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Algebra in the Stone-Čech compactification: applications to topologies on groups

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Abstract. For every discrete group G, the Stone-Čech compactification βG of G has a natural structure of compact right topological semigroup. Assume that G is endowed with some left invariant topology $\Im$ and let $\overline{\tau}$ be the set of all ultrafilters on G converging to the unit of G in $\Im$. Then $\overline{\tau}$ is a closed subsemigroup of βG. We survey the results clarifying the interplays between the algebraic properties of $\overline{\tau}$ and the topological properties of $(G, \Im)$ and apply these results to solve some open problems in the topological group theory.

The paper consists of 13 sections: Filters on groups, Semigroup of ultrafilters, Ideals, Idempotents, Equations, Continuity in βG and G^*, Ramsey-like ultrafilters, Maximality, Refinements, Resolvability, Potential compactness and ultraranks, Selected open questions.

Introduction

Let G be a discrete group with the unit e and let βG be the Stone-Čech compactification of G. We take the points of βG to be the ultrafilters on G with the points of G identified with the principal ultrafilters. Using the universal property of βG, we extend the group multiplication on G to the semigroup operation on βG. Formally, the product of ultrafilters p and q is defined by the rule

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$$A \in pq \Longleftrightarrow \{g \in G: \ g^{-1}A \in q\} \in p.$$

With this operation, βG is a right topological semigroup, i.e. the mapping $x \longmapsto xs$ is continuous for every $s \in \beta G$. The semigroup βG is interesting because of its applications to combinatorial number theory and to topological dynamics as well as for its own sake [10].

Now assume that G is endowed with some left invariant topology $\Im$, i.e. all shifts $x \longmapsto gx$, $g \in G$ are continuous. Then the set

$$\overline{\tau} = \{p \in \beta G: \ p \text{ converges to } e \text{ in } \Im\}$$

is a closed subsemigroup of βG. Thus, we get general problem of study the interplays between the properties of left invariant (in particular, group) topology $\Im$ on G and the algebraic structure of corresponding semigroup of ultrafilters $\overline{\tau}$. The following striking results were obtained on this way.

• There exists in ZFC a maximal regular left topological group (in particular, a maximal regular homogeneous space).

• Every non-discrete left topological group of second category can be partitioned into countably many dense subsets.

• Every countable Abelian group G with only finite numbers of elements of order 2 can be partitioned into countably many subsets that are dense in every group topology on G.

• If there exists a maximal topological group, then there exists a P-point in ω^*. If there exists a non-discrete irresolvable topological group which is either Abelian or countable, then there exists a P-point in ω^*.

• If G is a countable discrete group and F is a finite subgroup from βG, then there exist a finite subgroup H from G and an idempotent $p \in \beta G$ such that $F = Hp$ and p commutes with every element of H. In particular, every finite subgroup of $\beta\mathbb{Z}$ is a singleton.

In the first part (sections 1-6) we present the results on algebraic structure of $\overline{\tau}$ which are necessary for applications in the second part (sections 7-12). We conclude the paper with the list of selected open questions (section 13).

1. Filters on groups

Let φ be filter on a group G with the unit e, $A \subseteq G$.

A subset

$$cl(A, \varphi) = \{g \in G: \ gF \cap A \neq \emptyset \text{ for every } F \in \varphi\}$$

is called a *closure* of A by φ.

A subset

$$int\ (A, \varphi) = \{g \in G : \ gF \subseteq A \text{ for some } F \in \varphi\}$$

is called an *interior* of A with respect to φ.

Let φ, ψ be filters on G. We define a *product* φ, ψ by the rule: $A \in \varphi, \psi$ if and only if $int\ (A, \psi) \in \varphi$. By this definition, the family

$$\{\bigcup_{g \in F} gH_g : \ F \in \varphi, \ \ Hg \in \psi \text{ for every } g \in F\}$$

forms a base for the filter $\varphi\psi$. The following basic identity

$$int\ (int\ (A, \psi), \varphi) = int\ (A, \varphi\psi)$$

shows that the product is associative.

A topology $\Im$ on a group G is called *left invariant* if the mapping $x \longmapsto gx$ is continuous for every $g \in G$. A group G endowed with a left invariant topology is called *left topological.* Since every left invariant topology $\Im$ on G is uniquely determined by the filter τ of neighborhoods of e, G endowed with $\Im$ will be denoted by (G, τ). A filter φ on a group G is called *left topological* if φ is the filter of neighborhoods of e for some left invariant topology on G.

For characterization of left topological filters we need some more definitions. Given an arbitrary filter φ on G, we denote by $int\ (\varphi)$ the filter on G with the base $\{int\ (F, \varphi) : \ F \in \varphi\}$, and by Cl_φ the mapping, determined by the rule $Cl_\varphi(A) = cl\ (A, \varphi)$ for every $A \subseteq G$.

Theorem 1.1. *For every filter φ on a group G, the following statements are equivalent*

(i) φ is left topological,

(ii) $\varphi\varphi = \varphi$ and $e \in F$ for every $F \in \varphi$,

(iii) $\varphi = int\ (\varphi)$,

(iv) Cl_φ is a closure operator.

Theorem 1.2. *For every filter φ on a group G, the filter $int\ (\varphi)$ is left topological.*

Theorem 1.3. *For every ultrafilter φ on a group G, the left topological group $(G,\ int\ (\varphi))$ is Hausdorff and zero-dimensional. If $\varphi\varphi = \varphi$ then $(G,\ int\ (\varphi))$ is extremely disconnected.*

For every filter φ on a group G, we denote by $hull\ (\varphi)$ the maximal (by inclusion) left topological filter on G such that $hull\ (\varphi) \subseteq \varphi$. In other words, $hull\ (\varphi)$ is the filter of neighborhoods of e for the strongest left invariant topology in which φ converges to e.

A topological space X is called *strongly extremally disconnected* if, for every open non-closed subset U of X, there exists an element $x \in\ cl\ U \setminus U$

such that $\{x\}\bigcup U$ is a neighborhood of x. Every strongly extremally disconnected space is extremally disconnected.

Theorem 1.4. *For every group G and every ultrafilter φ on G, the left topological group $(G,\ hull\ (\varphi))$ is strongly extremally disconnected.*

Now let G be a discrete group, X be a topological space and $(g,x)\longmapsto g(x)$ be a continuous action of G on X. We fix an arbitrary element $x_0\in X$ and denote by φ the filter on G with the base consisting of the subsets of the form $\{g\in G:\ g(x_0)\in U\}$ where U runs over all neighborhoods of x_0. It is easy to check that φ is left topological. On the other hand, if φ is a left topological filter on G, then φ arises in this way from the left action of G on $X=(G,\varphi)$ with $x_0=e$.

Comments. For proofs of these statements see [20] and [41]. Every countable group G admits a Hausdorff topology in which all mapping $x\longmapsto gx$, $x\longmapsto xg$, $g\in G$ and $x\longmapsto x^{-1}$ are continuous. For every countable zero-dimensional homogeneous space X and every countable group G, there is a left invariant topology on G such that G is homeomorphic to X [61].

2. Semigroup of ultrafilters

Let X be a discrete space and let βX be the Stone-Čech extension of X. We take the point of βX to be the ultrafilters on X with the points of X identified with the principal ultrafilters. We put $X^*=\beta X\setminus X$. For every subset $A\subseteq X$, let $\bar{A}=\{q\in\beta X:\ A\in q\}$. The topology of βX can be defined by stating that the family $\{\bar{A}:\ A\subseteq X\}$ is a base for the open sets. For every filter φ on X, the subset $\bar{\varphi}=\bigcap\{\bar{A}:\ A\in\varphi\}$ is closed in βX, and, for every nonempty closed subset $K\subseteq\beta X$, there exists a filter φ on X such that $K=\bar{\varphi}$. We put $\varphi^*=\bar{\varphi}\bigcap X^*$. Let Y be a compact Hausdorff space. For every mapping $f:\ X\longrightarrow Y$, we denote by f^β the Stone-Čech extension of f onto βX.

