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RESEARCH ARTICLE

# Tiled orders of width 3

**Viktor Zhuravlev, Dmytro Zhuravlyov**

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Abstract. We consider projective cover over tiled order and calculate the kernel of epimorphism from direct sum of submodules of distributive module to their sum.

## 1. Tiled orders over discrete valuation rings

Recall [1] that a *semimaximal ring* is a semiperfect semiprime right Noetherian ring $A$ such that for each primitive idempotent $e \in A$ the ring $eAe$ is a discrete valuation ring (not necessarily commutative).

Denote by $M_n(B)$ the ring of all $n \times n$ matrices over a ring $B$.

**Theorem 1** (see [1]). *Each semimaximal ring is isomorphic to a finite direct product of prime rings of the following form:*

$$\Lambda = \begin{pmatrix} \mathcal{O} & \pi^{\alpha_{12}}\mathcal{O} & \dots & \pi^{\alpha_{1n}}\mathcal{O} \\ \pi^{\alpha_{21}}\mathcal{O} & \mathcal{O} & \dots & \pi^{\alpha_{2n}}\mathcal{O} \\ \dots & \dots & \dots & \dots \\ \pi^{\alpha_{n1}}\mathcal{O} & \pi^{\alpha_{n2}}\mathcal{O} & \dots & \mathcal{O} \end{pmatrix}, \qquad (1)$$

*where $n \geq 1$, $\mathcal{O}$ is a discrete valuation ring with a prime element $\pi$, and $\alpha_{ij}$ are integers such that*

$$\alpha_{ij} + \alpha_{jk} \geq \alpha_{ik}, \qquad \alpha_{ii} = 0$$

*for all $i, j, k$.*

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The ring $\mathcal{O}$ is embedded into its classical division ring of fractions $\mathcal{D}$, and (1) is the set of all matrices $(a_{ij}) \in M_n(\mathcal{D})$ such that

$$a_{ij} \in \pi^{\alpha_{ij}}\mathcal{O} = e_{ii}\Lambda e_{jj},$$

where $e_{11}, \ldots, e_{nn}$ are the matrix units of $M_n(\mathcal{D})$. It is clear that $Q = M_n(\mathcal{D})$ is the classical ring of fractions of $\Lambda$.

Obviously, the ring $A$ is right and left Noetherian.

**Definition 1.** *A module $M$ is* distributive *if its lattice of submodules is distributive, i.e.,*

$$K \cap (L + N) = K \cap L + K \cap N$$

*for all submodules $K$, $L$, and $N$.*

Clearly, any submodule and any factormodule of a distributive module are distributive modules.

A *semidistributive module* is a direct sum of distributive modules. A ring $A$ is *right* (*left*) *semidistributive* if it is semidistributive as the right (left) module over itself. A ring $A$ is *semidistributive* if it is both left and right semidistributive (see [7]).

**Theorem 2** (see [6]). *The following conditions for a semiperfect semiprime right Noetherian ring $A$ are equivalent:*

- *$A$ is semidistributive;*
- *$A$ is a direct product of a semisimple artinian ring and a semimaximal ring.*

By a *tiled order* over a discrete valuation ring, we mean a Noetherian prime semiperfect semidistributive ring $\Lambda$ with nonzero Jacobson radical. In this case, $\mathcal{O} = e\Lambda e$ is a discrete valuation ring with a primitive idempotent $e \in \Lambda$.

**Definition 2.** *An integer matrix $\mathcal{E} = (\alpha_{ij}) \in M_n(\mathbb{Z})$ is called*

- *an* exponent matrix *if $\alpha_{ij} + \alpha_{jk} \geq \alpha_{ik}$ and $\alpha_{ii} = 0$ for all $i, j, k$;*
- *a* reduced exponent matrix *if $\alpha_{ij} + \alpha_{ji} > 0$ for all $i, j$, $i \neq j$.*

We use the following notation: $\Lambda = \{\mathcal{O}, \mathcal{E}(\Lambda)\}$, where $\mathcal{E}(\Lambda) = (\alpha_{ij})$ is the exponent matrix of the ring $\Lambda$, i.e.

$$\Lambda = \sum_{i,j=1}^{n} e_{ij}\pi^{\alpha_{ij}}\mathcal{O},$$

in which $e_{ij}$ are the matrix units. If a tiled order is *reduced*, i.e., $\Lambda/R(\Lambda)$ is the direct product of division rings, then $\alpha_{ij} + \alpha_{ji} > 0$ if $i \neq j$, i.e., $\mathcal{E}(\Lambda)$ is reduced.

We denote by $\mathcal{M}(\Lambda)$ the poset (ordered by inclusion) of all projective right $\Lambda$-modules that are contained in a fixed simple $Q$-module $U$. All simple $Q$-modules are isomorphic, so we can choice one of them. Note that the partially ordered sets $\mathcal{M}_l(\Lambda)$ and $\mathcal{M}_r(\Lambda)$ corresponding to the left and the right modules are anti-isomorphic.

The set $\mathcal{M}(\Lambda)$ is completely determined by the exponent matrix $\mathcal{E}(\Lambda) = (\alpha_{ij})$. Namely, if $\Lambda$ is reduced, then

$$\mathcal{M}(\Lambda) = \{p_i^z \mid i = 1, \dots n, \text{and } z \in \mathbb{Z}\},$$

where

$$p_i^z \leq p_j^{z'} \quad \Longleftrightarrow \quad \begin{cases} z - z' \geq \alpha_{ij} & \text{if } \mathcal{M}(\Lambda) = \mathcal{M}_l(\Lambda), \\ z - z' \geq \alpha_{ji} & \text{if } \mathcal{M}(\Lambda) = \mathcal{M}_r(\Lambda). \end{cases}$$

Obviously, $\mathcal{M}(\Lambda)$ is an infinite periodic set.

