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Some classes of directoid groups

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Abstract. Directoid groups are 2-torsion-free abelian groups with an extra binary operation assigning upper bounds. They thus constitute a generalization of abelian l-groups and an equational substitute for directed abelian groups. We discuss some classes of directoid groups, including varieties.

Introduction

A *directoid* is a groupoid satisfying the identities

$$xx \approx x, xy \approx yx, (xy)x \approx xy, x((xy)z) \approx (xy)z.$$

A *directoid group* is a 2−torsion-free abelian group with a directoid operation $\cdot$ (which we often indicate by juxtaposition) connected with the group addition by the identity

$$x + yz \approx (x + y)(x + z).$$

The absence of 2-torsion is forced by the other conditions. For this as well as background information and a list of references concerning related topics we refer to [8]. For completeness we note another pertinent reference [3] which appeared subsequent to the submission of [8].

A directoid becomes an up-directed set if we define $a \leq b$ to mean $ab = b$ and with this notion of order a directoid group becomes a directed abelian group. These processes are reversible: an order of either kind gives rise to a binary operation (generally many such) if ab is defined to

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be an upper bound of $\{a, b\}$ subject to some constraints, including the requirement that $ab = b$ if $a \leq b$. Again, see [8] for details.

Directoid groups can be viewed both as a generalization of abelian l-groups and an attempt at equationalizing directed abelian groups, so comparisons and contrasts between these three structures are of interest. Here we study some classes of directoid groups, including varieties. Whereas there are no proper varieties of abelian l-groups, we find several families of varieties of directoid groups and show that the lattice of all varieties is quite complicated. Just *how* complicated remains to be seen: we make no attempt at an exhaustive listing. There are at least two types of classes which we might informally, though reasonably, regard as "varieties of directed abelian groups": varieties of directoid groups which contain, along with any member, all its order-isomorphic copies, and, in imitation of the $e-$ *varieties* of regular semigroups [9], [13] (and some other structures [10]), classes of directed abelian groups closed under order-homomorphic images, products and directed subgroups. We shall see that there are no proper subclasses of the second kind but we present a family of non-trivial varieties of the first kind. Some other rather interesting classes of directoid groups, for instance those in which $ab = a \vee b$ whenever $a \vee b$ exists, fail to be varieties. We discuss a number of such classes and their interdependence.

Our notation follows that of [8] and is generally consistent with that of [1] and [5]. For the reader's convenience we note two items: $\circ$ denotes the binary operation in a directoid group given by $a \circ b = -((-a) \cdot (-b))$ and $\|$ indicates incomparability in any partially ordered system.

1. Non-varieties

It is easy to perturb the lattice operation on an $l-$group to produce a new directoid group - there are examples in [7], while **1.1**(iii) of [8] provides an easy procedure for making more. To the extent that directoid groups are seen as a generalization of abelian l-groups, such examples may appear a little artificial, inasmuch as an arbitrary upper bound is substituted for a "natural" one. On the other hand, it does seem worth knowing that if desired, a (2-torsion-free abelian) directed group can be made into a directoid group whose operation coincides with the supremum whenever the latter exists. We shall prove this in **1.2** below.

In what follows, $a \vee b, a \wedge b$ will denote, respectively, the supremum, infimum of a, b whenever they exist. Clearly if $a \vee b$ exists then so does $(-a) \wedge (-b)$, and then $(-a) \wedge (-b) = -(a \vee b)$ and thus in a directoid group if $a \cdot b = a \vee b$ then $(-a) \circ (-b) = -(a \cdot b) = -(a \vee b) = (-a) \wedge (-b)$, and so on.

We define a *t.m. directoid group* to be a directoid group satisfying the (equivalent) conditions

$$a \vee b \text{ exists} \Rightarrow a \cdot b = a \vee b;$$
$$a \wedge b \text{ exists} \Rightarrow a \circ b = a \wedge b,$$

and denote the class of *t.m.* directoid groups by $\mathcal{TM}$. (*t.m.* stands for "treillis manqué".)

Lemma 1.1. *For elements a, b, c of a partially ordered group we have $(a+c) \vee (b+c) = a \vee b + c$, in the sense that LHS exists if and only if RHS exists and when they exist they are equal.*

Proof. If $a \vee b$ exists then $a, b \leq a \vee b$ so $a+c, b+c \leq a \vee b + c$. If $a+c, b+c \leq y$ then $a, b \leq y - c$ so $a \vee b \leq y - c$ and so $a \vee b + c \leq y$. Thus $a \vee b + c = (a+c) \vee (b+c)$. The other part is proved similarly. □

Theorem 1.2. *Every 2-torsion-free abelian directed group has an operation making it a t.m. directoid group.*