Now let G be a discrete group. For every element $g\in G$, we extend the left shift $l_g:\ G\longrightarrow G$, $l_g(x)=gx$ to the mapping $l_g^\beta:\ \beta G\longrightarrow\beta G$. Clearly, for every ultrafilter $q\in\beta G$, $l_g^\beta=\{gQ:\ Q\in q\}$, so we write $l_g^\beta(q)=gq$. Further, for each $q\in\beta G$, we extend the right shift $r_q:\ G\longrightarrow\beta G$, $r_q(x)=xq$ to the mapping $r_q^\beta:\ \beta G\longrightarrow\beta G$. For every $p\in\beta G$, we put $r_q^\beta(p)=pq$. It is easy to verify that this product of ultrafilters coincides with the product defined in Section 1. By the construction, the mapping $x\longmapsto xq$ is continuous for every $q\in\beta G$, and the mapping $x\longmapsto qx$ is continuous for every $q\in G$. In particular, βG

is a right topological semigroups. For the structure of βG and plenty of its combinatorial application we address to the books [10] and [27].

For every left topological filter τ on G, $\overline{\tau}$ and τ^* are closed subsemigroups of βG. The semigroup $\overline{\tau}$ is called the *the semigroup of ultrafilters* of the left topological group (G, τ). In what follows we use the semigroups of ultrafilters to explore the topological properties of left topological and topological groups.

Let S be a closed subsemigroup of βG and let φ be a filter on G such that $S = \overline{\varphi}$. We say that S is *uniform* if φ is a left topological filter. Equivalently, S is uniform if, for every $U \in \varphi$, there exists $V \in \varphi$, such that $V\overline{\varphi} \subseteq \overline{U}$. Every finite subsemigroup of βG is uniform. For examples of non-uniform subsemigroups see [20] and [11].

By definition, for every left topological group (G, τ), $\overline{\tau}$ is the set of all ultrafilters on G converging to e. Now we assume that (G, τ) is Hausdorff and denote by C_τ the set of all converging ultrafilters on (G, τ). For $q \in C_\tau$ and the limit g of q, we put $\pi_1(q) = g, \pi_2(q) = g^{-1}q$. Thus, we have defined the bijection $\pi = \pi_1 \times \pi_2$ between (G, τ) and the direct product $(G, \tau) \times \overline{\tau}$ of the left topological group (G, τ) and the right topological semigroup $\overline{\tau}$. In the following theorems βG means the Stone-$\check{C}$ech compactification of a discrete group G.

Theorem 2.1. *For every Hausdorff left topological group (G, τ), the following statements are equivalent*

(i) C_τ is a subsemigroup of βG and π_1 is a homomorphism,

(ii) the mapping $x \longmapsto g^{-1}xg$ is continuous at e in (G, τ) for every $g \in G$.

Theorem 2.2. *For every Hausdorff left topological group (G, τ), the following statements are equivalent*

(i) C_τ is a subsemigroup of βG and π_1, π_2 are a homomorphisms,

(ii) for every element $g \in G$, there exists $U \in \tau$ such that $gx = xg$ for each $x \in U$.

Theorem 2.3. *For every Hausdorff left topological group (G, τ), the following statements hold*

(i) π_1 is continuous if and only if (G, τ) is regular,

(ii) π_2 is continuous if and only if (G, τ) is discrete.

Theorem 2.4. *Let (G, τ) be a Hausdorff semitopological group. Then C_τ is a subsemigroup of βG and the following statements hold*

(i) (G, τ) is a homomorphic image of C_τ,

(ii) (G, τ) is a continuous homomorphic image of C_τ provided that (G, τ) is regular,

(iii) C_τ is algebraically isomorphic to $G \times \overline{\tau}$.

Corollary. *Let (G,τ) be a compact Hausdorff topological group. Then (G,τ) is a continuous homomorphic image of βG. If G is commutative, βG is algebraically isomorphic to $G\times\overline{\tau}$.*

Comments. Most results of this section are from [29].

3. Ideals

Every compact Hausdorff right topological semigroup S contains at least one minimal left ideal. Every minimal left ideal is closed and the union $M(S)$ of all minimal left ideals of S is the minimal ideal of S. For algebraic structure of $M(S)$ see [10, Chapter 1] or [47, Chapter 1]. Note that every minimal left ideal L of S is of the form $L = Sx$ for every $x\in L$, so the following theorem describes all minimal left ideal of closed subsemigroups of βS.

Theorem 3.1. *Let τ be a filter on a group G such that $\overline{\tau}$ is a subsemigroup of βG, and let p be an ultrafilter on G such that $p\in\overline{\tau}$ and $p\in\overline{\tau}p$. Then $\overline{\tau}p$ is a minimal left ideal of $\overline{\tau}$ if and only if, for every $A\in p$ and every $U\in\tau$, there exists a finite subset $F\subseteq U$ such that cl $(F^{-1}A,p)\in\tau$.*

Corollary 1. *Let (G,τ) be a left topological group and let $p\in\overline{\tau}$. Then $\overline{\tau}p$ is a minimal ideal of semigroup $\overline{\tau}$ if and only if, for every $A\in p$ and every neighborhood U of e, there exists a finite subset $F\subseteq U$ such that cl $(F^{-1}A,p)$ is a neighborhood of e.*

Corollary 2. *Let (G,τ) be a left topological group and let V be a neighborhood of the unit e. For every finite partition $V = A_1\bigcup\ldots\bigcup A_n$ and every neighborhood U of e, there exist A_i and a finite subset $F\subseteq U$ such that $F^{-1}A_iA_i^{-1}$ is a neighborhood of e.*

Corollary 3. *For every group G and every finite partition $G = A_1\bigcup\ldots\bigcup A_n$, there exist A_i and a finite subset F of G such that $G = FA_iA_i^{-1}$ and $A_iA_i^{-1}A_iA_i^{-1}\in p$ for every $p\in\beta G$ such that $pp = p$.*

If L is a minimal left ideal of βG, then G acts continuously on L by the rule $x\longmapsto gx$, $x\in L$, $g\in G$, and L is the minimal universal G-space (see [18] and [53]).

Now we define some ideals and describe some ideal decomposition of the semigroup of ultrafilters of left topological group.

Let (G,τ) be a left topological group and let φ be a filter on G such that $\tau\subseteq\varphi$. We say that φ is an *o-filter* if φ has a base consisting of the open subsets of (G,τ). A filter that is maximal in the class of all o-filters is called an *o-ultrafilter.* By Zorn Lemma, every o-filter is contained in some o-ultrafilter, but o-ultrafilter needs not to be an ultrafilter.

If φ is a filter on (G,τ), $\tau \subseteq \varphi$ and $\varphi\tau = \varphi$, we say that $\overline{\varphi}$ is a uniform right ideal of the semigroup $\overline{\tau}$. It is easy to check that $\overline{\varphi}$ is a uniform right ideal if and only if φ is an o-filter.

Given an arbitrary filter φ on (G,τ) such that $\tau \subseteq \varphi$, there exists the maximal o-filter $o(\varphi)$ satisfying $o(\varphi) \subseteq \varphi$, which is called the *open hull* of φ. Clearly, φ is an o-filter if and only if $\varphi = o(\varphi)$. Moreover, φ is an o-filter if and only if $\varphi = o(p)$ for every ultrafilter p on G such that $\varphi \subseteq p$.

