Algebra and Discrete Mathematics
Number 1. **(2009).** pp. 74 – 82 RESEARCH ARTICLE

On action of outer derivations on nilpotent ideals of Lie algebras

Dmitriy V. Maksimenko

Communicated by A. P. Petravchuk

Abstract. Action of outer derivations on nilpotent ideals of Lie algebras are considered. It is shown that for a nilpotent ideal I of a Lie algebra L over a field F the ideal $I + D(I)$ is nilpotent, provided that $charF = 0$ or I nilpotent of nilpotency class less than $p - 1$, where $p = charF$. In particular, the sum $N(L)$ of all nilpotent ideals of a Lie algebra L is a characteristic ideal, if $charF = 0$ or $N(L)$ is nilpotent of class less than $p - 1$, where $p = charF$.

It is known that the nilradical of a finite dimensional Lie algebra over a field of characteristic 0 is characteristic, i.e. it is invariant under any derivation of the algebra. It was shown in [3], that for an arbitrary Lie algebra L (not necessarily finite dimensional) over a field of characteristic 0 the image $D(I)$ of a nilpotent ideal $I \subseteq L$ under derivation $D \in Der(L)$ lies in some nilpotent ideal of the algebra L. The restriction on characteristic of the ground field is essential while proving this assertion.

We use methods which are analogous to ones in [6] during the investigation of behavior of solvable ideals under outer derivations. It is shown in Theorem 1 of the paper that the image of a nilpotent ideal of nilpotency class n from a Lie algebra L over a field F under an outer derivation lies in a nilpotent ideal provided that $n < p-1$, where $p = charF$. The methods of research here are completely different from ones in [3] because it is impossible in general to construct automorphisms from nilpotent derivations of Lie algebras over fields of positive characteristic.

2000 Mathematics Subject Classification: *17B40.*
Key words and phrases: *Lie algebra, derivation, solvable radical, nilpotent ideal.*

The notations in the paper are standard. If T is an $F-$subspace of a Lie algebra L then we denote by $T^1 = T, T^2 = [T,T], \ldots, T^n = [T^{n-1}, T]$.

For elements $x_1, \ldots, x_n$ of a Lie algebra L we denote

$$[x_1, x_2, \ldots, x_n] = [[\ldots[x_1, x_2], \ldots x_{n-1}], x_n].$$

For a Lie algebra L we denote by $Der(L)$ the Lie algebra of all derivations of L. If $D \in Der(L)$ and T is an F-subspace of L we denote for convenience $D^0(T) = T,\ D^k(T) = D(D^{k-1}(T))$ for $k \geqslant 1$.

Further, for any elements $x_1, \ldots, x_m \in L$, any derivation $D \in Der(L)$ and an arbitrary natural $n \geqslant 1$ it holds (Leibniz's rule for differentiation of several multipliers):

$$D^n([x_1, \ldots, x_m]) = \sum_{k_1+\cdots+k_m=n} \frac{n!}{k_1! \ldots k_m!} [D^{k_1}(x_1), \ldots, D^{k_m}(x_m)] \quad (1)$$

(the summation is extended over all nonnegative $k_1, \ldots, k_m$). The special case of this formula is the usual Leibniz's rule

$$D^n([x,y]) = \sum_{k=0}^{n} \binom{n}{k} [D^k(x), D^{n-k}(y)]$$

for arbitrary elements $x, y \in L$ и $D \in Der(L)$.

Let L be a Lie algebra over an arbitrary field, let I be its ideal, $D \in Der(L)$. As for any $x \in I, y \in L$ it holds $[x, D(y)] = D([x,y]) - [D(x), y]$, than $I + D(I)$ is an ideal of the Lie algebra L. It is easy to see that the sum

$$I + D(I) + D^2(I) + \ldots + D^k(I)$$

is also an ideal for any natural $k > 1$.

We need some lemmas for proving the main theorem.