Proof. Let $M \subseteq G$ be as in **1.2** of [8]. If $a \in M$ let $a \cdot 0 = a \vee 0$ if this exists, otherwise anything suitable and $(-a) \cdot 0 = a \cdot 0 - a$ as usual. Clearly $(-a) \vee 0$ exists if and only if $a \vee 0$ exists and then $(-a) \vee 0 = a \vee 0 - a$. Let $\cdot$ be defined on all of G as in **1.1**(iii) of [8]. If $d \vee c$ exists then $(d-c) \vee 0$ exists and $d \cdot c = c + (d-c) \cdot 0 = c + (d-c) \vee 0 = d \vee c$. □

A directoid group will be called an *m-directoid group* if whenever its elements a, b have at least one minimal upper bound, $a \cdot b$ is minimal. We denote by $\mathcal{M}$ the class of m-directoid groups. If c is a minimal upper bound of a, b then for every d, $c+d$ is a minimal upper bound of $a+d, b+d$ and so on so by an argument similar to that used for **1.2** we get

Theorem 1.3. *Every 2-torsion-free abelian directed group has an operation making it an m-directoid group.*

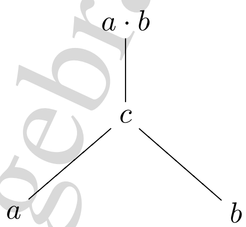
A *t.m.* directoid group is "close to" an $l-$group. The following have opposite behaviour in a sense.

A *c.l.directoid group* is a directoid group in which $a \cdot b = a \vee b$ if and only if a and b are comparable. We denote by $\mathcal{CL}$ the class of *c.l.* directoid groups. (*c.l.* stands for "contralattice".) An example of a *c.l.* directoid group was given by Jakubík [12] (p. 16, Example) but they can be built on any directed group.

Theorem 1.4. *Every 2-torsion-free abelian directed group has an operation making it a c.l. directoid group.*

Proof. Let G be a 2-torsion-free abelian directed group, M a set as in **1.2** of [8]. For $a \in M$ we define $a \cdot 0 = a$ if $a > 0$, $2(a \vee 0)$if $a \| 0$ but $a \vee 0$ exists and anything suitable otherwise. If $a > 0$ then $(-a) \cdot 0 = a \cdot 0 - a = 0 = (-a) \vee 0$. If $a \in M$ and $a \| 0$, then $-a \| 0$. If $(-a) \vee 0$ exists then $a \vee 0$ exists and thus $(-a) \cdot 0 = 0 \cdot a - a = 2(0 \vee a) - a = 2(a + (-a) \vee 0) - a = a + 2((-a) \vee 0)$ whence $(-a) \cdot 0 - (-a) \vee 0 = a + 2(-a) \vee 0) - (-a) \vee 0 = a + (-a) \vee 0 = 0 \vee a \neq 0$; thus $(-a) \cdot 0 \neq (-a) \vee 0$. If $a \in M$ and $(-a) \vee 0$ doesn't exist there is no problem. Now if $b \cdot c = b \vee c$ for some $b, c \in G$ then $(b - c) \vee 0 = b \vee c - c = b \cdot c - c = (b - c) \cdot 0$ so $b - c$ and 0 are comparable, whence b and c are comparable. □

We get some restriction if we specify a condition on the order rather than the directoid operation. A directoid group is a *multilattice group* (see [16]) if for every upper bound c of elements a, b there is a minimal upper bound m of a, b with $m \leq c$. For a *nr multilattice group* [16] we merely require at least one minimal upper bound for each a, b. (Somewhat cryptically, *nr* stands for "non relativiste" [16].) We'll call a directoid group *fork-free* if its underlying directoid conains no forks [12] i.e. its Hasse diagram does not contain a configuration



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Let $\mathcal{FF}$ denote the class of fork-free directoid groups. Applying **1.3** to an *nr* multilattice group, we get

Theorem 1.5. *Every 2-torsion-free abelian nr multilattice group has an operation making it a fork-free directoid group.*

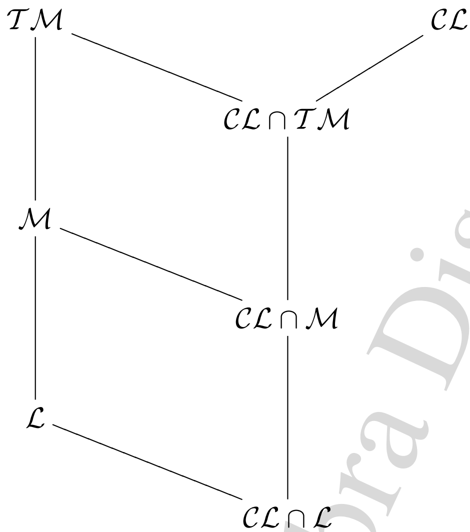
A directed group is an *antilattice*(Fuchs [6]) if it satisfies

(i) $a \vee b$ exists $\Rightarrow a \leq b$ or $b \leq a$ and

(ii) $a_1, a_2 \leq b_1, b_2 \Rightarrow \exists c$ with $a_1 \leq c \leq b_1$, $a_2 \leq c \leq b_2$

(Directed groups satisfying (ii) are called *Riesz groups* [6].) Every directoid group on an antilattice is a *c.l.* directoid group. But these are also *t.m.* directoid groups. Conversely, every *c.l.* directoid group which is also a *t.m.* directoid group must satisfy (i).