An ultrafilter $p \in \overline{\tau}$ is called *preopen* if *int* $(F,\tau) \in p$ for every closed in (G,τ) subset $F \in p$. An ultrafilter $p \in \overline{\tau}$ is preopen if and only if $o(p)$ is an o-ultrafilter. The set of all preopen ultrafilters from $\overline{\tau}$ forms the closed ideal of $\overline{\tau}$.

Theorem 3.2. *For every left topological group (G,τ), the ideal of all preopen ultrafilters is disjoint union of the uniform right ideals $\overline{\varphi}$, where φ runs over all o-ultrafilters from $\overline{\tau}$.*

We say that a filter $\varphi \subseteq \tau$ is a *c-filter* if φ has a base consisting of the subsets of $G \setminus \{e\}$ that are closed in $G \setminus \{e\}$. A filter that is maximal in the class of all c-filters is called *c-ultrafilter.* By Zorn Lemma, every c-filter is contained in some c-ultrafilter, but c-ultrafilter needs not to be an ultrafilter.

Theorem 3.3. *Let (G,τ) be a nondiscrete Hausdorff left topological group such that the space $G \setminus \{e\}$ is normal. Then the semigroup τ^* is disjoint union of the uniform right ideals $o(\varphi)$, where φ runs over all c-ultrafilters on (G,τ).*

A c-filter φ on (G,τ) is called *primitive* if, for any closed in $G \setminus \{e\}$ disjoint subsets A, B, $A \bigcup B \in \varphi$ implies either $A \in \varphi$ or $B \in \varphi$. Every c-ultrafilter is primitive.

Theorem 3.4. *For every nondiscrete Hausdorff left topological group (G,τ), the semigroup τ^* is a union of the uniform right ideals $o(\varphi)$, where φ runs over all primitive ultrafilters from $\overline{\tau}$.*

Note that the union in Theorem 3.3 could not be disjoint.

The applications of the above notions are based on the following observations.

- A left topological group (G,τ) is extremally disconnected if and only if $\overline{\tau}$ has only one uniform right ideal.
- A left topological group (G,τ) is nodec (= every nowhere dense subset is closed) if and only if every ultrafilter $p \in \overline{\tau}$ is preopen.

• A left topological group (G,τ) is irresolvable (= G can not be partitioned onto two dense subsets) if and only if every preopen ultrafilter is an o-ultrafilter.

Comments. The results of this section are from [19] and [21].

4. Idempotents

An element p of a semigroup S is called an idempotent if $pp = p$. Denote by $E(S)$ the set of all idempotents of S and define three natural preoders on $E(S)$

$$e \leq_l f \text{ if and only if } e = ef,$$
$$e \leq_r f \text{ if and only if } e = fe,$$
$$e \leq f \text{ if and only if } e = ef = fe.$$

Every compact right topological semigroup has an idempotent [10, Theorem 2.5]. In particular, for every discrete group G, every closed subsemigroup of βG has an idempotent. For combinatorial and dynamical applications of this theorem see [10] and [18], [53].

Every idempotent $p \in \beta G$ determines two topologies $(G, int\ (p))$ and $(G, hull\ p)$ which will be characterized in Section 9. Here we define two special types of idempotents of βG (namely, strongly right maximal and strongly summable) of great importance for constructions of extremal topologies on group.

By [10, Theorem 2.12], every compact right topological semigroup has a *right maximal idempotent*, i.e. an idempotent, which is maximal with respect to $\leq_r$. If G is a countable discrete group and p is a right maximal idempotent in βG, then the subsemigroup $\{q \in \beta G : \ qp = p\}$ is finite [10, Theorem 9.4].

Let G be an infinite discrete Abelian group, $p \in G^*$ be an idempotent. Then, for every subset $P \in p$, there exists an infinite subset $A \subseteq P$ such that $FS(A) \subseteq P$, where $FS(A)$ is the set of all finite sums of distinct elements of A. This observation easily implies the Hindman's Theorem: for every finite partition $G = A_1 \bigcup ... \bigcup A_n$, there exist A_i and an infinite subset $A \subseteq A_i$ such that $FS(A) \subseteq A_i$.

An idempotent $p \in G^*$ is called *strongly summable* if, for every $P \in p$, there exists an infinite subset $A \subseteq P$ such that $FS(A) \subseteq P$ and $FS(A) \in p$. By [12, Theorem 2.8], Martin's Axiom (MA) implies that there is a strongly summable ultrafilter $p \in G^*$. On the other hand [12, Theorem 3.6], the existence of strongly summable ultrafilters on G implies the existence of P-point in ω^*.

An element r of a semigroup S is called *right cancellable* if, for any $p, q \in S$, $pr = qr$ implies $p = q$. Let G be a countable discrete group, $r \in \beta G$. By [10, Theorem 8.18], p is right cancellable in βG if and only if there is no idempotent $p \in G^*$ such that $pr = r$.

5. Morphisms

A topological space X with the distinguished element e_X (the identity) and a partial binary operation (the multiplication) is called a *local left topological group* if there is a left topological group G such that

- e_X is the identity of G,
- X is an open neighborhood of e_X in G,
- the partial multiplication on X is precisely the partial operation induced on X by the multiplication on G.

Note that every left topological group is a local left topological group.

Let X, Y be local left topological groups. A mapping $f : X \longrightarrow Y$ is called a *local homomorphism* if $f(e_X) = e_Y$ and, for every $x \in X$, there exists a neighborhood U of e_X such that, for all $y \in U$, the products xy, $f(x)f(y)$ are defined and $f(xy) = f(x)f(y)$. If, in addition, f is a homeomorphism, then f is called a *local isomorphism.* If the left topological groups (G, τ) and (G', τ') are locally isomorphic, then the semigroups $\overline{\tau}$ and $\overline{\tau}'$ are topologically isomorphic.

Theorem 5.1 *Any two nondiscrete countable Hausdorff zero-dimensional local left topological groups of countable weight are locally isomorphic.*

A local automorphism f of a local left topological group X is called a homogeneous local automorphism of order m (m is a natural number) if the f-orbit $O(f, x) = \{f^n(x) : \ n < \omega\}$ of every element $x \in X$, $x \neq e_X$ is of cardinality m. We consider the following example. Let $\mathbb{Z}_{m+1} = \{0, 1, ..., m\}$ be the cyclic group and let $H(m+1) = \oplus_\omega \mathbb{Z}_{m+1}$ be the direct sum of ω discrete copies of $\mathbb{Z}_{m+1}$ endowed with the topology of pointwise convergence, so $H(m+1)$ is a topological group. Let s_m be the coordinate-wise substitution on H(m) induced by a substitution on $\mathbb{Z}_{m+1}$ which is the product of independent cycles $(1)(2, 3, ..., m+1)$. Clearly, f is a homogeneous local automorphism of order m.

Theorem 5.2 *Let X be a countable nondiscrete Hausdorff zero-dimensional local left topological group and let f be a homogeneous local automorphism of X of order m. Then there exists a local isomorphism $h : \ X \longrightarrow H(m+1)$ such that the following diagram is commutative*

$$\begin{array}{ccc} X & \xrightarrow{h} & H(m) \\ f\downarrow & & s_m\downarrow \\ X & \xrightarrow{h} & H(m) \end{array}$$

Theorem 5.3 *Let G, H be countable discrete groups and let $\varphi: \beta G \longrightarrow H^*$ be a continuous homomorphism. Then $\varphi(\beta G)$ is a finite group.*

Theorem 5.4 *Let G be a countable discrete groups, H be a finite discrete group and let $\varphi: G^* \longrightarrow H$ be a continuous homomorphism. Then there exists a homomorphism $f: G \longrightarrow H$ such that $\varphi = f^{\beta}|_{G^*}$.*

Theorem 5.5. *Let G, H be countable discrete groups, $\varphi: G^* \longrightarrow H^*$ be a continuous surjective homomorphism. Then there exists a homomorphism $f: G \longrightarrow H$ such that $\varphi = f^{\beta}|_{G^*}$.*

Comments. The notion of local left topological semigroup was introduced by E. Zelenyuk. Theorems 5.1 and 5.2 are parts of more general Theorem 3.1 from [60]. Originally, Theorem 5.1 arouse as the positive answer to the author's question: are the semigroups of ultrafilters of any two countable metrizable topological group topologically isomorphic? The most remarkable application of local left topological groups is Zelenyuk's Theorem, stating that every finite subgroup of βG is a singleton, where G is a countable discrete torsion-free group. For proof of this theorem see [57] or [10, Chapter 7]. The finite subgroups of βG, where G is a countable discrete group, were characterized in [31]. These all have the form Fp, where F is a finite subgroup of G and p is an idempotent which commutes with the elements of F.

