditions associated with these terms (see [4]). For $n \in \mathbb{N}_1$ we have the identity

$$xy^n = x. \tag{$*^n$}$$

The variety of groupoids defined by the identity $(*^n)$ is denoted by $\mathcal{Q}_n$. The elements of $\mathcal{Q}_n$ are called *groupoids of right exponent* $n$. We set

$$\mathcal{Q} := \{\mathcal{Q}_n \mid n \in \mathbb{N}_1\}, \quad \mathcal{Q}^* := \bigcup \mathcal{Q}. \tag{13}$$

The groupoids in $\mathcal{Q}^*$ are said to be of *finite right exponent*. As a generalization of $(*^n)$ we propose the following conditions:

$$\forall\, x, y\ \exists\, n \in \mathbb{N}_1\ \ xy^n = x. \tag{$*^t$}$$

We call $\mathfrak{G}$ *right-torsion* if the formula $(*^t)$ is satisfied in $\mathfrak{G}$. The class of right-torsion groupoids is denoted by $\mathcal{T}^*$ (in [4] this class is denoted by $\mathcal{QG}_t^*$).

Let $s \in \mathbb{S}$. The following formulas can be seen as generalizations of the identities $(*^n)$ for Steinitz numbers:

$$\exists\, n \in \mathbb{N}_1\ \forall\, x, y \quad n \mid s,\ xy^n = x. \tag{$*^s$}$$

Thus $\mathfrak{G}$ satisfies $(*^s)$ if and only if there exists $n \in \mathbb{N}_1$ such that $n \mid s$ and $\mathfrak{G}$ satisfies $(*^n)$. $\mathcal{Q}_{\bar{s}}$ denotes the class of groupoids satisfying $(*^s)$. The members of $\mathcal{Q}_{\bar{s}}$ are called *s-bounded right exponent groupoids*. If $m \in \mathbb{N}_1$ then $\mathcal{Q}_m = \mathcal{Q}_{\bar{m}}$. Thus the bar over $s$ in $\mathcal{Q}_{\bar{s}}$ can be omitted. Clearly if $s \in \mathbb{S}$ then

$$\mathcal{Q}_s = \bigcup_{\substack{n \in \mathbb{N}_1 \\ n \mid s}} \mathcal{Q}_n.$$

Thus

$$\bigcup_{s \in \mathbb{S}} \mathcal{Q}_s = \bigcup_{s \in \mathbb{S}} \Big( \bigcup_{\substack{n \in \mathbb{N}_1 \\ n \mid s}} \mathcal{Q}_n \Big) = \mathcal{Q}^*.$$

Let $s \in \mathbb{S}$. As a generalization of the condition $(*^t)$ for Steinitz numbers we have:

$$\forall\, x, y\ \exists\, n \in \mathbb{N}_1 \quad n \mid s,\ xy^n = x. \tag{$*_s^t$}$$

The class of all groupoids satisfying $(*_s^t)$ is denoted by $\mathcal{T}_s$. The elements of $\mathcal{T}_s$ are called *s-bounded right torsion groupoids*. We have the following equality:

$$\bigcup_{s \in \mathbb{S}} \mathcal{T}_s = \mathcal{T}^*.$$

To establish connections between the above classes we record the following proposition:

**Proposition 8.1** ([4], Proposition 14)**.**

$$\mathcal{Q}^* \subsetneq \mathcal{T}^* \subsetneq \mathcal{QG}^*. \tag{14}$$

## 9. Main results

We start by recalling a useful theorem from [4].

**Theorem 9.1** ([4], Theorem 5)**.**

(i) $\mathfrak{G} \in \mathcal{Q}_n$ *if and only if* $\mathfrak{G}$ *is a right quasigroup of finite cycle type and its cycle weight* $\omega^*_{\mathfrak{G}}$ *divides* $n$.

(ii) *Let* $\mathfrak{G}$ *be a right quasigroup of finite cycle type. Then*

$$\mathfrak{G} \in \mathcal{Q}_{\omega^*_{\mathfrak{G}}} - \bigcup_{i<\omega^*_{\mathfrak{G}}} \mathcal{Q}_i.$$

In the next theorem, for uniformity of notation, we apply among other things the map described in (1) and we identify $\mathbb{N}_1$ with its image under $\langle\ \rangle$ in $\mathbb{S}$ as well as $\mathbb{S}$ with its image under inclusion in $\bar{\mathbb{S}}$. We consequently identify $\mathfrak{N}_1$ with its isomorphic image in $\mathfrak{S}$ as well as $\mathfrak{S}$ with its embedding in $\bar{\mathfrak{S}}$. Thus

$$\mathbb{N}_1 \subseteq \mathbb{S} \subseteq \bar{\mathbb{S}}.$$

Also in the proof of this theorem for $n \in \mathbb{N}_1$ we write briefly $n$ instead of $\langle n \rangle$.