If G is a Riesz group and an *nr* multilattice group (in particular, a multilattice group) then every pair a, b has a minimal upper bound m. If $a, b \leq d$ then there exists c with $a \leq c \leq m$ and $b \leq c \leq d$. By minimality we then have $m = c \leq d$. Since d is any upper bound, it follows that $m = a \vee b$, so G is an $l-$group. Hence the fork-free directoid groups which are Riesz groups are precisely the l-groups. Let $\mathcal{R}$ denote the class of Riesz directoid groups. We shall now demonstrate that the classes we have been discussing are related by inclusion as shown in the following Hasse diagram, all inclusions being proper.



(Note that $\mathcal{CL} \cap \mathcal{L}$ is the class of linearly ordered (directoid) groups.)

First, let G be an l-group which is not linearly ordered. Then G, as a directoid group with respect to $\vee$, is in $\mathcal{TM}$ but not in $\mathcal{CL}$. Now let M be a subset of G as in **1.2** of [8]. For $a \in M$ and $a \| 0$, let $a \cdot 0 = 2(a \vee 0)$ and as usual $(-a) \cdot 0 = a \cdot 0 - a = 2(a \vee 0) - a = a \vee 0 + a \vee 0 - a = a \vee 0 + 0 \vee (-a)$. Then $(-a) \cdot 0 \neq (-a) \vee 0$ as $a \vee 0 \neq 0$. Since $a \| 0$ if and only if $-a \| 0$ we have $x \cdot 0 = x \vee 0$ if and only if x and 0 are comparable. If $c \cdot d = c \vee d$ then $(c-d) \cdot 0 = c \cdot d - d = c \vee d - d = (c-d) \vee 0$ so $c-d$ and 0 are comparable, i.e. c and d are comparable. Thus G, with $\cdot$, is in $\mathcal{CL}$ but not $\mathcal{TM}$, so that $\mathcal{CL} \| \mathcal{TM}$. Let$\mathbb{Z}^0$ denote the group of integers with the discrete order, $\mathbb{Z} * \mathbb{Z}^0$ the lexicographic product, where $\mathbb{Z}$ has its standard order and let $(0, a) \cdot (0, 0) = (a, 0)$ and $(0, -a) \cdot (0, 0) = (a, -a)$ for all positive a. This gives us the directoid group $\mathbb{Z} * \mathbb{Z}^0$ of **3.2** in [8]. If $(m, n) \| (k, l)$ then $m = k$ and $n \neq l$. Thus each $(m+1, p)$ is a minimal upper bound, but there is no supremum. It follows that $\mathbb{Z} * \mathbb{Z}^0 \in \mathcal{CL} \cap \mathcal{TM}$. But since, e.g., $(0, 2) \cdot (0, 0) = (2, 0)$ is not a minimal upper bound of $(0, 2)$ and $(0, 0)$, this is not an m-directoid group. Hence $\mathcal{CL} \cap \mathcal{M} \subset \mathcal{CL} \cap \mathcal{TM}$. Consequently $\mathcal{M} \subset \mathcal{TM}$. We can make $\mathbb{Q} * \mathbb{Q}^0$ into an m-directoid group by **1.3**. (Here, as in $\mathbb{Z}$ in the previous example, we are using the linear and discrete orders on $\mathbb{Q}$.) As before, every pair of incomparable elements must have the form $(r,s),(r,u)$, where $s \neq u$. Every $(r+\epsilon, a)$, where $\epsilon > 0$ and a is arbitrary, is an upper bound for such a pair, so the pair has no minimal upper bounds. Hence $\mathbb{Q} * \mathbb{Q}^0$ is in $\mathcal{M}$, and certainly in $\mathcal{CL}$. But it is not in $\mathcal{L}$, so $\mathcal{CL} \cap \mathcal{M} \not\subseteq \mathcal{L}$. Since $\mathcal{CL} \cap \mathcal{L}$ is the class of linearly ordered groups, we have $\mathcal{CL} \cap \mathcal{L} \subset \mathcal{L}$ and $\mathcal{L} \not\subseteq \mathcal{CL} \cap \mathcal{M}$. From the latter we get $\mathcal{L} \| \mathcal{CL} \cap \mathcal{M}$. Since $\mathcal{CL} \cap \mathcal{L} \subset \mathcal{CL} \cap \mathcal{M}$, we have $\mathcal{L} \subset \mathcal{M}$. Since $\mathcal{L} \not\subseteq \mathcal{CL} \cap \mathcal{M}$, so $\mathcal{M} \neq \mathcal{CL} \cap \mathcal{M}$. Since $\mathcal{CL} \cap \mathcal{L}$ is the class of linearly ordered groups, $\mathcal{L}$ is not contained in $\mathcal{CL} \cap \mathcal{TM}$, so neither is $\mathcal{M}$.