Lemma 1. *Let L be a Lie algebra over a field of characteristic $p \neq 2$, I be an abelian ideal of L, $D \in Der(L)$. Then $[D(I), D(I)] \subseteq I$.*

Proof. Take arbitrary elements $x, y \in I$. As the ideal I is abelian, then $[x,y] = 0$ and, therefore, $D^2([x,y]) = 0$. From the other hand, by Leibniz's rule we obtain the following:

$$0 = D^2([x,y]) = [D^2(x), y] + 2[D(x), D(y)] + [x, D^2(y)].$$

Since I is an ideal of L, it follows from the previous relation that

$[D(x), D(y)] \in I$. Since x, y are arbitrary elements from I then $[D(I), D(I)] \subseteq I$. □

Lemma 2. *Let L be a Lie algebra over an arbitrary field, I be an ideal of L, $D \in Der(L)$. Then for any $x_1, \dots, x_s \in I$ and any nonnegative number $m < s$ it holds:*

$$D^m([x_1, \dots, x_s]) \in I^{s-m}.$$

Proof. Denote by $l = s - m > 0$. Using the relation (1), we obtain:

$$D^m([x_1, \dots, x_s]) = \sum_{k_1+\dots+k_s=m} \frac{m!}{k_1! \dots k_s!} [D^{k_1}(x_1), \dots, D^{k_s}(x_s)] \qquad (2)$$

Since all $k_1, \dots, k_s$ are nonnegative and $k_1 + \dots + k_s = m < s$, then at least l of the numbers $k_1, \dots, k_s$ are equal to zero. As, by definition, $D^0(x) = x$ for all $x \in I$, then, as one can easily make sure, every summand $[D^{k_1}(x_1), D^{k_2}(x_2), \dots, D^{k_s}(x_s)]$ of this sum belongs to I^l. Hence, $D^m([x_1, \dots, x_s]) \in I^l = I^{s-m}$. □

Lemma 3. *Let I be a nilpotent ideal of nilpotency class n from a Lie algebra L over a field K of characteristic 0 or characteristic $p > n+1$, $D \in Der(L)$. Then $(I + D(I))^{n+1} \subseteq I$.*

Proof. To prove the statement of Lemma it is sufficient to show that

$$[\underbrace{D(I), \dots, D(I)}_{n+1}] \subseteq I \qquad (3)$$

Consider the equality (2) for $m = n+1$, $s = n+1$ and take into account that $[x_1, \dots, x_n, x_{n+1}] = 0$ for all elements $x_1, \dots, x_n, x_{n+1} \in I$:

$$D^{n+1}([x_1, \dots, x_{n+1}]) = \\ = \sum_{k_1+\dots+k_{n+1}=n+1} \frac{(n+1)!}{k_1! \dots k_{(n+1)}!} [D^{k_1}(x_1), \dots, D^{k_{n+1}}(x_{n+1})] = 0.$$

Since all $k_1, \dots, k_{n+1}$ are nonnegative, then the last relation can be written down in the form

$$\frac{(n+1)!}{1! \dots 1!} [D(x_1), \dots, D(x_{n+1})] + \\ + \sum_{k_1+\dots+k_{n+1}=n+1} \frac{(n+1)!}{k_1! \dots k_{n+1}!} [D^{k_1}(x_1), \dots, D^{k_{n+1}}(x_{n+1})] = 0,$$

where the summation is extended over all nonnegative $k_1, \dots, k_{n+1}$, at least one of which is more then 1. Since all numbers $k_1, \dots, k_{n+1}$ in the last sum are nonnegative, then at least, one of them is zero. Therefore all summands under the sign of sum in the last relation belong to the ideal I. But then, obviously, $(n+1)![D(x_1),\ldots,D(x_{n+1})] \in I$. As $n+1 < p$ or $charF = 0$, then it follows that $[D(x_1),\ldots,D(x_{n+1})] \in I$. Since the elements $x_1,\ldots,x_n,x_{n+1} \in I$ were chosen arbitrarily we obtain the relation (3). $\square$

Lemma 4. *Let I be a nilpotent ideal of nilpotency class n from a Lie algebra L over a field of characteristic 0 or characteristic $p > n+1$, $D \in Der(L)$. Then $[I, \underbrace{D(I),\ldots,D(I)}_{n+1}] \subseteq I^2$.*

Proof. Take arbitrary elements $x_1,\ldots,x_{n+2} \in I$. Denote for convenience:

$t_1 = [x_1, D(x_2), D(x_3), \ldots, D(x_{n+2})]$;
$t_2 = [D(x_1), x_2, D(x_3), \ldots, D(x_{n+2})]$;
$\ldots$
$t_{n+1} = [D(x_1), D(x_2), \ldots, x_{n+1}, D(x_{n+2})]$;
$t_{n+2} = [D(x_1), D(x_2), \ldots, D(x_{n+1}), x_{n+2}]$.