Let $P$ be an arbitrary poset. A subset of $P$ is called a chain if any two of its elements are related. A subset of $P$ is called a antichain if no two distinct elements of the subset are related.

**Definition 3.** *The maximal number $w(P)$ of elements in an antichain of $P$ is called the width of $P$.*

The width of $\mathcal{M}_r(\Lambda)$ is called the width of a tiled order $\Lambda$ and denotes by $w(\Lambda)$.

**Definition 4.** *A right (resp. left) $\Lambda$-module $M$ (resp. $N$) is called a right (resp. left) $\Lambda$-lattice if $M$ (resp. $N$) is a finitely generated free $\mathcal{O}$-module.*

Given a tiled order $\Lambda$ we denote $Lat_r(\Lambda)$ (resp. $Lat_l(\Lambda)$) the category of right (resp. left) $\Lambda$-lattices. We denote by $S_r(\Lambda)$ (resp. $S_l(\Lambda)$) the partially ordered by inclusion set, formed by all $\Lambda$-lattices contained in a fixed simple $M_n(\mathcal{D})$-module $W$ (resp. in a left simple $M_n(\mathcal{D})$-module $V$). Such $\Lambda$-lattices are called irreducible.

Let $\Lambda = \{\mathcal{O}, \mathcal{E}(\Lambda)\}$ be a tiled order, $W$ (resp. $V$) is a simple right (resp. left) $M_n(\mathcal{D})$-module with $\mathcal{D}$-basis $e_1, \dots, e_n$ such that $e_i e_{jk} = \delta_{ij} e_k$ ($e_{ij} e_k = \delta_{jk} e_i$).

Then any right (resp. left) irreducible $\Lambda$-lattice $M$ (resp. $N$), lying in $W$ (resp. in $V$) is a $\Lambda$-module with $\mathcal{O}$-basis $(\pi^{\alpha_1} e_1, \dots, \pi^{\alpha_n} e_n)$, while

$$\begin{cases} \alpha_i + \alpha_{ij} \geq \alpha_j, & \text{for the right case;} \\ \alpha_{ij} + \alpha_j \geq \alpha_i, & \text{for the left case.} \end{cases} \tag{2}$$

Thus, irreducible $\Lambda$-lattices $M$ can be identified with integer-valued vector $(\alpha_1, \ldots, \alpha_n)$ satisfying (2). We shall write $\mathcal{E}(M) = (\alpha_1, \ldots, \alpha_n)$ or $M = (\alpha_1, \ldots, \alpha_n)$.

The order relation on the set of such vectors and the operations on them corresponding to sum and intersection of irreducible lattices are obvious.

**Remark 1.** Obviously, irreducible $\Lambda$-lattices $M_1 = (\alpha_1, \ldots, \alpha_n)$ and $M_2 = (\beta_1, \ldots, \beta_n)$ are isomorphic if and only if $\alpha_i = \beta_i + z$ for $i = 1, \ldots, n$ and $z \in \mathbb{Z}$.

## 2. Kernel of epimorphism from direct sum of modules to their sum

**Proposition 1.** *Let $M$ be an irreducible and non-projective $\Lambda$-module, $X$ be a maximal submodule of $M$. Then there exists projective submodule of $M$, which is not submodule of $X$.*

*Proof.* Let

$$\mathcal{E}(M) = (\alpha_1, \ldots, \alpha_{i-1}, \alpha_i, \alpha_{i+1}, \ldots, \alpha_n) \quad \text{and}$$
$$\mathcal{E}(X) = (\alpha_1, \ldots, \alpha_{i-1}, \alpha_i + 1, \alpha_{i+1}, \ldots, \alpha_n).$$

Since $M$ is right $\Lambda$-module, $\alpha_i + \alpha_{ik} \geq \alpha_k$ for all $i, k$. Consider the projective module $\pi^{\alpha_i} P_i$ with

$$\mathcal{E}(\pi^{\alpha_i} P_i) = (\alpha_i + \alpha_{i1}, \ldots, \alpha_i + \alpha_{i,i-1}, \alpha_i + \alpha_{ii}, \alpha_i + \alpha_{i,i+1}, \ldots, \alpha_i + \alpha_{in}).$$

Obviously, $\pi^{\alpha_i} P_i \subset M$, but $\pi^{\alpha_i} P_i \nsubseteq X$.

Since $X$ is maximal submodule of $M$, then $X + \pi^{\alpha_i} P_i = M$. Besides,

$$\mathcal{E}(X \cap \pi^{\alpha_i} P_i) = (\alpha_i + \alpha_{i1}, \ldots, \alpha_i + \alpha_{i,i-1}, \alpha_i + 1, \alpha_i + \alpha_{i,i+1}, \ldots, \alpha_i + \alpha_{in}) = \mathcal{E}(\pi^{\alpha_i} R_i), \quad (3)$$

i. e. $X \cap \pi^{\alpha_i} P_i = R_i$, where $R_i = \text{rad } P_i$. $\square$

**Proposition 2.** *Let $X_1, \ldots, X_s$ be the set of all maximal submodules of irreducible and non-projective $\Lambda$-module $M$ with $\mathcal{E}(M) = (\alpha_1, \ldots, \alpha_n)$ and $\mathcal{E}(X_i) = \mathcal{E}(M) + e_{j_i}$, where $e_k = (\underbrace{0, \ldots, 0}_{k-1}, 1, 0, \ldots, 0)$. Then*

$$P(M) = \bigoplus_{i=1}^{s} \pi^{\alpha_{j_i}} P_{j_i} \qquad \text{and} \qquad M = \sum_{i=1}^{s} \pi^{\alpha_{j_i}} P_{j_i}.$$