For the case $G = H = \mathbb{Z}$, Theorem 5.3 was proved by D.Strauss [52] as the answer to question of van Douwen. Theorems 5.3, 5.4 and 5.5 are from [38]. For discontinues automorphisms see [29].

6. Equations

Some problems concerning joint continuity in βG (see Section 7) or resolvability of left topological groups (see Section 11) can be reduced to the equations in βG. Here we present some statements about corresponding equations.

Let G be an Abelian group, $p \in \beta G$. For every integer m, by mp we denote the ultrafilter on G with the base $\{mP: P \in p\}$.

Theorem 6.1. *Let G be an infinite discrete Abelian group, $p \in G^*$, $s \in \mathbb{Z}$, $s \neq 0, 1$. If the subgroup $\{g \in G: s(s-1)g = 0\}$ is finite, then the equation $x + p = y + sp$ is not solvable in βG.*

Theorem 6.2. *Let G be an infinite discrete Abelian group, $p \in G^*$, $s \in \mathbb{Z}$, $s \neq 0, 1$. Suppose that one of the following condition holds*

(i) there exists $P \in p$ such that every element of P has an infinite order,

(ii) for every $P \in p$, the set of all prime divisors of orders of elements of P is infinite.

Then the equation $x + p = y + sp$ is not solvable in βG.

In other words, these theorems states that, under corresponding conditions, $\beta G + p \bigcap \beta G + sp = \emptyset$.

Theorem 6.3. *If G is a discrete Abelian group, $p \in \beta G$ and $p + p = 2p$, then p is a principal ultrafilter .*

Theorem 6.4. *For every $p \in \mathbb{Z}^*$ and every integer $s, s \neq 0, 1$, the equation $p + x = sp + y$ is not solvable in $\beta\mathbb{Z}$.*

It follows from Theorem 6.4 that, for every idempotent $p \in \mathbb{Z}^*$ and every integer $s, s \neq 0, 1$, the subgroup of βG generated by p and sp is a free product of the subsemigroups $\{p\}$ and $\{sp\}$.

Comments. These results are from [23] and [24]. For generalizations see [13].

7. Continuity in βG and G^*

Let G be a discrete group, $p, q \in \beta G$. We say that (p, q) is a *point of joint continuity* if the multiplication $\beta G \times \beta G \longrightarrow \beta G$ is continuous at the point (p, q). It is easy to see that (p, q) is a point of joint continuity if and only if, for every $R \in p, q$, there exist $P \in p$, $Q \in q$ such that $PQ \subseteq R$. If either $p \in G$ or $q \in G$, then (p, q) is a point of joint continuity. These points of joint continuity are called *standard*.

Let G be a countable discrete Abelian group, $p, q \in G^*$. If (p, q) is a point of joint continuity, then there exists $r \in G^*$ such that the equation $x + r = y + 2r$ has a solution in βG. This reduction from [23] and Theorem 6.1 give us

Theorem 7.1. *If G is a countable discrete Abelian group with finite subsets of elements of order 2, then every point of joint continuity in $\beta G \times \beta G$ is standard.*

Let G be a discrete group. For every $p \in \beta G$, we define the mapping $\lambda_p : \beta G \longrightarrow \beta G$ by the rule $\lambda_p(q) = pq$. The set

$$\Lambda(\beta G) = \{p \in \beta G : \lambda_p \text{ is continuous at every point } q \in \beta G\}$$

is called the topological center of βG.

For every $p \in G^*$, we denote by λ_p^* the restriction of λ_p to G^*. The set

$$\Lambda(G^*) = \{p \in G^* : \ \lambda_p^* \text{ is continuous at every point } q \in G^*\}$$

is called the topological center of G^*.

Theorem 7.2. *For every discrete group G, $\Lambda(G^*) = \emptyset$. Moreover, if $|G|$ is nonmeasurable, then there exists $q \in G^*$ such that λ_p^* is discontinuous at every point $q \in G^*$.*

Corollary *For every discrete group G, $\Lambda(\beta G) = G$.*

Theorem 7.3. *Suppose that a countable discrete G is isomorphic to some subgroup of compact topological group. If $q \in G^*$ and, for every $p \in G^*$, λ_p^* is continuous at q, then q is a P-point in G^*.*

Theorem 7.4. *Let G be a countable discrete group and let $p \in G^*$ be either an idempotent or a right cancellable element of βG. If λ_p^* is continuous at p, then there exists a mapping $f : \ G \longrightarrow \omega$ such that $f^\beta(p)$ is a P-point in ω^*.*

Given an arbitrary discrete group G, we denote by $\mu : \ \beta G \longrightarrow \beta G$ the mapping defined by $\mu(q) = qq$. Let μ^* be the restriction of μ to G^*.

Theorem 7.5. *Let G be a countable discrete group and let $p \in G^*$ be either an idempotent or a right cancellable element of βG. If μ^* is continuous at p, then there exists a mapping $f : \ G \longrightarrow \omega$ such that $f^\beta(p)$ is a P-point in ω^*.*

Theorem 7.6. *Let G be a countable discrete Abelian group with finite set of elements of order 2. If $p \in G^*$ and μ^* is continuous at p, then p is a P-point in G^*.*

In spite of above theorems, the mappings λ_p^* and μ^* have some continuity-like property.

Let X, Y be topological spaces. A mapping $f : \ X \longrightarrow Y$ is called *quasi-continuous* at point $x \in X$ if, for every neighborhoods U and V of x and $f(x)$, there exists a nonempty open subset $W \subseteq U$, such that $f(W) \subseteq U$.

Theorem 7.7. *Let G be a discrete group and let $p \in G^*$. Then μ^* is quasi-continuous at p. If G is countable, then λ_p^* is quasi-continuous at p.*

Comments. Theorem 7.1 is from [23]. Under some additional to ZFC set-theoretical assumptions, there are non-standard points of joint continuity in βG [23].

Theorem 7.2 is from [28], Corollary from Theorem 7.2 and the first statement of this theorem in the case of countable groups is due to [9]. For generalization of Theorem 7.2 see [40].

The remaining theorems of this section are from [45].

8. Ramsey-like ultrafilters

It follows easily from the Ramsey theorem that, for every coloring $\chi : G \longrightarrow \{0,1\}$ of an infinite Abelian group G, there exists an infinite subset $A \subseteq G$ such that the subset $PS(A) = \{a+b : \ a,b \in A, \ a \neq b\}$ is monochrome. We say that an ultrafilter $p \in G^*$ is a *PS-ultrafilter* if, for every coloring $\chi : \ G \longrightarrow \{0,1\}$, there exists $A \in p$ such that $PS(A)$ is monochrome. Clearly, every Ramsey (=selective) ultrafilter on G is a PS-ultrafilter. An Abelian group G is called a Boolean group if $2g = 0$ for every $g \in G$. Every strongly summable ultrafilter P (see Section 4) on an infinite Boolean group is a PS-ultrafilter but p is not selective. If p is a PS-idempotent on a countable Boolean group G, then $(G, int\ p)$ is a maximal topological group (see Section 1).