Let us note, finally, that the above discussion (though not the diagram) yields the equation

$$\mathcal{R} \cap \mathcal{FF} = \mathcal{L}.$$

In the next section we shall see some examples of varieties of directoid groups. For the present we note

Proposition 1.6. *None of the classes $\mathcal{M}, \mathcal{TM}, \mathcal{CL}, \mathcal{R}, \mathcal{FF}$ is a variety of directoid groups.*

Proof. Jakubík [11] has shown that every directoid group is isomorphic to a directoid subgroup of an m- (whence $t.m.$) directoid group. This accounts for $\mathcal{M}$ and $\mathcal{TM}$. If we take the unique directoid structure on $\mathbb{Z}$ with its standard order and the $l-$group product $\mathbb{Z} \times \mathbb{Z}$, then $\mathbb{Z} \in \mathcal{CL}$ but $\mathbb{Z} \times \mathbb{Z} \notin \mathcal{CL}$, so $\mathcal{CL}$ is not a variety. The Jaffard group $J = \{(m,n) \in \mathbb{Z} \times \mathbb{Z} : m \equiv n \pmod 2\}$ is a directed subgroup of $\mathbb{Z} \times \mathbb{Z}$. We can make J a directoid group (e.g. as in Example 2.10 of [7]) and then by **2.9** of [8] we can make a directoid group G on $\mathbb{Z} \times \mathbb{Z}$ with J as a directoid subgroup. Now $J \notin \mathcal{R}$ as pairs of its elements can have more than one minimal upper bound. However G *is* in $\mathcal{R}$. Hence $\mathcal{R}$ is not a variety. (Note that $\mathcal{R}$ is closed under convex directoid subgroups and by Proposition 2.4 and Proposition 5.3 of Fuchs [6] also closed under direct products and homomorphic images.)

Finally, $\mathcal{FF}$ is not homomorphically closed. This is conveniently demonstrated by means of the example which follows. □

Example 1.7. Let $A = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ and let $(x_1, x_2, x_3) \leq (y_1, y_2, y_3)$ mean that $x_1 = y_1, x_2 = y_2$ and $x_3 = y_3$ or $x_1 \leq y_1, x_2 \leq y_2$ and $x_3 + 2 \leq y_3$ (cf. Example 2.7 of [7]). A little checking will show that we get a set M as in **1.2** of [8] by taking all (a,b,c) such that $(0,0,0) < (a,b,c)$ or $a \geq 0, b \geq 0, c = 1$ or $a \geq 0, b \geq 0, (a,b) \neq (0,0), c \leq 0$ or $a > 0, b < 0$. For $(a,b,c) \in M$ we define

(i) $(a,b,c) \cdot (0,0,0) = (a,b,c)$ if $(a,b,c) \geq (0,0,0)$,

(ii) $(a,b,1) \cdot (0,0,0) = (a,b,3)$ if $a, b \geq 0$,

(iii) $(a,b,c)\cdot(0,0,0)=(a,b,2)$ if $a\geq 0, b\geq 0, (a,b)\neq(0,0), c\leq 0$,
(iv) $(a,b,c)\cdot(0,0,0)=(a+2,0,0\vee c+2)$ if $a>0, b<0$,
and then, as usual, $(-a,-b,-c)\cdot(0,0,0)=(a,b,c)\cdot(0,0,0)-(a,b,c)$. If $(r,s,t)\|(0,0,0)$ we have $(r,s,t)\cdot(0,0,0)=(*,*,t\vee 0+2)$ and RHS is a minimal upper bound of (r,s,t) and $(0,0,0)$. But then if $x,y,z\in A$ and $x,y\leq z\leq x\cdot y$, then $0, y-x\leq z-x\leq x\cdot y-x=0\cdot(y-x)$ so $z-x=x\cdot y-x$ and hence $z=x\cdot y$. Thus z is a minimal upper bound, and A is fork-free.

Now let $B=\mathbb{Z}\times\mathbb{Z}$ with the product order. We make B a directoid group by defining

$$(a,b)\cdot(0,0)=\begin{cases}(a,b) & \text{if } a\geq 0, b\geq 0\\ \text{(a+2,0)} & \text{if } a>0, b<0,\end{cases}$$

and $(-s)\cdot 0=s\cdot 0-s$ in each case. The function $f:A\to B$ with $f(a,b,c)=(a,b)$ for all a,b,c is a surjective directoid group homomorphism. But if $a>0$ and $b<0$ then