*Proof.* Since rad $M = \bigcap_{i=1}^{s} X_i$, we have $\mathcal{E}(\text{rad } M) = \mathcal{E}(M) + \sum_{i=1}^{s} e_{j_i}$, $P(M) = \bigoplus_{i=1}^{s} \pi^{\alpha_{j_i}} P_{j_i}$. Besides, $\pi^{\alpha_{j_i}} P_{j_i} \subset M$ for each $i$, whereas $\sum_{i=1}^{s} \pi^{\alpha_{j_i}} P_{j_i} \subset M$. Suppose that $\sum_{i=1}^{s} \pi^{\alpha_{j_i}} P_{j_i} \neq M$. Then there is the maximal submodule $X_k$ such that $\pi^{\alpha_{j_i}} P_{j_i} \subseteq X_k$. This contradicts to inclusion $\pi^{\alpha_{j_k}} P_{j_k} \nsubseteq X_k$. □

**Lemma 1.** *Let $M_1$, $M_2$, $M_3$ be submodules of distributive module $M$ and $\varphi\colon M_1 \oplus M_2 \oplus M_3 \to M_1 + M_2 + M_3$ be epimorphism of their direct sum on their sum defined by the rule $(x_1, x_2, x_3) \mapsto x_1 + x_2 + x_3$. Then* $\ker\varphi = \{(m_{12} - m_{31}, m_{23} - m_{12}, m_{31} - m_{23}) \mid m_{12} \in M_1 \cap M_2, m_{23} \in M_2 \cap M_3, m_{31} \in M_3 \cap M_1\}$.

*Proof.* Let us calculate the kernel of homomorphism $\varphi$. By definition $\ker\varphi = \{(x_1, x_2, x_3) \in M_1 \oplus M_2 \oplus M_3 \mid x_1 + x_2 + x_3 = 0\}$.

Hence $x_1 = -(x_2 + x_3)$ and $x_1 \in (M_2 + M_3) \cap M_1$. Similarly, $x_2 \in (M_1 + M_3) \cap M_2$ and $x_3 \in (M_2 + M_1) \cap M_3$. Since modules $M_1$, $M_2$, $M_3$ are distributive, we have $(M_i + M_j) \cap M_k = (M_i \cap M_k) + (M_j \cap M_k)$. Therefore $x_1 = x_{12} + x_{13}$, $x_2 = x_{21} + x_{23}$, $x_3 = x_{31} + x_{32}$, where $x_{ij} \in M_i \cap M_j$. Since $x_1 + x_2 = (x_{12} + x_{21}) + (x_{13} + x_{23}) \in M_3$ and $x_{13} + x_{23} \in M_3$, then $x_{12} + x_{21} \in M_3$. Given that $x_{12}, x_{21} \in M_1 \cap M_2$, we have $x_{12} + x_{21} \in M_1 \cap M_2 \cap M_3$. Similarly $x_{23} + x_{32} \in M_1 \cap M_2 \cap M_3$, $x_{31} + x_{13} \in M_1 \cap M_2 \cap M_3$.

Therefore $x_{12} + x_{21} = t_3 \in M_1 \cap M_2 \cap M_3$, $x_{23} + x_{32} = t_1 \in M_1 \cap M_2 \cap M_3$, $x_{31} + x_{13} = t_2 \in M_1 \cap M_2 \cap M_3$. Hence $x_{21} = t_3 - x_{12}$, $x_{32} = t_1 - x_{23}$, $x_{13} = t_2 - x_{31}$. Then $(x_1, x_2, x_3) = (x_{12} + t_2 - x_{31}, x_{23} + t_3 - x_{12}, x_{31} + t_1 - x_{23})$. From the equality $x_1 + x_2 + x_3 = 0$ implies that $t_1 + t_2 + t_3 = 0$. Therefore

$$(x_1, x_2, x_3) = (x_{12} + t_2 - x_{31}, x_{23} - (t_1 + t_2) - x_{12}, x_{31} + t_1 - x_{23}) = \\ = ((x_{12} + t_2) - x_{31}, (x_{23} - t_1) - (t_2 + x_{12}), x_{31} - (x_{23} - t_1)).$$

Denoting by $x_{12} + t_2 = y_{12} \in M_1 \cap M_2$, $x_{23} - t_1 = y_{23} \in M_2 \cap M_3$, $x_{31} = y_{31}$, we obtain $\ker\varphi = \{(y_{12} - y_{31}, y_{23} - y_{12}, y_{31} - y_{23}) \mid y_{12} \in M_1 \cap M_2, y_{23} \in M_2 \cap M_3, y_{31} \in M_3 \cap M_1\}$. □

If $M_1 \cap M_2 \subset M_3$, then $x_{12} + x_{21} \in M_3$ for any $x_1 \in M_1$, $x_2 \in M_2$. Therefore $\ker\varphi = \{(x_1, x_2, -(x_1 + x_2)) \mid x_1 \in (M_2 + M_3) \cap M_1, x_2 \in (M_3 + M_1) \cap M_2\}$. Hence $\ker\varphi \simeq ((M_2 + M_3) \cap M_1) \oplus ((M_3 + M_1) \cap M_2)\}$. Since $(M_2 + M_3) \cap M_1 = M_2 \cap M_1 + M_3 \cap M_1 = M_3 \cap M_1$ and $(M_3 + M_1) \cap M_2 = M_3 \cap M_2 + M_1 \cap M_2 = M_3 \cap M_2$, then $\ker\varphi \simeq (M_3 \cap M_1) \oplus (M_3 \cap M_2)$.