A subset A of an Abelian group G is called 2-*independent* or a Sidon set if, for any two-elements subsets $\{a,b\}$, $\{c,d\}$ of A, $a+b = c+d$ implies $\{a,b\} = \{c,d\}$. A PS-ultrafilter is selective if and only if there exists a 2-independent subset $A \in p$.

Theorem 8.1. *Every PS-ultrafilter on an infinite Abelian group without elements of order 2 is selective.*

Theorem 8.2. *Let G be an infinite Abelian group, $p \in G^*$, $H = \{g \in G : \ 2^n g = 0$ for some $n \in \mathbb{N}\}$. If p is a PS-ultrafilter then either p is selective or there exists $g \in G$ such that $g + H \in p$.*

An ultrafilter p on a set X is called a *Q-point*, if, for every partition of X onto finite subsets, there exists $P \in p$ such that $|P \bigcap A| \leq 1$ for every cell A of the partition. It follows from Theorem 8.2 that if PS-ultrafilter p is a Q-point then p is selective.

Theorem 8.3. *Let G be an infinite Abelian group, p be a PS-ultrafilter on G such that, for every $P \in p$, there exist $a,b \in P$, $a \neq b$ such that $a+b \in P$. Then $p+p = p$ and $2p = 0$.*

Theorem 8.4. *If p is a PS-ultrafilter on a countable Abelian group G, then either p is right cancellable or there exist $p' \in G^*$, $g \in G$ such that $p = g + p'$, $p' + p' = p'$.*

Theorem 8.5. *Let G be a countable Abelian group, p be a PS-ultrafilter on G. Then there exists a mapping $f : \ G \longrightarrow \omega$ such that $f^\beta(p)$ is a P-point in ω^*.*

Let H be a subgroup of an infinite Abelian group G. We say that an ultrafilter $p \in G^*$ is *H-selective* if either there exists $g \in G$ such that $g + H \in p$ or there exists $P \in p$ such that $|P \bigcap (g + H)| \leq 1$ for every $g \in G$. We say that p is *subselective* if p is H-selective for every subgroup H of G. Every PS-ultrafilter is subselective, but the class of subselective ultrafilters is much more wider than the class of PS-ultrafilters.

Theorem 8.6. *For every infinite Abelian group G, there exists a subselective ultrafilter $p \in G^*$ such that $p + p = p$ and $\|p\| = |G|$ where $\|p\| = min\{|P| : P \in p\}$.*

Theorem 8.7. *Every free ultrafilter on an infinite Abelian group G is subselective if and only if every infinite subgroup H of G has a finite index in G.*

Comments. The PS-ultrafilters were introduced in [25], all results of this section are from [39].

9. Maximality

All topological spaces in this section are suppose to be Hausdorff. A topological space X is called *maximal* if X has no isolated points but X has an isolated points in every stronger topology. We say that a left topological group is maximal if it is maximal as a topological space.

Let G be an infinite discrete group with the identity e. For every idempotent $p \in G^*$, we put $\tau_p = \{P \bigcup \{e\} : P \in p\}$. Then τ_p is a left topological filter and the left topological group (G, τ_p) is maximal. In the notations of Section 1, $\tau_p = hull(p)$ so τ_p is the strongest left invariant topology on G in which p converges to e. In what follows, we write $G(p)$ instead of (G, τ_p).

On the other hand, if (G, τ) is a maximal left topological group, then $\tau^* = \{p\}$, p is an idempotent of G^* and $(G, \tau) = G(p)$. Thus, there is a bijection between the set off all maximal left invariant topologies on G and the set of all idempotents of G^*. It follows that, for every infinite group G, there are $2^{2^{|G|}}$ maximal left invariant topologies on G.

For every filter φ on a set X, we put $\|\varphi\| = min\{|F| : F \in \varphi\}$. If (G, τ) is a nondiscrete left topological group, then there exists an idempotent $p \in \tau^*$ such that $\|p\| = \|\tau\|$. In other words, the topology of (G, τ) can be strengthened to the maximal left invariant topology (G, τ') such that $\|\tau'\| = \|\tau\|$.

As distinct from the topological group case, a left invariant topology on a group needs not to be regular. Given an idempotent $p \in G^*$, let φ_p be the filter of neighborhoods of the identity e of the strongest regular left invariant topology on G in which p converges to e. In the notations of Section 1, it can be shown that $\varphi_p = int(p)$ so $\overline{\varphi_p} = \{q \in \beta G : \ qp = p\}$. In what follows we write $G[p]$ instead of $(G, int\ p)$. By Theorem 1.3, $G[p]$ is zero-dimensional and extremally disconnected.

Theorem 9.1. *Let (G, τ) be a countable nondiscrete regular left topological group of countable weight. Then there exists an idempotent $p \in \tau^*$ such that $G[p] = G(p)$.*

Corollary. *Every infinite group G admits a regular maximal left invariant topology. In particular, there exists (in ZFC) a regular maximal homogeneous space.*

The proof of Theorem 9.1 uses the right maximal idempotent and Theorem 5.1.

Theorem 9.2. *Every maximal left topological group is a union of a countable family of closed discrete subspaces.*

It follows from Theorem 9.2 that every maximal left topological group has countable pseudocharacter, i.e. the identity e of G is an intersection of some countable family of open subsets. Equivalently, every countable complete idempotent from βG is a principal ultrafilter.

Let (G, τ) be an infinite left topological group. The cardinal κ is called the *index of boundedness* of (G, τ) if κ is the minimal cardinal such that, for every $U \in \tau$, there exists $F \subseteq G$ such that $G = FU$ and $|F| < \kappa$. If the index of boundedness of (G, τ) is $\aleph_0$, (G, τ) is called totally bounded.

Theorem 9.3. *The index of boundedness of every maximal left topological group (G, τ) is greater than the cofinality of $|G|$.*

Theorem 9.4. *Let G be an infinite group, p be an idempotent from the minimal ideal of the semigroup βG. Then $G[p]$ is totally bounded.*

It follows from Theorem 9.4, that every infinite group admits a zero-dimensional totally bounded left invariant topology.

By [43, Theorem 2], for every free group G and every idempotent $p \in G^*$, the cellularity of the left topological group $G[p]$ is $|G|$. In view of Theorem 9.4, there are the totally bounded left topological groups of arbitrarily large cellularity.

Now we present some results concerning the maximal topological groups.

Theorem 9.5. *Let G be an infinite group, $p \in G^*$. If $G[p]$ is a topological group, then $G[p]$ contains a countable open Boolean subgroup.*

If (G, τ) is a maximal topological group, then $(G, \tau) = G(p)$ for some idempotent $p \in G^*$. Since every group topology is regular, we have

$G(p) = G[p]$. Hence, Theorem 9.5 is a generalization of Malykhin's theorem [16]: every maximal topological group has a countable open Boolean subgroup.

Under MA, an example of maximal topological group was constructed by Malykhin in [15].

Let G be a countable Boolean group, p be an idempotent from G^*. Then $G(p)$ is a topological group if and only if p is a PS-ultrafilter (see Section 8). In particular, every strongly summable ultrafilter on G determines the maximal group topology. It was shown in [20] that the existence of a maximal topological group implies the existence of P-point in ω^*. The following theorem is a generalization of this statement.

Theorem 9.6. *Let G be a group, p be an idempotent from G^* such that $G[p]$ is a topological group. Then there exists a mapping $f : G \longrightarrow \omega$ such that $f^\beta(p)$ is a P-point in ω^*.*

Theorem 9.7. *Every maximal topological group is complete in the left uniformity.*

It is interesting that if there exists a maximal topological group then there exists a maximal topological group with distinct left and right uniformities.