**Theorem 9.2.** *Let* $\mathfrak{G}$ *be a right quasigroup. Then*

$$\mathfrak{G} \in \mathcal{Q}^* \iff \langle \mathfrak{G} \rangle \in \mathbb{N}_1. \tag{15}$$

$$\mathfrak{G} \in \mathcal{T}^* - \mathcal{Q}^* \iff \langle \mathfrak{G} \rangle \in \mathbb{S} - \mathbb{N}_1. \tag{16}$$

$$\mathfrak{G} \in \mathcal{QG}^* - \mathcal{T}^* \iff \langle \mathfrak{G} \rangle \in \bar{\mathbb{S}} - \mathbb{S}. \tag{17}$$

*Proof.* (15), $\Rightarrow$. Let $\mathfrak{G} \in \mathcal{Q}^*$. Then using (13) and next Theorem 9.1 we find that $\mathfrak{G} \in \mathcal{Q}_{\omega^*_{\mathfrak{G}}}$ (where $\omega^*_{\mathfrak{G}} \in \mathbb{N}_1$). Hence by (12) the equality $\langle \mathfrak{G} \rangle = \omega^*_{\mathfrak{G}}$ is satisfied.

(15), $\Leftarrow$. Let $\langle \mathfrak{G} \rangle = n$ for some $n \in \mathbb{N}_1$. Then using Theorem 9.1 and next (13) we infer that $\mathfrak{G} \in \mathcal{Q}^*$.

(16), $\Rightarrow$. Let $\mathfrak{G} \in \mathcal{T}^* - \mathcal{Q}^*$. Evidently $\mathfrak{G} \in \mathcal{QG}^*$. By assumption $\mathfrak{G}$ is a right torsion quasigroup, i.e. $\mathfrak{G}$ satisfies $(*^t)$. Thus there is no element $a \in G$ such that $r_a$ has an infinite cycle in the disjoint cycle decomposition. This means that $\aleph_0 \notin \Sigma^*_{\mathfrak{G}}$, i.e. $\Sigma^*_{\mathfrak{G}} \subseteq \mathbb{N}_1$. Let $s \in \bar{\mathbb{S}}$ be such that $\langle \mathfrak{G} \rangle = s$ (recall that for every right quasigroup this number exists: cf. Section 5.1). Thus $s_{\aleph_0} = 0$ and $\langle \mathfrak{G} \rangle \in \mathbb{S}$. By assumption, $\mathfrak{G} \notin \mathcal{Q}^*$. Theorem 7.1 implies that (11) is not satisfied in $\mathfrak{G}$. Because the second condition of (11) is satisfied, the first does not hold in $\mathfrak{G}$. This means that $|\Sigma^*_{\mathfrak{G}}| = \aleph_0$. Hence $\langle \mathfrak{G} \rangle \in \mathbb{S} - \mathbb{N}_1$.

(16), $\Leftarrow$. Let $\langle \mathfrak{G} \rangle = s$ for some $s \in \mathbb{S} - \mathbb{N}_1$. Thus for every $i \in \mathbb{N}_1$ there exists $j \in \mathbb{N}_1$ such that $i \leq j$, $s_j \neq 0$. Moreover, $s_{\aleph_0} = 0$. In this situation:

(a) There is no $a \in G$ such that $r_a$ has an infinite cycle as factor in the disjoint cycle decomposition.

(b) For every $i \in \mathbb{N}_1$ there exists $a \in G$ such that in the disjoint cycle decomposition of $r_a$ there exists a factor of length greater than $i$.

Hence, $\mathfrak{G} \in \mathcal{T}^*$ (by (a)) and $\mathfrak{G} \notin \mathcal{Q}^*$ (by (b)).