Let us write formally the expression for the kernel of the homomorphism $\varphi$ in the other way

$$(y_{12}-y_{31}, y_{23}-y_{12}, y_{31}-y_{23}) = \\ = y_{12}(1,-1,0)+y_{23}(0,1,-1)+y_{31}(-1,0,1).$$

Note that $((M_1\cap M_2)(1,-1,0))\cap((M_2\cap M_3)(0,1,-1))=0$, but

$$(((M_1\cap M_2)(1,-1,0))+((M_2\cap M_3)(0,1,-1)))\cap((M_3\cap M_1)(-1,0,1))\neq 0.$$

Therefore, the sum of modules is not direct.

Consider the epimorphism $\psi\colon (M_1\cap M_2)\oplus(M_2\cap M_3)\oplus(M_3\cap M_1)\to \ker\varphi$, defined by the equality

$$\psi(x_{12},x_{23},x_{31}) = (x_{12}-x_{31}, x_{23}-x_{12}, x_{31}-x_{23}).$$

Then $\ker\psi = \{(x_{12},x_{23},x_{31}) \mid x_{12}-x_{31} = x_{23}-x_{12} = x_{31}-x_{23} = 0\} = \{(x_{12},x_{23},x_{31}) \mid x_{12}=x_{23}=x_{31}\}$. By the fundamental theorem on homomorphism of modules we have

$$\ker\varphi \simeq ((M_1\cap M_2)\oplus(M_2\cap M_3)\oplus(M_3\cap M_1)/\ker\psi,$$

i. e.

$$\ker\varphi \simeq ((M_1\cap M_2)\oplus(M_2\cap M_3)\oplus(M_3\cap M_1))\,/(M_1\cap M_2\cap M_3).$$

Note that in the general case

$$\ker\varphi \simeq \{(y_{12}-y_{31}, y_{23}-y_{12}) \mid y_{31}\in M_3\cap M_1, \\ y_{23}\in M_2\cap M_3, y_{12}\in M_1\cap M_2\}$$

or

$$\ker\varphi \simeq (M_3\cap M_1)\oplus(M_2\cap M_3)+(M_1\cap M_2)(1,-1).$$

Let $M_1,\dots,M_n$ be submodules of $M$ such that $M_i \not\subseteq \sum\limits_{j\neq i} M_j$ for all $i=1,\dots,n$ and $I_1$, $I_2$ be nonempty subsets of the set $I=\{1,\dots,n\}$ such that $I_1\cup I_2 = I$, $I_1\cap I_2=\emptyset$. We have the following exact sequences

$$0\to K\to \underset{i\in I}{\oplus} M_i \to \sum_{i\in I} M_i \to 0,$$

$$0\to K_1\to \underset{i\in I_1}{\oplus} M_i \to \sum_{i\in I_1} M_i \to 0,$$

$$0\to K_2\to \underset{i\in I_2}{\oplus} M_i \to \sum_{i\in I_2} M_i \to 0,$$

where $K, K_1, K_2$ are the kernels of epimorphisms from direct sum on the sum of modules. Next commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_{I_1} \oplus K_{I_2} & \longrightarrow & K_{I_1} \oplus K_{I_2} & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_I & \longrightarrow & \bigoplus\limits_{i=1}^{n} M_i & \longrightarrow & \sum\limits_{i=1}^{n} M_i & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \left(\sum\limits_{j\in I_1} M_j\right) \cap \left(\sum\limits_{j\in I_2} M_j\right) & \to & \left(\sum\limits_{j\in I_1} M_j\right) \oplus \left(\sum\limits_{j\in I_2} M_j\right) & \to & \sum\limits_{i=1}^{n} M_i & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

has exact rows and two columns exact. Therefore by lemma $3 \times 3$ first column

$$0 \to K_{I_1} \oplus K_{I_2} \to K_I \to \Big(\sum_{j\in I_1} M_j\Big) \cap \Big(\sum_{j\in I_2} M_j\Big) \to 0.$$

is also exact.

In particular, if $I_2 = \{k\}$, $I_1 = I \setminus \{k\}$, then $K_{I_2} = 0$ and we have from the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_{I_1} & \longrightarrow & K_{I_1} & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_I & \longrightarrow & \bigoplus\limits_{i=1}^{n} M_i & \longrightarrow & \sum\limits_{i=1}^{n} M_i & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \left(\sum\limits_{i\neq j} M_i\right) \cap M_j & \longrightarrow & \left(\sum\limits_{i\neq j} M_i\right) \oplus M_j & \longrightarrow & \sum\limits_{i=1}^{n} M_i & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

the exact sequence

$$0 \to K_{I_1} \to K_I \to \left(\sum_{j\in I_1} M_j\right) \cap M_k \to 0.$$

**Theorem 3.** *Let $M_1, \ldots, M_n$ be submodules of distributive module $M = \sum\limits_{i=1}^{n} M_i$ and epimorphism $\varphi\colon \overset{n}{\underset{i=1}{\oplus}} M_i \mapsto M$ operates by the rule $\varphi(m_1, \ldots, m_n) = m_1 + \ldots + m_n$. Then* $\ker\varphi = \{(y_1, \ldots, y_n) \mid y_i = \sum\limits_{j\neq i} sign(j-i)\cdot m_{ij},\ m_{ij} \in M_i \cap M_j\}$.

*Proof.* We use induction by $n$. It is well known that the kernel of epimorphism equals to $\{m_{12}, -m_{12}\}$, where $m_{12} \in M_1 \cap M_2$, that implies the base of induction for $n = 2$.