A group G provided with a topology is called *semitopological* if G is left and right topological. A semitopological group is called *quasitopological* if the mapping $x \longmapsto x^{-1}$ is continuous. A group G provided with a topology is called *paratopological* if the multiplication $(x, y) \longrightarrow xy$ is jointly continuous.

Theorem 9.8. *For every group G provided with a topology, the following statements are equivalent*

(i) G is a maximal semitopological group,

(ii) G is a maximal left topological group and, for every element $g \in G$, there exists a neighborhood U of the identity such that $xg = gx$ for every $x \in U$.

Theorem 9.9. *For every group G provided with a topology, the following statements are equivalent*

(i) G is a maximal quasitopological group,

(ii) G is a maximal semitopological group and there exists a neighborhood U of the identity e such that $x^2 = e$ for every $x \in U$.

Corollary. *Let G be a maximal quasitopological group. Then every neighborhood of the identity contains an infinite Boolean subgroup.*

As distinct from the topological groups case, the maximal quasitopological groups can be easily constructed in ZFC. Indeed, let G be an infinite Boolean group and let p be an idempotent from G^*. Then $G(p)$ is the maximal quasitopological group. In view of Corollary from Theorem 9.9 it is naturally to ask (see [34] or [3 , Question 3)]: does there exist an open Abelian subgroup in every maximal quasitopological group? The negative answer (in ZFC) to this question was done in [44].

Theorem 9.10. *Every maximal paratopological group is a topological group.*

Comments. All results of this section are from [33] and [34]. For proof of Theorem 9.1 see also [10, Chapter 9]. The topologies on semigroups determined by idempotents were considered in [11].

10. Refinements

As we have seen in the previous section, maximality is a very exotic phenomenon in the category of topological groups because there exist ZFC-models without maximal topological groups. In contrast to topological groups every nondiscrete left topological group has the left topological refinements that are maximal. But from the topological point of view these refinements could be very far from the original left invariant topology on a group. In this section we consider some new kinds of refinements of the left invariant topologies. On one hand, these refinements are very close to the original topology. On the other hand, they have a wide spectrum of extremal topological properties. The results presented in this section witness that the category of left topological groups is much more appropriate environment for life of extremal objects than the category of topological groups. Moreover, the extremal objects are not exotic in this category because they can be constructed from any left topological groups in some very natural ways.

Let τ, φ be left topological filters on a group G. We say that (G, φ) is an *open refinement* of (G, τ) if $\tau \subseteq \varphi$ and every nonempty open subset of (G, φ) contains a non-empty open subset of (G, τ). If in addition $\tau \neq \varphi$, (G, φ) is called a *proper* open refinement of (G, τ). A left topological group (G, τ) is called *o-maximal* if it has no proper open refinements. By Zorn Lemma, for every left topological group, there exists a maximal open refinement. Every maximal open refinement is o-maximal. To characterize these refinements we use the o-ultrafilters from Section 3.

Theorem 10.1. *Let τ, φ be left topological filters on a group G, $\tau \subseteq \varphi$. Then (G, φ) is a maximal open refinement of (G, τ) if and only if there exists an o-ultrafilter ψ on (G, τ) such that $\varphi = \{F \bigcup \{e\} : \ F \in \omega\}$, where e is the identity of G.*

Theorem 10.2. *A left topological group* (G, τ) *is o-maximal if and only if, for every open subset* U *of* (G, τ) *such that* $e \in cl\ (U, \tau)$*, the subset* $\{e\} \bigcup U$ *is a neighborhood of* e*.*

It follows from above theorems that every left topological group has a strongly extremally disconnected (see Section 1) open refinement. We note also that some basic cardinal invariants of left topological groups (density, cellularity, index of boundedness) are stable under the open refinements.

Let τ, φ be left topological filters on a group G, $\tau \subseteq \varphi$. We say that (G, φ) is a *dense refinement* of (G, τ) if $cl\ (U, \tau) \in \tau$ for every $U \in \varphi$. If in addition $\tau \neq \varphi$, (G, φ) is called a *proper dense refinement.* A left topological group is called *d-maximal* if it has no proper dense refinement. By Zorn Lemma, every left topological group has a maximal dense refinement and every maximal dense refinement is d-maximal.

Let (G, τ) be a left topological group and let φ be a filter on G such that $\tau \subseteq \varphi$. We put $(\tau\varphi)_e = \{F \bigcup \{e\} : F \in \tau\varphi\}$ where e is the identity of G. Then $(G, (\tau\varphi)_e)$ is a dense refinement of (G, τ). If (G, τ) is d-maximal, then $(\tau p)_e = \tau$ for every ultrafilter $p \in \overline{\tau}$. It follows that $\overline{\tau} = L \bigcup \{e\}$ where L is the minimal left ideal of $\overline{\tau}$.

Let τ, φ be left topological filters on a group $G, \tau \subseteq \varphi$. We say that (G, φ) is an *open-dense refinement* of (G, τ) if it is both open and dense refinement. If in addition $\tau \neq \varphi$, (G, φ) is called a *proper* open-dense refinement of (G, τ). A left topological group is called *od-maximal* if it has no proper open-dense refinements.

Given any left topological filter τ on a group G, consider the strongest filter ψ on G containing all o-filters on (G, τ). We call ψ the *open core* of τ. Clearly, ψ is the strongest filter containing all o-ultrafilters on (G, τ), so $p \in \overline{\psi}$ if and only if p is preopen in (G, τ). By Theorem 1.1, ψ_e is a left topological filter. For every subset $A \subseteq G$, we have $A \in \psi$ if and only if *int* $(A \bigcap U, \tau) \neq \emptyset$ for every open subset U in (G, τ) such that $e \in cl(U, \tau)$.

Theorem 10.3. *Let* (G, τ) *be a left topological group and let* ψ *be an open core of* τ*. Then* (G, ψ_e) *is the unique maximal open-dense refinement of* (G, τ)*.*

A left topological group (G, τ) is called *nodec* if every nowhere dense subset in (G, τ) is closed. Note that every nowhere dense subset of a nodec group is discrete.

Theorem 10.4. *For every left topological group* (G, τ)*, the following statements are equivalent*

(*i*) (G, τ) *is nodec,*

(ii) every ultrafilter $p \in \tau^*$ is preopen,
(iii) (G, τ) is od-maximal.

By Theorems 10.3 and 10.4, every left topological group has an open-dense refinement which is nodec.

Theorem 10.5. *A nondiscrete left topological group is maximal if and only if it is o-maximal and nodec.*

Now we give an ultrafilter characterization of extremally disconnected left topological groups.

Theorem 10.6. *Let (G, τ) be a left topological group and let φ be an o-ultrafilter on (G, τ). Then the left topological group $(G, int\ \varphi)$ is zero-dimensional and extremally disconnected.*

Theorem 10.7. *Let (G, τ) be a left topological group and let φ be an o-ultrafilter on (G, τ). Then (G, τ) is extremally disconnected if and only if $int(\varphi) \subseteq \tau$. If (G, τ) is regular, then (G, τ) is extremally disconnected if and only if $int\ (\varphi) = \tau$.*

By Theorem 10.6 and 10.7, every regular left topological group has a zero-dimensional extremally disconnected open refinements.

A regular left topological group (G, φ) is called a *bounded refinement* of a regular left topological group (G, τ) if $\tau \subseteq \varphi$ and, for every subset $U \in \varphi$, there exists a finite subset $K \subseteq G$ such that $KU \in \tau$. If in addition $\tau \neq \varphi$, (G, φ) is called a proper bounded refinement of (G, τ).We say that (G, τ) is *b-maximal* if it has no proper bounded refinements. By Zorn Lemma, every regular left topological group has a maximal bounded refinement. Clearly, every maximal bounded refinement is b-maximal.