$$(a,b),(0,0)\leq(a+1,0)<(a+2,0)=(a,b)\cdot(0,0).$$

We end this section with a further observation about nr multilattice groups. Vaida [16] proved (in a more general setting, without commutativity) that a partially ordered group is an *nr* multilattice group if and only if it has a *generalized Jordan decomposition*, i.e. every element t can be written as $u-v$ for some positive elements u,v for which 0 is a maximal lower bound. In the case of 2-torsion-free abelian groups we can prove one implication by a directoid group argument which produces a generalized Jordan decomposition which mimics that in an $l-$group. We can make an **nr** multilattice group G into a fork-free directoid group (**1.5**) and then $a\circ b$ is always a maximal lower bound of a,b. For every $g\in G$ we have (using Proposition 2.6(ii) of [7]) $g=g+0=g\cdot 0+g\circ 0=g\cdot 0-(-g)\cdot 0$ and $(g\cdot 0)\circ(-g)\cdot 0=0$ by Proposition 2.6(iv) of [7] so 0 is a maximal lower bound of $g\cdot 0$ and $(-g)\cdot 0$.

2. Varieties

There are no proper subvarieties of abelian $l-$groups: free abelian $l-$groups are subdirect products of copies of $\mathbb{Z}$ with its linear order [17] and every non-zero abelian $l-$group clearly contains an isomorphic copy of $\mathbb{Z}$. The situation is considerably more complicated when we come to directoid groups. We shall present a few examples to demonstrate this without attempting a classification. As well, we show the existence of classes with some claim to being viewed as "varieties of directed groups".

We shall deal with the following classes.

$\mathcal{L}$: The class of abelian $l-$groups.

$\mathcal{A}_n$ $(n \in \mathbb{Z}^+, n > 1)$: The class of directoid groups satisfying $(nx \cdot ny) \cdot nz \approx nx \cdot (ny \cdot nz)$.

$\mathcal{B}_n$ $(n \in \mathbb{Z}^+, n > 1)$: The class of directoid groups satisfying $nx \cdot ny \approx n(x \cdot y)$.

$\mathcal{I}_n$ $(n \in \mathbb{Z}, n > 1)$: The class of $n-$*isolated* directoid groups; those satisfying $nx \geq 0 \Rightarrow x \geq 0$.

$\mathcal{D}$: The class of directoid groups satisfying $(mx \cdot nx) \cdot kx \approx mx \cdot (nx \cdot kx)$ for all $m, n, k \in \mathbb{Z}$.

(Note that for $n = 1$, $\mathcal{A}_n$ would be $\mathcal{L}$ and $\mathcal{B}_n$ and $\mathcal{I}_n$ would be the class of all directoid groups.) The directoid groups in $\mathcal{D}$ are called *alternating* by Kopytov and Dimitrov [14].

With the exception of the $\mathcal{I}_n$, all of these classes are obviously varieties. It follows from the next result (which generalizes Proposition 2.6 (vi) of [7]) that the $\mathcal{I}_n$ are also varieties.

Proposition 2.1. *The following conditions are equivalent for a directoid group G and a positive integer n.*

(i) $(n+1)x \geq 0 \Rightarrow x \geq 0$.

(ii) $x \cdot (-nx) \geq 0$ *for all* x.

Proof. (i)⇒(ii). We have $x \cdot (-nx) \geq x$, so $-x \geq -(x \cdot (-nx))$. Hence $x \cdot (-nx) \geq -nx \geq -n(x \cdot (-nx))$, so $(n+1)(x \cdot (-nx)) \geq 0$, whence $x \cdot (-nx) \geq 0$.

(ii) ⇒(i). If $(n+1)x \geq 0$ then $-(n+1)x \leq 0$, so $0 = (-(n+1)x) \cdot 0$. But then $x = x + 0 = x + (-(n+1)x) \cdot 0 = (-nx) \cdot x \geq 0$. □

Corollary 2.2. *For $n > 0$, $\mathcal{I}_{n+1}$ is the variety defined by*

$$(x \cdot (-nx)) \cdot 0 \approx x \cdot (-nx).$$

Theorem 2.3. *For every integer $n > 1$ we have*

$$\mathcal{B}_n \subset \mathcal{I}_n, \quad \mathcal{B}_n \| \mathcal{A}_n, \quad \mathcal{A}_n \| \mathcal{I}_n.$$

Proof. If $a \in A \in \mathcal{B}_n$ and $na \geq 0$, then $na = na \cdot 0 = n(a \cdot 0)$ so $n(a \cdot 0 - a) = 0$. Now $a \cdot 0 \geq a$ so $a \cdot 0 - a \geq 0$. Since positive elements can't have finite order, we have $a \cdot 0 = a$, i.e. $a = a \cdot 0 \geq 0$. Thus A is in $\mathcal{I}_n$, so $\mathcal{B}_n \subseteq \mathcal{I}_n$.