Suppose that the kernel of epimorphism $\varphi(m_1, \ldots, m_{n-1}) = m_1 + \ldots + m_{n-1}$ is $K_n = \{(y_1, \ldots, y_{n-1}) \mid y_i = \sum\limits_{j\neq i} sign(j-i)\cdot m_{ij},\ m_{ij} \in M_i \cap M_j\}$. Denote by $L = \{(y_1, \ldots, y_n) \mid y_i = \sum\limits_{j\neq i} sign(j-i)\cdot m_{ij},\ m_{ij} \in M_i \cap M_j\}$. Obviously, $K_n \simeq \{(y_1, \ldots, y_{n-1}, 0)\} \subset L$. Then

$$\overset{n}{\underset{i=1}{\oplus}} M_i/L \simeq \left(\overset{n}{\underset{i=1}{\oplus}} M_i/K_n\right)\Big/(L/K_n).$$

By assumption we have $\overset{n}{\underset{i=1}{\oplus}} M_i/K_n \simeq \left(\sum\limits_{i\neq n} M_i\right) \oplus M_n$.

**Proposition 3.** $L/K_n \simeq \left(\sum\limits_{i\neq n} M_i\right) \cap M_n$.

*Proof.* Indeed,

$$\begin{aligned} L/K_n &= \{(y_1, \ldots, y_n) + K_n\} = \\ &= \{(m_{1n}, m_{2n}, \ldots, m_{n-1n}, -(m_{1n} + m_{2n} + \ldots + m_{n-1n})) + K_n\}. \end{aligned}$$

Consider epimorphism $\psi\colon L/K_n \mapsto \left(\sum\limits_{i\neq n} M_i\right) \cap M_n$, for which

$$\begin{aligned} &\psi\left((m_{1n}, m_{2n}, \ldots, m_{n-1n}, -(m_{1n} + m_{2n} + \ldots + m_{n-1n})) + K_n\right) = \\ &\qquad = m_{1n} + m_{2n} + \ldots + m_{n-1n}. \end{aligned}$$

The kernel of this epimorphism

$$\ker \psi = \{(m_{1n}, m_{2n}, \ldots, m_{n-1n}, 0) + K_n, \text{ where } m_{1n} + m_{2n} + \ldots + m_{n-1n} = 0\} \simeq K_n.$$

Therefore $\psi$ is isomorphism. $\square$

Hence, $\overset{n}{\underset{i=1}{\oplus}} M_i/L \simeq \left(\sum\limits_{i\neq n} M_i\right) \oplus M_n \Big/ \left(\sum\limits_{i\neq n} M_i\right) \cap M_n \simeq \sum\limits_{i=1}^{n} M_i$. On the other hand $\overset{n}{\underset{i=1}{\oplus}} M_i/K \simeq \sum\limits_{i=1}^{n} M_i$. Therefore $K \simeq L$. Obviously, $L \subseteq K = \ker \varphi$. Hence, $L = K$. $\square$

**Corollary 1.** *Let $M$ be irreducible $\Lambda$-module and $P(M) = \overset{s}{\underset{i=1}{\oplus}} \pi^{\alpha_{j_i}} P_{j_i}$, $M = \sum\limits_{i=1}^{s} \pi^{\alpha_{j_i}} P_{j_i}$. Then the kernel of epimorphism $\varphi\colon P(M) \mapsto M$ equals to* $\ker \varphi = \{(y_1, \ldots, y_n) \mid y_i = \sum\limits_{k\neq i} sign(k-i) \cdot m_{ik},\ m_{ik} \in P_{j_i} \cap P_{j_k}\}$.

*Proof.* Tiled order $\Lambda$ is semidistributive ring. Therefore every irreducible $\Lambda$-module is distributive. According to preliminary theorems core epimorphism has specified above form. $\square$

The kernel $K$ as submodule in $\overset{n}{\underset{i=1}{\oplus}} M_i$ can be formally written as вигляді

$$K = \sum_{i<j} M_i \cap M_j (e_i - e_j), \text{where } e_k = (\underbrace{0, \ldots, 0}_{k-1}, 1, 0, \ldots, 0).$$

## 3. Tiled order of width 3

**Proposition 4.** *Modules $P\Big(M_i \cap (\sum\limits_{k\neq i} M_k)\Big)$, $i = 1, \ldots, n$, have a common direct summand $P'$ if and only if the modules $P(M_i \cap M_j)$, $i, j = 1, \ldots, n$, also have common direct summand $P'$.*

*Proof.* Let modules $P\Big(M_i \cap (\sum\limits_{k\neq i} M_k)\Big)$, $i = 1, \ldots, n$, have a common direct summand $P'$. This is equivalent to the fact that module $M_i \cap (\sum\limits_{k\neq i} M_k)$ has the maximal submodules $X_i$ with $\mathcal{E}(X_i) = \mathcal{E}(M_i \cap (\sum\limits_{k\neq i} M_k)) + e'$. Since $M_i \cap M_j = \Big(M_i \cap (\sum\limits_{k\neq i} M_k)\Big) \cap \Big(M_j \cap (\sum\limits_{k\neq j} M_k)\Big)$ for $i \neq j$, then the module

$M_i \cap M_j$ also have maximal submodules $N_{ij}$ with $\mathcal{E}(N_{ij}) = \mathcal{E}(M_i \cap M_j) + e'$. Therefore, the modules $P(M_i \cap M_j)$, $i, j = 1, \dots, n$, have also common summand $P'$.