Since $I^{n+1} = 0$, we can write down the following equalities:

$$u_s = [x_1, x_2, \ldots, D(x_s), \ldots, x_{n+2}] = 0$$

for $s = 1,\ldots,n+2$. Applying the Leibniz's rule (1) for computation $0 = D^n(u_s) = D^n([x_1, x_2, \ldots, D(x_s), \ldots, x_{n+2}])$ we obtain

$$D^n(u_s) = \sum_{k_1+\cdots+k_{n+2}=n} \frac{n!}{k_1!\ldots k_{n+2}!}[D^{k_1}(x_1),\ldots$$
$$\ldots, D^{k_s+1}(x_s), \ldots, D^{k_{n+2}}(x_{n+2})].$$

Since all k_j are nonnegative and $k_1+\cdots+k_{n+2} = n$, then at least two numbers among $k_1,\ldots,k_{n+2}$ are equal to 0. If at least three numbers among $k_1,\ldots,k_{n+2}$ are equal to 0 then the summand of this sum of the form

$$\frac{n!}{k_1!\ldots k_{n+2}!}[D^{k_1}(x_1),\ldots D^{k_s+1}(x_s),\ldots,D^{k_{n+2}}(x_{n+2})]$$

lies obviously in I^2. Let now exactly two numbers k_i, k_j are equal to 0 in this summand. If $i \neq s$ и $j \neq s$, then, as above, one can show that the summand

$$\frac{n!}{k_1!\ldots k_{n+2}!}[D^{k_1}(x_1),\ldots D^{k_s+1}(x_s),\ldots,D^{k_{n+2}}(x_{n+2})]$$

lies in the ideal I^2. So, we have to consider only the case when one of the indices i, j, for instance, i coincides with s. Then $k_s = 0$, $k_j = 0, j \neq$

s. Since all other numbers k_m are equal to 1, then we obtain that the summand

$$\frac{n!}{k_1!\dots k_{n+2}!}[D^{k_1}(x_1),\dots D^{k_s+1}(x_s),\dots,D^{k_{n+2}}(x_{n+2})]$$

is equal to

$$\frac{n!}{1!\dots 1!}[D(x_1),\dots D(x_{j-1}),D^0(x_j),D(x_{j+1})\dots,D^{k_{n+2}}(x_{n+2})]=n!t_j.$$

Therefore, having fixed $i=s$ and arbitrarily chosen j, not equal to s, we obtain that

$$D^n(u_s)=n!(t_1+\dots t_{s-1}+t_{s+1}+\dots+t_{n+2})+z_s \tag{4}$$

for some $z_s\in I^2$. Denote by $v_s=D^n(u_s)/n!$ for $s=1,\dots,n+2$. Then taking into account the relation $char K=p>n+1$ we see that

$$v_s=t_1+\dots t_{s-1}+t_{s+1}+\dots+t_{n+2}\in I^2$$

for arbitrary $s=1,\dots,n+2$. Consider the sum $v=\sum_{s=1}^{n+2}v_s$. It is easy to see that $v=(n+1)\sum_{k=1}^{n+2}t_k,\quad v\in I^2$. Because of the restriction on characteristic of the ground field it holds the relation $t=t_1+t_2+\dots+t_{n+2}\in I^2$. But then the element $t_1=t-v_1$ belongs to the ideal I^2. As elements $x_1,\dots,x_{n+2}$ were chosen arbitrarily and

$$t_1=[x_1,D(x_2),D(x_3),\dots,D(x_{n+2})]$$

we have that $[I,\underbrace{D(I),\dots,D(I)}_{n+1}]\subseteq I^2$.