Let m be a positive integer relatively prime to n, ${\mathbb{Z}_m}^0$ the group of integers modulo m with the discrete order, and let $H^{(m)} = \mathbb{Z} * {\mathbb{Z}_m}^0$ (lexicographic product) where $\mathbb{Z}$ has its standard order (cf. **3.3** of [8]). Let $\mathbb{Z}_m = \{\overline{0}, \overline{1}, \ldots, \overline{m-1}\}$, and write $H^{(m)}$ as an internal sum. If for $\overline{a} \in \mathbb{Z}_m, k \in \mathbb{Z}$ we have $n(k + \overline{a}) \geq 0$, i.e. $nk + n\overline{a} \geq 0$, then either $nk = 0$

and $n\overline{a} = 0$ or $nk > 0$. In the former case, $k = 0$ and $\overline{a} = 0$; in the latter, $k > 0$. Thus $k+\overline{a} \geq 0$. It follows that with any directoid operation, $H^{(m)}$ is in $\mathcal{I}_n$. Let $rn + sm = 1$ $(r, s \in \mathbb{Z})$ so that $n\overline{r} = \overline{nr} = \overline{1}$. We can make $H^{(m)}$ a directoid group with $\overline{1} \cdot 0 = 1$. Then $n\overline{r} \cdot n\overline{2r} = \overline{1} \cdot \overline{2nr} = \overline{1} \cdot \overline{2} = \overline{1} + 0 \cdot \overline{1} = \overline{1} + 1$ and $n(\overline{r} \cdot \overline{2r}) = n(\overline{r} + 0 \cdot \overline{r}) = n\overline{r} + n(0 \cdot \overline{r}) = \overline{1} + n(0 \cdot \overline{r})$. Now $\overline{r}$ has finite order and (as $n\overline{r} = \overline{1}$) it is non-zero, so that $\overline{r}\|0$ whence $0 \cdot \overline{r} > 0$. Thus $0 \cdot \overline{r} = k + \overline{a}$ for some $\overline{a} \in \mathbb{Z}_m, k > 0$. But then $n(\overline{r} \cdot \overline{2r}) = \overline{1} + n(0 \cdot \overline{r}) = nk + (\overline{1} + n\overline{a}) \neq \overline{1} + 1 = n\overline{r} \cdot n\overline{2r}$, so $H^{(m)} \notin \mathcal{B}_n$. We have proved that $\mathcal{B}_n \subset \mathcal{I}_n$. We next examine $\mathcal{A}_n$.

If $m \neq 3$, so that $\overline{1} \neq -2\overline{1}$, we can choose our directoid operation in $H^{(m)}$ so that $\overline{1} \cdot 0 = 1$ and $\overline{2} \cdot 0 = 2$. Then $(n\overline{r} \cdot \overline{2r}) \cdot 0 = (\overline{1} \cdot \overline{2}) \cdot 0 = (\overline{1} + \overline{1} \cdot 0) \cdot 0 = (\overline{1} + 1) \cdot 0 = \overline{1} + 1$ (as $1 > 0$), while $n\overline{r} \cdot (n\overline{2r} \cdot 0) = \overline{1} \cdot (\overline{2} \cdot 0) = \overline{1} \cdot 2 = 2$, so $H^{(m)} \notin \mathcal{A}_m$. (For completeness we note that if $m = 3$ then $2\overline{1} = -\overline{1}$ so we can make $\overline{1} \cdot 0 = 1$ and $\overline{2} \cdot 0 = \overline{1} \cdot 0 - \overline{1} = 1 - \overline{1}$ and then $(n\overline{r} \cdot n\overline{2r}) \cdot 0 = (\overline{1} \cdot \overline{2}) \cdot 0 = (\overline{1} + 0 \cdot \overline{1}) \cdot 0 = (\overline{1} + 1) \cdot 0 = \overline{1} + 1$, while $n\overline{r} \cdot (n\overline{2r} \cdot 0) = \overline{1} \cdot (\overline{2} \cdot 0) = \overline{1} \cdot (1 - \overline{1}) = 1 - \overline{1}$ so $H^{(3)}$(with this directoid operation) $\notin \mathcal{A}_n$.) Thus $H^{(m)}$(with a suitable directoid operation) $\notin \mathcal{A}_n$ whenever m and n are relatively prime, so that under this condition $H^{(m)} \in \mathcal{I}_n \setminus \mathcal{A}_n$.

As in **3.1** of [8], let $\mathbb{Z}^{(n)} = \mathbb{Z}$ with $u \preceq v$ if and only if $v = u+nc+(n+1)d$ for some $c, d \in \mathbb{Z}^+ \cup \{0\}$. Now if $g, h \in \mathbb{Z}$ we can assume $g \leq h$ and then $nh-ng = n(h-g)$ so $ng \preceq nh$. It follows that $\preceq$ is linear on $n\mathbb{Z}$ and for any directoid operation $\cdot$ we have $na{\cdot}nb = na{\vee}nb$ for all $a, b \in \mathbb{Z}$. Thus for all $a, b, c \in \mathbb{Z}$ we have $(na \cdot nb) \cdot nc = (na \vee nb) \cdot nc = n(a \vee b) \cdot nc = n(a \vee b) \vee nc = (na \vee nb) \vee nc = na \vee (nb \vee nc) = na \vee (n(b \vee c)) = na \cdot (n(b \vee c)) = na \cdot (nb \vee nc) = na \cdot (nb \cdot nc)$, so $\mathbb{Z}^{(n)} \in \mathcal{A}_n$ (for any directoid operation). On the other hand in $\mathbb{Z}^{(n)}$ we have $n1 \succeq 0$ but $1 \not\succeq 0$, so $\mathbb{Z}^{(n)} \notin \mathcal{I}_n$. We thus have $\mathbb{Z}^{(n)} \in \mathcal{A}_n \setminus \mathcal{I}_n$, and this with the result above gives us $\mathcal{A}_n\|\mathcal{I}_n$ for all n.