Now let the modules $P(M_i \cap M_j)$, $i, j = 1, \dots, n$, have a common summand $P'$. This means that the module $P(M_i \cap M_j)$ has the maximal submodule $N_{ij}$ with $\mathcal{E}(N_{ij}) = \mathcal{E}(M_i \cap M_j) + e'$. Therefore, module $M_k \cap (M_i + M_j) = M_i \cap M_k + M_j \cap M_k$ has maximal submodule $X_{ijk}$ with $\mathcal{E}(X_{ijk}) = \mathcal{E}(M_k \cap (M_i + M_j)) + e'$. Similarly we get that the module $M_k \cap (M_i + \dots + M_j) = M_i \cap M_k + \dots + M_j \cap M_k$ has maximal submodule $X_k$ with $\mathcal{E}(X_k) = \mathcal{E}(M_k \cap (M_i + \dots + M_j)) + e'$. In particular, the module $M_i \cap (\sum\limits_{k \neq i} M_k)$ has the maximal submodule $Y_i$ with $\mathcal{E}(Y_i) = \mathcal{E}(M_i \cap (\sum\limits_{k \neq i} M_k)) + e'$. This is equivalent to the fact that modules $P\Big(M_i \cap (\sum\limits_{k \neq i} M_k)\Big)$, $i = 1, \dots, n$, have a common direct summand $P'$. □

Let module $M$ with $\mathcal{E}(M) = (\alpha_1, \dots, \alpha_n)$ has a projective cover $P(M) = \pi^{\alpha_i} P_i \oplus \pi^{\alpha_j} P_j \oplus \pi^{\alpha_k} P_k$ and $M = \pi^{\alpha_i} P_i + \pi^{\alpha_j} P_j + \pi^{\alpha_k} P_k$. Then

$$K = (\pi^{\alpha_i} P_i \cap \pi^{\alpha_j} P_j)(e_i - e_j) + (\pi^{\alpha_j} P_j \cap \pi^{\alpha_k} P_k)(e_j - e_k) + \\ + (\pi^{\alpha_k} P_k \cap \pi^{\alpha_i} P_i)(e_k - e_i).$$

Suppose that $\pi^{\alpha_i} P_i \cap \pi^{\alpha_j} P_j = \pi^{\alpha_j} P_j \cap \pi^{\alpha_k} P_k$. Then

$$(\pi^{\alpha_i} P_i + \pi^{\alpha_k} P_k) \cap \pi^{\alpha_j} P_j = \pi^{\alpha_i} P_i \cap \pi^{\alpha_j} P_j$$

and

$$(\pi^{\alpha_i} P_i + \pi^{\alpha_j} P_j) \cap \pi^{\alpha_k} P_k = (\pi^{\alpha_j} P_j + \pi^{\alpha_k} P_k) \cap \pi^{\alpha_i} P_i.$$

From the equality $(\pi^{\alpha_i} P_i + \pi^{\alpha_k} P_k) \cap \pi^{\alpha_j} P_j = \pi^{\alpha_i} P_i \cap \pi^{\alpha_j} P_j$ we get

$$(\pi^{\alpha_i} P_i + \pi^{\alpha_j} P_j) \cap \pi^{\alpha_k} P_k = \\ = ((\pi^{\alpha_i} P_i + \pi^{\alpha_j} P_j) \cap \pi^{\alpha_k} P_k) \cap ((\pi^{\alpha_j} P_j + \pi^{\alpha_k} P_k) \cap \pi^{\alpha_i} P_i) = \\ = \pi^{\alpha_i} P_i \cap \pi^{\alpha_k} P_k.$$

So we have two exact sequences

$$0 \to \pi^{\alpha_i} P_i \cap \pi^{\alpha_j} P_j \to K \to (\pi^{\alpha_i} P_i + \pi^{\alpha_j} P_j) \cap \pi^{\alpha_k} P_k \to 0,$$
$$0 \to \pi^{\alpha_i} P_i \cap \pi^{\alpha_k} P_k \to K \to (\pi^{\alpha_i} P_i + \pi^{\alpha_k} P_k) \cap \pi^{\alpha_j} P_j \to 0.$$

Whereas $(\pi^{\alpha_i} P_i + \pi^{\alpha_k} P_k) \cap \pi^{\alpha_j} P_j = \pi^{\alpha_i} P_i \cap \pi^{\alpha_j} P_j$ and $(\pi^{\alpha_i} P_i + \pi^{\alpha_j} P_j) \cap \pi^{\alpha_k} P_k = \pi^{\alpha_i} P_i \cap \pi^{\alpha_k} P_k$, then the exact sequence splits:

$$K \simeq (\pi^{\alpha_i} P_i \cap \pi^{\alpha_j} P_j) \oplus (\pi^{\alpha_i} P_i \cap \pi^{\alpha_k} P_k).$$

Let the width of tiled order do not exceed 3.

**Proposition 5.** *Let irreducible $\Lambda$-module $M$ have exactly two maximal non-projective submodules $X$ and $Y$ with*

$$\mathcal{E}(M) = (\alpha_1, \dots, \alpha_{i-1}, \alpha_i, \alpha_{i+1}, \dots, \alpha_n),$$
$$\mathcal{E}(X) = (\alpha_1, \dots, \alpha_{j-1}, \alpha_j + 1, \alpha_{j+1}, \dots, \alpha_n) \text{ and}$$
$$\mathcal{E}(Y) = (\alpha_1, \dots, \alpha_{i-1}, \alpha_i + 1, \alpha_{i+1}, \dots, \alpha_n).$$

*Then $P(M) = \pi^{\alpha_i} P_i \oplus \pi^{\alpha_j} P_j$ and we have the exact sequence*

$$0 \to \pi^{\alpha_i} P_i \cap \pi^{\alpha_j} P_j \to \pi^{\alpha_i} P_i \oplus \pi^{\alpha_j} P_j \to M \to 0.$$

*Proof.* We have $M = X + \pi^{\alpha_i} P_i = Y + \pi^{\alpha_j} P_j$, $\pi^{\alpha_i} P_i + \pi^{\alpha_j} P_j \subseteq M$, but $\pi^{\alpha_i} P_i + \pi^{\alpha_j} P_j$ does not belong to any maximal submodule $X$ or $Y$. Then $\pi^{\alpha_i} P_i + \pi^{\alpha_j} P_j = M$.