Theorem 10.8. *A regular left topological group (G, τ) is a maximal bounded refinement of a regular left topological group (G, τ) if and only if there exists an idempotent p from the minimal ideal of the semigroup $\overline{\tau}$ such that $\varphi = int\ (p)$. A regular left topological group (G, τ) is b-maximal if and only if $\varphi = int\ (p)$ for every idempotent p from the minimal ideal of $\overline{\tau}$.*

By Theorem 1.3, for every infinite group G and every idempotent $p \in G^*$, $(G, int\ (p))$ is zero-dimensional and extremally disconnected. Applying Theorem 10.8, we conclude that every regular left topological group admits a zero-dimensional extremally disconnected bounded refinement.

Theorem 10.9. *Let (G, τ) be a regular nondiscrete left topological group. A left topological group (G, φ) is a maximal regular nondiscrete refinement of (G, τ) if and only if there exists a right maximal idempotent p of the semigroup τ^* such that $\varphi = int\ (p)$.*

By Theorem 1.4, for every group G and every ultrafilter p on G, the left topological group $(G, hull\ (p))$ is strongly extremally disconnected.

Theorem 10.10. *Let G be an infinite Boolean group and let p be a Ramsey ultrafilter on G. Then the left topological group $(G, int\ (p))$ is a topological group.*

Given any group G and any $p, q \in \beta G$, we say that p, q are *locally isomorphic* if the corresponding left topological group $(G, hull\ (p))$ and $(G, hull\ (q))$ are locally isomorphic (see Section 5). The problem of classification of ultrafilter on a group up to the local isomorphism is very difficult. The next theorem gives a solution in one special case.

Theorem 10.11. *Let G be a countable group and let p, q be the right cancellable ultrafilters from G^*. Then p, q are locally isomorphic if and only if there exists a bijection $f: G \longrightarrow G$ such that $f^{\beta}(p) = q$.*

We note that a free ultrafilter p on a countable group G is right cancellable if and only if there exists a subset $P \in p$ such that the identity of G is the only limit point of p is $(G, hull\ (p))$. For every right cancellable $p \in G^*$, the left topological group $(G, hull\ (p))$ is Hausdorff and zero-dimensional.

Comments. All results of this section are from [41]. It is still unknown: does there exist in ZFC a nondiscrete nodec topological group or a nondiscrete extremally disconnected topological group? Under MA, every countable Abelian group admits a nondiscrete nodec group topology (see [58] or [35, Chapter 2]). In the case of countable groups, the second question is an old problem of Arhangel'skiĭ [2]. Under CH, the first example of nondiscrete countable extremally disconnected group was constructed in [51]. Theorem 10.10 is a generalization of the Sirota's results. Some partial results incline us to the negative answer to the Arhangel'skiĭ question. Thus, Theorem 9.6 can be re-formulated as follows. If there exists a countable nondiscrete extremally disconnected topological group G such that the topology of G is maximal in the class of all regular left invariant topologies on G, then there is a P-point in ω^*. Another statement of this type was proved in [59]. If there exists a countable nondiscrete extremally disconnected topological group which has a nonclosed discrete subsets then there exists a P-point of ω^*. For some partial cases of Theorem 10.11 see [54].

11. Resolvability

Given a cardinal κ, a Hausdorff topological space X is called *κ-resolvable* if X can be partitioned into κ dense subset. In the case $\kappa = 2$ we say that

X is *resolvable*. If X is not κ-resolvable, we say that X is *κ-irresolvable.*

Theorem 11.1. *Every nondiscrete left topological group of second category is $\aleph_0$-resolvable.*

The following construction is crucial for the proof of Theorem 11.1. Let G be an uncountable group of cardinality γ with the unit e. Using a minimal well-ordering of G we can construct inductively the chain $\{G_\alpha : \alpha < \gamma\}$ of subgroups of G such that

(*i*) $G_0 = \{e\}$, $G = \bigcup\{G_\alpha : \alpha < \gamma\}$,

(*ii*) $G_\alpha \subset G_\beta$ for any $\alpha < \beta < \gamma$,

(*iii*) $\bigcup\{G_\alpha : \alpha < \gamma\} = G_\beta$ for every limit ordinal $\beta < \gamma$,

(*iv*) $|G_\alpha| < \gamma$ for every $\alpha < \gamma$.

For every ordinal $\alpha < \gamma$, we decompose $G_{\alpha+1} \setminus G_\alpha$ into the right cosets by the subgroup G_α and fix some subset X_α of representatives of cosets. Thus $G_{\alpha+1} \setminus G_\alpha = G_\alpha X_\alpha$. Take arbitrary element $g \in G$, $g \neq e$ and choose the smallest subgroup G_α such that $g \in G_\alpha$. By (*iii*), $\alpha = \alpha_1 + 1$ for some ordinal $\alpha_1 < \gamma$. Then $g \in G_{\alpha_1+1} \setminus G_{\alpha_1}$ and $g = g_1\, x_{\alpha_1}$, $g_1 \in G_{\alpha_1}$. If $g_1 \neq e$, we choose α_2, $g_2 \in G_{\alpha_2}$, $x_{\alpha_2} \in X_{\alpha_2}$ such that $g_1 = g_2 x_{\alpha_2}$. After the finite number $s(g)$ of steps we get the representation

$$g = x_{\alpha_{s(g)}}\ x_{\alpha_{s(g)-1}} \ldots x_{\alpha_2}\ x_{\alpha_1},$$

$$\alpha_{s(g)} < \alpha_{s(g)-1} < \ldots < \alpha_2 < \alpha_1, \quad x_{\alpha_i} \in X_{\alpha_i}.$$

Using this representation we define the partition

$$G \setminus \{e\} = \bigcup_{n \in \omega} S_{n+1}$$

where $S_{n+1} = \{g \in G \setminus \{e\} : s(g) = n+1\}$.

Now let (G, τ) be a nondiscrete left topological group, φ be an o-ultrafilter on (G, τ). Then (G, τ) is $\aleph_0$-irresolvable if and only if $\overline{\varphi}$ is finite. By means of this criterion, the general case of Theorem 11.1 can be reduced to the case in which every subset S_n is dense.

Theorem 11.2. *Every infinite group G can be partitioned into $|G|$ subsets dense in every left invariant left totally bounded topology on G.*

Theorem 11.3. *If (G, τ) is a nondiscrete $\aleph_0$-irresolvable topological Abelian group, then the subgroup $\{g \in G : 2g = 0\}$ is countable and open.*

Theorem 11.4. *Let (G, τ) be a nondiscrete left topological Abelian group, m be integer such that the mapping $x \longmapsto mx$ is continuous and the subgroup $\{g \in G : m(m-1)g = 0\}$ is finite. If $|m| > 1$, then (G, τ) is $\aleph_0$-irresolvable. If $m = -1$, then (G, τ) is resolvable.*

Theorem 11.5. *Let (G,τ) be a countable regular left topological group such that the mapping $x \longmapsto x^{-1}$ is continuous. If (G,τ) is non-discrete and $\aleph_0$-irresolvable, then the subgroup $\{g \in G: \ 2g = 0\}$ is open.*

Theorem 11.6 *Let G be a countable group with only finite numbers of elements of order 2. Assume that G admits a totally bounded group topology. Then G can be partitioned into countable many subsets dense in every non-discrete group topology on G.*

Theorem 11.7. *Let (G,τ) be a non-discrete $\aleph_0$-irresolvable topological group. If G is either Abelian or countable, then there exists a P-point in ω^*.*

Comments. The concept of resolvability of topological space was introduced by E.Hewitt in 1944. Theorem 11.1 is from [37] and [42]. It answers the following question posed by A.V.Arhangel'skiĭ and P.Collins [4]: is every submaximal topological group of first category? The examples of non-discrete irresolvable topological spaces of second category under some set-theoretical assumptions are given in [49] and [50].