□

Lemma 5. *Let I be a nilpotent ideal of nilpotency class n from a Lie algebra L over a field of characteristic 0 or characteristic $p>n+1$, $D\in Der(L)$. Then there exits a function $f_n(m)$ of a natural argument m such that $f_n(m)=f_n(m-1)+n-m+1, f_n(1)=n+1$ and*

$$[I^m,\underbrace{D(I),\dots,D(I)}_{f_n(m)}]\subseteq I^{m+1} \tag{5}$$

for $m=1,\dots,n$.

Proof. Let n be a fixed natural number. Then for $m=1$ we have by Lemma 4 the relation $[I,\underbrace{D(I),\dots,D(I)}_{n+1}]\subseteq I^2$ and therefore one can take $f_n(1)=n+1$.

Assume that it is already proved that the function $f_n(t)$ satisfies the condition

$$[I^{m-1}, \underbrace{D(I), \ldots, D(I)}_{f_n(m-1)}] \subseteq I^m.$$

Let us show that the following inclusion holds:

$$[I^m, \underbrace{D(I), \ldots, D(I)}_{f_n(m-1)+n-m+1}] \subseteq I^{m+1}.$$

We denote for convenience $N = f_n(m-1)+n-m+2$ and take arbitrary elements $x_1 \in I^m, x_2, \ldots, x_N \in I$. Denote by $s = f_n(m-1)+1, t = n-m+1$. Then $N = t+s$.

It is easy to see that the following equality holds:

$$[x_1, D(x_2), \ldots, D(x_s), x_{s+1}, \ldots, x_N] = 0 \tag{6}$$

Really, $[x_1, D(x_2), \ldots, D(x_s)] \in I^m$ and, as $x_{s+1}, \ldots, x_N \in I$, then

$$[x_1, D(x_2), \ldots, D(x_s), \underbrace{x_{s+1}, \ldots, x_N}_{n-m+1}] \in I^{m+(n-m+1)} = I^{n+1} = 0.$$

Apply now the derivation D to the equality (6) $n-m+1$ times. Using Leibniz's rule (1), we obtain:

$$\sum \frac{t!}{k_1! \ldots k_N!} [D^{k_1}(x_1), D^{k_2+1}(x_2), \ldots$$
$$\ldots, D^{k_s+1}(x_s), D^{k_{s+1}}(x_{s+1}), \ldots, D^{k_N}(x_N)] = 0 \tag{7}$$

where the summation is extended over all nonnegative $k_1, \ldots, k_N$ such that $k_1 + \cdots + k_N = t = n-m+1$.

Since the sum of all numbers $k_1, \ldots, k_N$ is t, and their quantity is $N = s+t$, then obviously there are at least s numbers from the set $\{k_1, \ldots, k_N\}$ which are equal to 0. Let's prove that in the sum (7) all summands except maybe the summand

$$t![D^0(x_1), D(x_2), \ldots, D(x_s), D(x_{s+1}), \ldots, D(x_N)], \tag{8}$$

that corresponds to $k_1 = 0, k_2 = 0, \ldots, k_s = 0, k_{s+1} = 1, \ldots, k_N = 1$, lie in I^{m+1}.

Consider the possible cases:

a) There are exactly s numbers among $k_1, \ldots, k_N$ which are equal to 0. If these numbers are $k_1, \ldots, k_s$, then $k_{s+1} = \cdots = k_N = 1$ and we obtain the exceptional element (8). So we assume that at least one of the numbers $k_1, \dots, k_s$ is nonzero. Then at least one of the numbers $k_{s+1}, \dots, k_N$ is 0.

At first assume that $k_1 = 0$. Then $D^{k_1}(x_1) = x_1 \in I^m$ and if at least one of numbers $k_{s+1}, \dots, k_N$ is 0, then the summand

$$t! \cdot [D^{k_1}(x_1), D^{k_2+1}(x_2), \dots, D^{k_s+1}(x_s), D^{k_{s+1}}(x_{s+1}), \dots, D^{k_N}(x_N)] \quad (9)$$

belongs to the ideal I^{m+1}. Let now all the numbers $k_{s+1}, \dots, k_N$ be nonzero. Then $k_2 = \dots = k_s = 0$ and we obtain the exceptional element (8).