Since $M/X \simeq U_j$ та $M/Y \simeq U_i$, then $M/(\mathrm{rad} M) = M/(X \cap Y) \simeq U_i \oplus U_j$. Therefore $P(M) \simeq P(M/\mathrm{rad} M) \simeq P(U_i \oplus U_j) \simeq P(U_i) \oplus P(U_j) \simeq P_i \oplus P_j$. Obviously, the kernel of epimorphism $\varphi \colon \pi^{\alpha_i} P_i \oplus \pi^{\alpha_j} P_j \to M$ coincides with $\pi^{\alpha_i} P_i \cap \pi^{\alpha_j} P_j = \pi^{\alpha_i} R_i \cap \pi^{\alpha_j} R_j$. □

Consider the case when the module $M = (\alpha_1, \dots, \alpha_n)$ has exactly three maximal submodules $X = (\alpha_1, \dots, \alpha_{i-1}, \alpha_i + 1, \alpha_{i+1}, \dots, \alpha_n)$, $Y = (\alpha_1, \dots, \alpha_{j-1}, \alpha_j + 1, \alpha_{j+1}, \dots, \alpha_n)$ and $Z = (\alpha_1, \dots, \alpha_{k-1}, \alpha_k + 1, \alpha_{k+1}, \dots, \alpha_n)$.

Let module $M$ with $\mathcal{E}(M) = (\alpha_1, \dots, \alpha_n)$ have a projective cover

$$P(M) = \pi^{\alpha_i} P_i \oplus \pi^{\alpha_j} P_j \oplus \pi^{\alpha_k} P_k$$

and $M = \pi^{\alpha_i} P_i + \pi^{\alpha_j} P_j + \pi^{\alpha_k} P_k$. Then

$$K = (\pi^{\alpha_i} P_i \cap \pi^{\alpha_j} P_j)(e_i - e_j) + (\pi^{\alpha_j} P_j \cap \pi^{\alpha_k} P_k)(e_j - e_k) + \\ + (\pi^{\alpha_k} P_k \cap \pi^{\alpha_i} P_i)(e_k - e_i).$$

Also we have three exact sequences

$$0 \to \pi^{\alpha_i} P_i \cap \pi^{\alpha_j} P_j \to K \to (\pi^{\alpha_i} P_i + \pi^{\alpha_j} P_j) \cap \pi^{\alpha_k} P_k \to 0,$$
$$0 \to \pi^{\alpha_j} P_j \cap \pi^{\alpha_k} P_k \to K \to (\pi^{\alpha_j} P_j + \pi^{\alpha_k} P_k) \cap \pi^{\alpha_i} P_i \to 0,$$
$$0 \to \pi^{\alpha_k} P_k \cap \pi^{\alpha_i} P_i \to K \to (\pi^{\alpha_k} P_k + \pi^{\alpha_i} P_i) \cap \pi^{\alpha_j} P_j \to 0.$$

Let modules $\pi^{\alpha_i} P_i \cap \pi^{\alpha_j} P_j$, $\pi^{\alpha_j} P_j \cap \pi^{\alpha_k} P_k$ and $\pi^{\alpha_k} P_k \cap \pi^{\alpha_i} P_i$ are pairwise different.

Projective cover $P(K)$ of module $K$ is a direct summand of each of the the direct sums $P(\pi^{\alpha_i} P_i \cap \pi^{\alpha_j} P_j) \oplus P((\pi^{\alpha_i} P_i + \pi^{\alpha_j} P_j) \cap \pi^{\alpha_k} P_k)$, $P(\pi^{\alpha_j} P_j \cap \pi^{\alpha_k} P_k) \oplus P((\pi^{\alpha_j} P_j + \pi^{\alpha_k} P_k) \cap \pi^{\alpha_i} P_i)$, $P(\pi^{\alpha_k} P_k \cap \pi^{\alpha_i} P_i) \oplus P((\pi^{\alpha_k} P_k + \pi^{\alpha_i} P_i) \cap \pi^{\alpha_j} P_j)$.

Suppose that the module $P(K)$ contains 2 isomorphic direct summand $P'$. Since modules $\pi^{\alpha_i}P_i \cap \pi^{\alpha_j}P_j$ and $(\pi^{\alpha_i}P_i + \pi^{\alpha_j}P_j) \cap \pi^{\alpha_k}P_k$ are irreducible, their projective coverings do not contain isomorphic direct summands. Therefore, the module $P'$ is a direct summand of modules $P(\pi^{\alpha_i}P_i \cap \pi^{\alpha_j}P_j)$, $P((\pi^{\alpha_i}P_i + \pi^{\alpha_j}P_j) \cap \pi^{\alpha_k}P_k)$. The module $P'$ is a direct summand of modules $P((\pi^{\alpha_i}P_i + \pi^{\alpha_j}P_j) \cap \pi^{\alpha_k}P_k)$, $P((\pi^{\alpha_j}P_j + \pi^{\alpha_k}P_k) \cap \pi^{\alpha_i}P_i)$, $P((\pi^{\alpha_k}P_k + \pi^{\alpha_i}P_i) \cap \pi^{\alpha_j}P_j)$.