The start for study resolvability of topological groups was done by W.W.Comfort and J. van Mill [6]: every non-discrete topological Abelian group with only finite set of elements of order 2 is resolvable. Theorem 11.3 sharpens this result and was proved in [26]. Theorem 11.2 is from [17], Theorem 11.4 was proved in [36] by means of equations in βG.

Theorem 1.5 and 1.6 were proved by E.Zelenyuk [60] by means of automorphisms of local left topological groups. Under MA Zelenyuk constructed an irresolvable topological group which is not maximal [55].

For Abelian groups, Theorem 11.7 was proved in [37], the countable case is reduced to the Abelian case by means of Theorem 11.5.

More on resolvability of topological groups can be found in the surveys [7], [8] and [30]. For combinatorial aspects of resolvability see [5] and [46].

12. Potential compactness and ultra-rank

A group G is called *potentially compact* if, for every ultrafilter p on G, there exists a Hausdorff group topology on G with respect to which p converges to some point of G. Clearly, every group which admits a compact group topology is potentially compact.

Theorem 12.1. *Every potentially compact group is a subgroup of some compact topological group.*

To give a criterion of potential compactness of Abelian group G we use the Bohr's compactification $G^\sharp$ of discrete group G.

Theorem 12.2. *Abelian group G is potentially compact if and only if, for every element $h \in G^\sharp$, there exists $g \in G$ such that $\overline{< gh >} \bigcap G = \{e\}$, where $\overline{< gh >}$ is the smallest closed subgroup of G containing gh.*

Applying this criterion we get the following statements.

Theorem 12.3. *A free Abelian group G is potentially compact if and only if rank $G > 1$.*

Theorem 12.4. *An Abelian torsion group G is potentially compact if and only if it is reduced and the number of its non-trivial p-Sylow subgroups is finite.*

The concept of potential compactness can be generalized via the following cardinal invariant of group.

Let G be a group. A family $\Im$ of ultrafilters on G is called *suitable* if there exists a Hausdorff group topology on G in which all ultrafilters from $\Im$ converge. In particular, G admits a compact group topology if and only if βG is suitable. If G does not admit the compact group topologies, we denote by $inc(G)$ the minimal cardinality of non-suitable families of ultrafilters on G.

Let κ be a cardinal and let G be a subgroup of a topological group H. We say that G is *topologically κ-servant* in H if, for every family $\{h_\alpha :\ \alpha \in J\}$ of elements of H such that $|J| < \kappa$ there exists the family $\{g_\alpha :\ \alpha \in J\}$ of elements of G such that $G \bigcap \overline{<< X >>} = \{e\}$, where $X = \{g_\alpha h_\alpha :\ \alpha \in J\}$, $\overline{<< X >>}$ is the smallest closed invariant subgroup of H containing X.

Theorem 12.5. *If G is a topologically κ-servant subgroup of compact topological group H, then inc $(G) \geq \kappa$.*

Theorem 12.6. *Assume that a group G is a subgroup of some compact topological group. If G is not topologically κ-servant in the Bohr's compactification $G^\sharp$ of discrete group G, then inc $(G) < \kappa$.*

Applying Theorem 12.5 and 12.6 we get the following statements.

Theorem 12.7. *Every free group of rank > 0 is not potentially compact.*

Theorem 12.8. *If inc $(G) > \aleph_0$, then G is algebraically compact.*

Let (G, τ) be a topological group, φ be a set of converging ultrafilters on G. We say that φ determines the topology of G if τ is the strongest group topology on G in which every ultrafilter from φ converges. We denote by u-rank (G, τ) the minimal cardinality of the sets of ultrafilters on G which determine the topology of G.

To calculate the ultra-rank of topological group we use the preodering on the set of all group topologies on G defined by the rule: $\tau_1 \leq \tau_2$ if and only if, for every $V \in \tau_1$, there exists a finite subset $F \subseteq G$ such that $FV \in \tau_2$. It is easy to check that $\tau_1 \leq \tau_2$ if and only if every Cauchy ultrafilter in (G, τ_2) is a Cauchy ultrafilter in (G, τ_1). For every topological group G there exists the unique maximal topology τ_0 such that $\tau_0 \leq \tau$. We say that (G, τ_0) is the *totally bounded modification* of (G, τ).

Let (G, τ) be a topological group and let (G, τ_0) be the totally bounded modification of (G, τ). We consider $\widehat{G}$, the completion of (G, τ_0) in the two-sided group uniformity and extend the identity mapping $f : (G, \tau_0) \longrightarrow (G, \tau)$ to the continuous homomorphism $\widehat{f} : \widehat{G} \longrightarrow (\widehat{G}, \tau)$. We put $F_\tau = Ker\ \widehat{f}$ and say that the subset $X \subseteq F_\tau$ topologically generates F_τ as the normal divisor of $\widehat{G}$ if F_τ is the smallest closed invariant subgroup of $\widehat{G}$ containing X.

Theorem 12.9. *Let (G, τ) be a topological group. Then u-rank (G, τ) is the minimal cardinality of subsets of F_τ which topologically generate F_τ as the normal divisor of $\widehat{G}$.*

Applying Theorem 12.9 we conclude that ultra-rank of every infinite compact Abelian group G is $2^{2^{|G|}}$. On the other hand, ultra-rank of $\mathbb{Z}$ endowed with the topology of finite indices is 1.

Comments. All results of this section are from [19] and [62]. The preodering on the set of group topologies was defined independently in [19] and [48].

13. Selected open questions

Question 1. *Has every maximal topological group a base of neighborhoods of the unit consisting of subgroups? Equivalently, is every PS-ultrafilter on the countable Boolean group strongly summable?*

Question 2. *Let G be a countable Abelian group, q be a PS-ultrafilter on G. Is it true that q is either selective or $q = a + p$ for some $a \in G$ and some idempotent $p \in G^*$?*

Question 3. *Does there exist in ZFC a maximal regular homogeneous space X such that every non-empty open subset of X is uncountable?*

Question 4. *Let X be a non-discrete homogeneous space of second category. Is X resolvable? $\aleph_0$ - resolvable?*

Question 5. *Let (G, τ) be a left topological group of uncountable pseudo-character. Is (G, τ) resolvable? This is so if either cardinality of G is non-measurable or G is Abelian* [37].

Question 6. *Let (G, τ) be a topological group such that every non-empty open subset of G is uncountable. Is (G, τ) resolvable?*

If the answer to this question is positive then, by Theorem 11.7, it is impossible to construct a non-discrete irresolvable topological group in ZFC.

Question 7. *Let G be an infinite Abelian group, m be integer, $|m| > 1$. Assume that the subgroup $\{g \in G : m(m-1)g = 0\}$ is finite. Is it possible to decompose G into countably many subsets that are dense in any topology with continuous shifts and continuous mapping $x \longmapsto mx$.*

Question 8 (V. Malykhin). *Let (G, τ) be a countable non-discrete irresolvable topological group. Is $\bar{\tau}$ finite?*

Question 9 (I. Guran). *Let G, τ be an extremally disconnected paratopological group. Is (G, τ) a topological group?*

Question 10. *Let G be a countable discrete group, $q \in G^*$. Suppose that the mapping $\lambda_p^* : G^* \longrightarrow G^*$, $\lambda_p^*(x) = px$ is continuous at the point q for every $p \in G^*$. Is q a P-point in G^*.*

Question 11. *Does there exist (in ZFC) a countable group G and $p, q \in G^*$ such that λ_p^* is continuous at q?*

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