In $\mathbb{Z}^{(n)}$ (as $n > 1$) $2 - 1$ can't be positive, so $1\|2$, whence $1 \cdot 2 \neq 1, 2$ (for any operation $\cdot$) and hence $n(1 \cdot 2) \neq n, 2n$. But $n \preceq 2n$ so $n1 \cdot n2 = 2n$. Since $n(1 \cdot 2) \neq n1 \cdot n2$, $\mathbb{Z}^{(n)}$ is not in $\mathcal{B}_n$ so it is in $\mathcal{A}_n \setminus \mathcal{B}_n$. Let $\mathbb{Z} * \mathbb{Z}^0$ be given the directoid group structure used for the discussion of the previous Hasse diagram of classes. If $(k, a) \geq (0, 0)$ then for $n \geq 2$, $n(k, a) = (nk, na) \geq (0, 0)$ while if $(k, a) \leq (0, 0)$, then $n(k, a) = (nk, na) \leq (0, 0)$, as in the first case $k > 0$ or $k = 0$ and $a = 0$ and in the second $k < 0$ or $k = 0$ and $a = 0$. If $a \in \mathbb{Z}^+$ then $n(0, a) \cdot (0, 0) = (0, na) \cdot (0, 0) = (na, 0) = n((a, 0) \cdot (0, 0))$ and $n(0, -a) \cdot (0, 0) = (0, -na) \cdot (0, 0) = (na, -na) = n(a, -a)$
$= n((0, -a) \cdot (0, 0))$. This proves that in $\mathbb{Z} * \mathbb{Z}^0$ we have $nu \cdot 0 = n(u \cdot 0)$ for all u. If now $u, v \in \mathbb{Z} * \mathbb{Z}^0$ then $nu \cdot nv = (nu - nv) \cdot 0 + nv = (n(u - v)) \cdot 0 + nv = n((u - v) \cdot 0) + nv = n((u - v) \cdot 0 + v) = n(u \cdot v)$.

Thus $\mathbb{Z} * \mathbb{Z}^0 \in \mathcal{B}_n$ for every $n \geq 2$. But in the same group we have $((0,n)\cdot(0,2n))\cdot(0,3n) = ((0,-n)\cdot(0,0)+(0,2n))\cdot(0,3n) = ((n,-n)+(0,2n))\cdot(0,3n) = (n,n)\cdot(0,3n) = (n,n)$, while $(0,n)\cdot((0,2n)\cdot(0,3n)) = (0,n)\cdot((0,-n)\cdot(0,0)+(0,3n)) = (0,n)\cdot((n,-n)+(0,3n)) = (0,n)\cdot(n,2n) = (n,2n)$, so $\mathbb{Z} * \mathbb{Z}^0$ does not satisfy $(nx\cdot ny)\cdot nz \approx nx\cdot(ny\cdot nz)$ and so is in $\mathcal{B}_n \setminus \mathcal{A}_n$. Thus $\mathcal{B}_n \| \mathcal{A}_n$. □

Theorem 2.4. *For any integer $n > 1$,*
(i) the groups in $\mathcal{B}_n$ satisfy $nx = 0 \Rightarrow x = 0$ and
(ii) $\mathcal{B}_n \cap \mathcal{A}_n = \mathcal{L} \subset \mathcal{I}_n \cap \mathcal{A}_n$.

Proof. (i) Non-zero elements of finite order are incomparable with 0. If $nx = 0$ then $n(x\cdot 0) = nx\cdot n0 = 0\cdot 0 = 0$, so (as $x\cdot 0 \geq 0$) $x\cdot 0 = 0$. But then $x \leq 0$, whence $x = 0$.

(ii) For $a,b,c \in G \in \mathcal{A}_n \cap \mathcal{B}_n$ we have $n((a\cdot b)\cdot c) = n(a\cdot b)\cdot nc = (na\cdot nb)\cdot nc = na\cdot(nb\cdot nc) = na\cdot n(b\cdot c) = n(a\cdot b\cdot c)$ so $n((a\cdot b)\cdot c - a\cdot(b\cdot c)) = 0$, whence $(a\cdot b)\cdot c = a\cdot(b\cdot c)$.