Module $P(K)$ contains as direct summand each of the modules $P((\pi^{\alpha_i}P_i + \pi^{\alpha_j}P_j) \cap \pi^{\alpha_k}P_k)$, $P((\pi^{\alpha_j}P_j + \pi^{\alpha_k}P_k) \cap \pi^{\alpha_i}P_i)$, $P((\pi^{\alpha_k}P_k + \pi^{\alpha_i}P_i) \cap \pi^{\alpha_j}P_j)$.

Hence, we obtain that $P(K)$ with a pairwise different modules $\pi^{\alpha_i}P_i \cap \pi^{\alpha_j}P_j$, $\pi^{\alpha_j}P_j \cap \pi^{\alpha_k}P_k$, $\pi^{\alpha_k}P_k \cap \pi^{\alpha_i}P_i$ has at least four non-isomorphic direct summands.

Thus, $P(K)$ contains only non-isomorphic direct summands. So $P(K) = \pi^{\alpha_a}P_a \oplus \pi^{\alpha_b}P_b \oplus \pi^{\alpha_c}P_c$.

Now we have 2 exact sequences

$$0 \to L \to P(K) \to K \to 0$$

$$0 \to K \to P(M) \to M \to 0$$

**Theorem 4.** $L \simeq \pi^{\alpha_a}P_a \cap \pi^{\alpha_b}P_b \cap \pi^{\alpha_c}P_c$.

*Proof.* Consider the homomorphism $\varphi\colon P(K) \mapsto P(M)$ with the image $K$. For corresponding to $\varphi$ matrix $[\varphi]$ we have

$$[\varphi] \in \begin{pmatrix} \mathrm{Hom}_\Lambda(\pi^{\alpha_a}P_a, \pi^{\alpha_i}P_i) & \mathrm{Hom}_\Lambda(\pi^{\alpha_b}P_b, \pi^{\alpha_i}P_i) & \mathrm{Hom}_\Lambda(\pi^{\alpha_c}P_c, \pi^{\alpha_i}P_i) \\ \mathrm{Hom}_\Lambda(\pi^{\alpha_a}P_a, \pi^{\alpha_j}P_j) & \mathrm{Hom}_\Lambda(\pi^{\alpha_b}P_b, \pi^{\alpha_j}P_j) & \mathrm{Hom}_\Lambda(\pi^{\alpha_c}P_c, \pi^{\alpha_j}P_j) \\ \mathrm{Hom}_\Lambda(\pi^{\alpha_a}P_a, \pi^{\alpha_k}P_k) & \mathrm{Hom}_\Lambda(\pi^{\alpha_b}P_b, \pi^{\alpha_k}P_k) & \mathrm{Hom}_\Lambda(\pi^{\alpha_c}P_c, \pi^{\alpha_k}P_k) \end{pmatrix}.$$

Since $\mathrm{Hom}_\Lambda(\pi^{\alpha_a}P_a, \pi^{\alpha_i}P_i) \simeq \pi^{\alpha_i-\alpha_a}e_i\Lambda e_a = \pi^{\alpha_i-\alpha_a} \cdot \pi^{\alpha_{ia}}\mathcal{O}$, then

$$[\varphi] = (\varphi_{ml}) \in \begin{pmatrix} \pi^{\alpha_i-\alpha_a+\alpha_{ia}}\mathcal{O} & \pi^{\alpha_i-\alpha_b+\alpha_{ib}}\mathcal{O} & \pi^{\alpha_i-\alpha_c+\alpha_{ic}}\mathcal{O} \\ \pi^{\alpha_j-\alpha_a+\alpha_{ja}}\mathcal{O} & \pi^{\alpha_j-\alpha_b+\alpha_{jb}}\mathcal{O} & \pi^{\alpha_j-\alpha_c+\alpha_{jc}}\mathcal{O} \\ \pi^{\alpha_k-\alpha_a+\alpha_{ka}}\mathcal{O} & \pi^{\alpha_k-\alpha_b+\alpha_{kb}}\mathcal{O} & \pi^{\alpha_k-\alpha_c+\alpha_{kc}}\mathcal{O} \end{pmatrix}.$$

Let $m_1 \in \pi^{\alpha_a}P_a$, $m_2 \in \pi^{\alpha_b}P_b$, $m_3 \in \pi^{\alpha_c}P_c$. Then

$$\varphi(m_1, m_2, m_3) = (m_1\varphi_{11} + m_2\varphi_{12} + m_3\varphi_{13}, m_1\varphi_{21} + \\ + m_2\varphi_{22} + m_3\varphi_{23}, m_1\varphi_{31} + m_2\varphi_{32} + m_3\varphi_{33}).$$

Since $K = \{(y_1, y_2, -(y_1 + y_2)\}$, the rank of $[\varphi]$ is 2. So the kernel of $\ker\varphi$ is obtained from the system of equations

$m_1\varphi_{11} + m_2\varphi_{12} + m_3\varphi_{13} = 0$, $m_1\varphi_{21} + m_2\varphi_{22} + m_3\varphi_{23} = 0$.

Hence, $m_1$, $m_2$ are expressed by $m_3$, and then $\ker\varphi$ is isomorphic to $\pi^{\alpha_a}P_a \cap \pi^{\alpha_b}P_b \cap \pi^{\alpha_c}P_c$. □

## Conclusion

The results obtained in sections 2, 3, to build a projective resolution of irreducible modules over tiled order of width 3 and calculate the global dimension of the order.

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### Contact information

**V. Zhuravlev** Department of Mechanics and Mathematics,
Kiyv National Taras Shevchenko University,
Volodymyrska, 64, Kyiv 01033, Ukraine
*E-Mail:* vshur@univ.kiev.ua

**D. Zhuravlyov** Department of Mechanics and Mathematics,
Kiyv National Taras Shevchenko University,
Volodymyrska, 64, Kyiv 01033, Ukraine

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