(17), $\Rightarrow$. Let $\mathfrak{G} \in \mathcal{QG}^* - \mathcal{T}^*$. This means that $\mathfrak{G}$ is a right quasigroup not satisfying $(*^t)$. Thus

$$\exists y \, \exists x \, \forall n \in \mathbb{N}_1 \;\; xy^n \neq x.$$

Hence for some $a \in G$ the following formula is satisfied:

$$\exists x \, \forall n \in \mathbb{N}_1 \;\; xa^n \neq x.$$

Thus for some $a \in G$, $r_a$ has an infinite cycle in the disjoint cycle decomposition. Hence $\langle \mathfrak{G} \rangle \in \bar{\mathbb{S}} - \mathbb{S}$.

(17), $\Leftarrow$. Let $\langle \mathfrak{G} \rangle \in \bar{\mathbb{S}} - \mathbb{S}$. Thus $\langle \mathfrak{G} \rangle_{\aleph_0} \neq 0$. Thus for some $a \in G$, $r_a$ has an infinite cycle in the disjoint cycle decomposition. Hence $\mathfrak{G}$ does not satisfy $(*^t)$ and $\mathfrak{G} \in \mathcal{QG}^* - \mathcal{T}^*$. □

## 10. Examples and applications

### 10.1. Examples

#### $s$-bounded right exponent groupoids

Let $(\mathfrak{H}_n)_{n \in \mathbb{N}_1}$ be a sequence of finite right quasigroups such that $\mathfrak{H}_n$ is a right quasigroup associated with the family of permutations $(f_i)_{i \in n}$ defined by

$$f_i := \begin{cases} (0 \ldots 2^j - 1) \cup \mathrm{id}_{n - \{0, \ldots, 2^j - 1\}} & \text{if } i = 2^j - 1 \text{ for some } j, \\ \mathrm{id}_n & \text{otherwise.} \end{cases}$$

Using Cayley tables we can illustrate this sequence as follows:

| $\mathfrak{H}_1$ | 0 |
|---|---|
| 0 | 0 |

,

| $\mathfrak{H}_2$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | 0 |

,

| $\mathfrak{H}_3$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 1 | 0 |
| 1 | 1 | 0 | 1 |
| 2 | 2 | 2 | 2 |

,

| $\mathfrak{H}_4$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 0 | 3 |
| 1 | 1 | 0 | 1 | 0 |
| 2 | 2 | 2 | 2 | 1 |
| 3 | 3 | 3 | 3 | 2 |

, … ,

| $\mathfrak{H}_{2^n}$ | 0 | 1 | 2 | 3 | 4 | $\cdots$ | $2^n-1$ |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 0 | 3 | 0 | $\cdots$ | $2^n-1$ |
| 1 | 1 | 0 | 1 | 0 | 1 | $\cdots$ | 0 |
| 2 | 2 | 2 | 2 | 1 | 2 | $\cdots$ | 1 |
| 3 | 3 | 3 | 3 | 2 | 3 | $\cdots$ | 2 |
| 4 | 4 | 4 | 4 | 4 | 4 | $\cdots$ | 3 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $2^n-1$ | $2^n-1$ | $2^n-1$ | $2^n-1$ | $2^n-1$ | $2^n-1$ | $\cdots$ | $2^n-2$ |

$, \ldots .$

Evidently the sequence $(\mathfrak{H}_n)_{n\in\mathbb{N}_1}$ is strictly increasing. More precisely, we have the following embeddings:

$$\mathfrak{H}_1 \xrightarrow{\text{id}} \mathfrak{H}_2 \xrightarrow{\text{id}} \mathfrak{H}_3 \xrightarrow{\text{id}} \mathfrak{H}_4 \ \ldots\ \mathfrak{H}_{2^n-1} \xrightarrow{\text{id}} \mathfrak{H}_{2^n} \ \ldots .$$

One can check that:

$$\begin{aligned}
&\langle \mathfrak{H}_1 \rangle = \langle 1 \rangle = (0, \overset{\infty}{\ldots})(0), \quad \mathfrak{H}_1 \in \mathcal{Q}_1,\\
&\langle \mathfrak{H}_2 \rangle = \langle 2 \rangle = (1, 0, \overset{\infty}{\ldots})(0), \quad \mathfrak{H}_2 \in \mathcal{Q}_2,\\
&\langle \mathfrak{H}_3 \rangle = \langle 2 \rangle = (1, 0, \overset{\infty}{\ldots})(0), \quad \mathfrak{H}_3 \in \mathcal{Q}_2,\\
&\langle \mathfrak{H}_4 \rangle = \langle 4 \rangle = (2, 0, \overset{\infty}{\ldots})(0), \quad \mathfrak{H}_4 \in \mathcal{Q}_4,\\
&\ldots\\
&\langle \mathfrak{H}_{2^n} \rangle = \langle 2^n \rangle = (n, 0, \overset{\infty}{\ldots})(0), \quad \mathfrak{H}_{2^n} \in \mathcal{Q}_{2^n}.
\end{aligned}$$