Consider now the case $k_1 = 1$. If $k_2 = \dots = k_s = 0$, then $D(x_1) \in I^{m-1}$ by Lemma 2 and therefore $[D(x_1), \underbrace{D(x_2), \dots, D(x_s)}_{f_n(m-1)}] \in I^m$. Since at least one of the numbers $k_{s+1}, \dots, k_N$ is 0, then the element of the form (9) lies in I^{m+1}. Suppose now that at least one of the numbers $k_2, \dots, k_s$ is equal to 1. Then at least two of the numbers $k_{s+1}, \dots, k_N$ are 0 and therefore again the element of the form (9) lies in I^{m+1}.

So, in case a) either the element (9) is of the exceptional form (8) or it lies in I^{m+1}.

b) There are exactly $s+i$ numbers among $k_1, \dots, k_N$ which are equal to 0, where $i \geqslant 1$. Show that we can suppose in this case that at least $i+1$ of the numbers $k_{s+1}, \dots, k_N$ are equal to 0. Really, since $N = s+t$ then we have that at least i of numbers $k_{s+1}, \dots, k_N$ are equal to 0. Assume that there are exactly i such numbers. Then all the numbers $k_1, \dots, k_s$ are equal to 0 and therefore $t! \cdot [x_1, D(x_2), \dots, D(x_s)] \in I^m$. Since $i \geqslant 1$, then at least one of the numbers $k_{s+1}, \dots, k_N$ is equal to 0 and $t! \cdot [x_1, D(x_2), \dots, D(x_s), D^{k_{s+1}}(x_{s+1}), \dots, D^{k_N}(x_N)] \in I^{m+1}$.

So, we will suppose further that there are at least $i+1$ of the numbers $k_{s+1}, \dots, k_N$ which are equal to 0. Denote the quantity of such numbers by r. Then according to our assumption $r \geqslant i+1$. Hence, the quantity of non-zero numbers among $k_{s+1}, \dots, k_N$ is equal to $t-r$ and for their sum it holds $\geqslant t-r$. But then the sum of all non-zero numbers among $k_1, \dots, k_s$ is less or equal $t-(t-r) = r$ and, therefore $k_1 \leqslant r$.

At first let the sum of all nonzero numbers among $k_1, \dots, k_s$ be less than r. Then $k_1 \leqslant r-1$ and therefore $D^{k_1}(x_1) \in I^{m-r+1}$. It follows from here that

$$[D^{k_1}(x_1), D^{k_2+1}(x_2), \dots$$
$$\dots, D^{k_s+1}(x_s), D^{k_{s+1}}(x_{s+1}), \dots, D^{k_N}(x_N)] \in I^{m-r+1+r} = I^{m+1}$$

since there are at least r elements among $D^{k_{s+1}}(x_{s+1}), \dots, D^{k_N}(x_N)$ which lie in I.

Let now the sum of all nonzero numbers among $k_1, \dots, k_s$ be equal to r. If $k_1 \leqslant r-1$, then, as above, one can show that the element of the form (8) lies in I^{m+1}. Let now $k_1 = r$. Then $k_2 = \cdots = k_s = 0$ and by the inductive assumption (since by Lemma 2 it holds $D^r(x_1) \in I^{m-r}$) we have the inclusion $[D^r(x_1), \underbrace{D(x_2), \ldots, D(x_s)}_{f_n(m-1)}] \in I^{m-r+1}$. But then $[D^r(x_1), D(x_2), \ldots, D(x_s), D^{k_{s+1}}(x_{s+1}), \ldots, D^{k_N}(x_N)] \in I^{m+1}$, since at least r elements among $D^{k_{s+1}}(x_{s+1}), \ldots, D^{k_N}(x_N)$ belong to I.

So, all summands in the relation (7), except maybe of the form (8) lie in I^{m+1}. But then in view of equality (7) the exceptive summand (8) lies in I^{m+1}. As the characteristic of the ground field does not divide $t = n - m + 1$, we obtain $[x_1, D(x_2), \ldots, D(x_N)] \in I^{m+1}$. Since the elements $x_1 \in I^m, x_2, \ldots, x_N \in I$ can be chosen arbitrarily we obtain