It is clear that

$$\forall\, n \in \mathbb{N} \quad \langle n \rangle \mid (\aleph_0, 0, \overset{\infty}{\ldots})(0).$$

Thus

$$\mathfrak{H}_1, \mathfrak{H}_2, \ldots, \mathfrak{H}_{2^n}, \ldots \in \mathcal{Q}_{2^{\aleph_0}}.$$

### $s$-bounded right torsion groupoids

Let us consider the right quasigroup $\mathfrak{G}_{2^{\aleph_0}}$ associated with the family of permutations $(f_i)_{i\in\mathbb{N}}$ defined by

$$f_i := \begin{cases} (0 \ldots 2^j - 1) \cup \text{id}_{\mathbb{N}-\{0,\ldots,2^j-1\}} & \text{if } i = 2^j - 1 \text{ for some } j,\\ \text{id}_{\mathbb{N}} & \text{otherwise.} \end{cases}$$

The upper left corner of the Cayley table of $\mathfrak{G}_{2^{\aleph_0}}$ looks as follows:

| $\mathfrak{H}_{2^{\aleph_0}}$ | $0$ | $1$ | $2$ | $3$ | $4$ | $\cdots$ | $2^n-1$ | $2^n$ | $\cdots$ |
|---|---|---|---|---|---|---|---|---|---|
| $0$ | $0$ | $1$ | $0$ | $3$ | $0$ | $\cdots$ | $2^n-1$ | $0$ | $\cdots$ |
| $1$ | $1$ | $0$ | $1$ | $0$ | $1$ | $\cdots$ | $0$ | $1$ | $\cdots$ |
| $2$ | $2$ | $2$ | $2$ | $1$ | $2$ | $\cdots$ | $1$ | $2$ | $\cdots$ |
| $3$ | $3$ | $3$ | $3$ | $2$ | $3$ | $\cdots$ | $2$ | $3$ | $\cdots$ |
| $4$ | $4$ | $4$ | $4$ | $4$ | $4$ | $\cdots$ | $3$ | $4$ | $\cdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\cdots$ |
| $2^n-1$ | $2^n-1$ | $2^n-1$ | $2^n-1$ | $2^n-1$ | $2^n-1$ | $\cdots$ | $2^n-2$ | $2^n-1$ | $\cdots$ |
| $2^n$ | $2^n$ | $2^n$ | $2^n$ | $2^n$ | $2^n$ | $\cdots$ | $2^n$ | $2^n$ | $\cdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ . |

For this right quasigroup we have

$$\langle \mathfrak{H}_{2^{\aleph_0}} \rangle = (\aleph_0, 0, \overset{\infty}{\ldots})(0),\ \mathfrak{H}_{2^{\aleph_0}} \in \mathcal{T}_{2^{\aleph_0}}.$$

## 10.2. Some applications

Using the notions of number codes, some theorem of [4] can be slightly enhanced by adding a new fourth item. Below we give two examples of such enhancements. The proofs are omitted because the equivalence of the first three items was proved in [4]. The proof of the equivalence of the first three items with the fourth item is not hard.

**Theorem 10.1** (cf. [4], Theorem 9)**.** *The following conditions are equivalent:*

(i) $\mathfrak{G}$ *is a groupoid of right exponent* $\omega^*_{\mathfrak{G}}$.

(ii) $\mathfrak{G}$ *is a groupoid of finite right exponent.*

(iii) $\mathfrak{G}$ *is a right quasigroup of finite cycle type.*

(iv) $\mathfrak{G}$ *is a right quasigroup with* $\langle \mathfrak{G} \rangle = \langle n \rangle$ *for some positive integer* $n$.

**Theorem 10.2** (cf. [4], Theorem 15)**.** *The following conditions are equivalent:*

(i) $\mathfrak{G}$ *is a right-torsion groupoid.*

(ii) $\mathfrak{G}$ *is a right quasigroup satisfying* $(*^t)$.

(iii) $\mathfrak{G}$ *is a right quasigroup of countable cycle type.*

(iv) $\mathfrak{G}$ *is a right quasigroup with* $\langle \mathfrak{G} \rangle = s$ *for some closed Steinitz number* $s$.

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