$$[I^m, \underbrace{D(I), \ldots, D(I)}_{f_n(m-1)+n-m+1}] \subseteq I^{m+1}$$

It means that one can put $f_n(m) = f_n(m-1) + n - m + 1$. Lemma is proved. □

Remark 1. The relation for the function $f_n(m)$ obtained while proving the previous lemma is an inhomogeneous recurrence relation of the 1-st order. Its solution (see, for example [2], §3.3.3) can be written down as a sum $f_n = f_n^h + f_n^p$, where f_n^h is the general solution of the homogeneous recurrence relation $f_n(m) - f_n(m-1) = 0$, and f_n^p is a particular solution for the inhomogeneous relation

$$f_n(m) = f_n(m-1) + n - m + 1 \tag{10}$$

The single characteristic root of the corresponding homogeneous relation is 1. So, its general solution is $f_n^h = C$, where C is an arbitrary constant. We find a particular solution in the form $f_n^p = m(A_1 m + A_0)$, where A_0, A_1 are indeterminate coefficients. Substituting f_n^p into the relation (10), we get $A_1 = -\frac{1}{2}, A_0 = n + \frac{1}{2}$. So, the general solution of the inhomogeneous relation (10) can be presented as $f_n(m) = C - \frac{1}{2}m^2 + (n + \frac{1}{2})m$. One finds the coefficient $C = 1$ from the initial condition $f_n(1) = n + 1$. Finally, we have $f_n(m) = m(n+1) - (m-1)(m+2)/2$.

Theorem 1. *Let I be a nilpotent ideal of nilpotency class n of a Lie algebra L over a field of characteristic 0 or characteristic $p > n + 1$, $D \in Der(L)$. Then $I + D(I)$ is a nilpotent ideal of the Lie algebra L of nilpotency class at most $n(n+1)(2n+1)/6 + 2n$.*

Proof. Denote by $k = \sum_{m=1}^{n} f_n(m)$. Using Lemma 5 one can easily show that $[I, \underbrace{D(I), \ldots, D(I)}_{k}] \subseteq I^{n+1} = 0$. Further, by Lemma 3 we have

$(I + D(I))^{k+n+1} = 0$. So, the ideal $I + D(I)$ is nilpotent of nilpotency class at most $k + n$. Direct calculation yields $k + n = n + \sum_{m=1}^{n} m(n + 1) - \sum_{m=1}^{n}(m-1)(m+2)/2 = n(n+1)(2n+1)/6 + 2n$.

□

Corollary 1. *Let L be a Lie algebra (not necessarily finite dimensional) over a field F, let $N(L)$ be the sum of all nilpotent ideals of L. If the ideal $N(L)$ is nilpotent, then it is a characteristic in the following cases: a) $charF = 0$; b) $charF = p > 0$ and nilpotency class of $N(L)$ is less than $p - 1$.*

Remark 2. We should note that the estimation of nilpotency class of the ideal $I + D(I)$ from Theorem 1 is rather rough. For example, for an ideal I of nilpotency class 2 of a Lie algebra over a field of characteristic $p > 3$ Theorem 1 gives the estimation 9, but direct calculation shows that nilpotency class of $I + D(I)$ does not exceed 8.

References

[1] N.A. Jacobson, *Lie algebras* . Interscience tracts, no. 10 (New York, 1962).

[2] Kenneth H. Rosen (Editor-in-Chief), *Handbook of Discrete and Combinatorial Mathematics*, CRC Press, (2000).

[3] B. Hartley, *Locally nilpotent ideals of a Lie algebra*, Proc. Cambridge Phil. Soc., **63** (1967), 257–272.

[4] G. Letzter, *Derivations and nil ideals* , Rendicotti del Circolo Matematico di Palermo, **37**, no.2 (1988), 174-176.

[5] V.S.Luchko, A.P.Petravchuk, *On one-sided ideals of associative rings*, Algebra and Discrete Mathematics, no.3 (2007), P.1-6.

[6] A.P.Petravchuk, *On behavior of solvable ideals of Lie algebras under outer derivations*, Communications in Algebra, to appear.

Contact information

Dmitriy V. Maksimenko
Kiev Taras Shevchenko University, Faculty of Mechanics and Mathematics, 64, Volodymyrska street, 01033 Kyiv, Ukraine
E-Mail: dm.mksmk@gmail.com

Received by the editors: 24.09.2007
and in final form 14.04.2009.