Now we consider the group $J_n = \{(a,b) \in \mathbb{Z}\times\mathbb{Z} : a \equiv b(mod n)\}$ of **3.4** in [8] with a directoid group structure such that $a\cdot b$ is the least upper bound of a and b whenever the latter exists in J_n, i.e. we are dealing with a *t.m.* directoid group. If $a,b,c,d \in \mathbb{Z}$ then $(na,nb),(nc,nd) \in J_n$ and (calculating in the $l-$group $\mathbb{Z}\times\mathbb{Z}$) $(na,nb)\vee(nc,nd) = (na\vee nc, nb\vee nd) = (n(a\vee c), n(b\vee d))$. This last is in J_n and is clearly the least upper bound *there*. Thus we have, in J_n, $(na,nb)\cdot(nc,nd) = (n(a\vee c), n(b\vee d) = n(a\vee c, b\vee d)$. In particular, $nr\cdot ns = n(r\cdot s)$ for all $r,s \in J_n$. If now $r,s,t \in J_n$, then $(nr\cdot ns)\cdot nt = n(r\vee s)\cdot nt = n((r\vee s)\vee t)$ and $nr\cdot(ns\cdot nt) = nr\cdot n(s\vee t) = n(r\vee(s\vee t)$. This proves that $J_n \in \mathcal{A}_n$. As J_n is a subgroup of an $l-$group, its order is *isolated*, (i.e. it satisfies the condition $\forall k \in \mathbb{Z}^+, kx \geq 0 \Rightarrow x \geq 0$), so certainly $J_n \in \mathcal{I}_n$. (See also **2.7** below.) But J_n is not a lattice as, e.g., $(n+1,1),(2,n+2) \in J_n$, but they have $(n+2,n+2)$ and $(2n+1,n+1)$ as mininal upper bounds. □

The variety $\mathcal{D}$ consists of those directoid groups for which every one-generator directoid subgroup is an $l-$group. This is asserted without proof in [14]. To prove it we make use of the fact that in any directoid group $G \in \mathcal{D}$, if $a \in G, n,k \in \mathbb{Z}^+$ and $n > k$, then $na\cdot 0 \geq ka$ and $n(a\cdot 0) = (na)\cdot 0$. It is shown in [5], p.75, that these are true in $l-$groups but only the associativity of $\cdot$ on integer multiples of x is needed. Now $\mathbb{Z}\times\mathbb{Z}$ with its standard product $l-$group structure is a free (abelian) $l-$group on $(-1,1)$ [2] and the cited results enable us to define a directoid group homomorphism $\mathbb{Z}\times\mathbb{Z} \to G$ under which $(-1,1) \mapsto a$. This being so for each a, every one-generator directoid subgroup of G is an $l-$group. The converse is obvious.

As noted above, if $a \in G \in \mathcal{D}$, then $n(a \cdot 0) = (na) \cdot 0$ for each $n \in \mathbb{Z}^+$, and we saw in the proof of **2.3** that this property implies $G \in \mathcal{B}_n$. We also saw that $\mathbb{Z} * \mathbb{Z}^0$, with the operation used earlier, does have the above property and hence is in $\mathcal{B}_n$. However, we also have $((0,1) \cdot (0,2)) \cdot (0,3) = ((0,1) + (0,0) \cdot (0,1)) \cdot (0,3) = ((0,1) + (1,0)) \cdot (0,3) = (1,1) \cdot (0,3) = (1,1)$, while $(0,1) \cdot ((0,2) \cdot (0,3)) = (0,1) \cdot ((0,2) + (0,0) \cdot (0,1)) = (0,1) \cdot ((0,2) + (1,0)) = (0,1) \cdot (1,2) = (1,2)$, so $\mathbb{Z} * \mathbb{Z}^0 \notin \mathcal{D}$. Hence $\mathcal{D} \subset \mathcal{B}_n$.

It was shown in [14] (5.4 Corollary) that if $A, B \in \mathcal{D}$, then the lexicographic product $A * B$, with $(a,b) \cdot (a',b') = (a \cdot a', b \cdot b')$ if $(a,b) \| (a',b')$, is also in $\mathcal{D}$. Hence if we take the natural orders, $(\mathbb{Q} \times \mathbb{Q}) * \mathbb{Q}$ is in $\mathcal{D}$. It is not in $\mathcal{L}$: it is noted in [14] that $(\mathbb{Z} \times \mathbb{Z}) * \mathbb{Z} \in \mathcal{D} \setminus \mathcal{L}$ but we are going to use the group based on $\mathbb{Q}$ for another purpose as well. For completeness we give an example to show that $(\mathbb{Q} \times \mathbb{Q}) * \mathbb{Q} \notin \mathcal{L}$. We have

$$(1,2,5) \cdot ((1,2,-1) \cdot (3,-1,-6)) = (1,2,5) \cdot (3,2,-1) = (3,2,-1)$$

and

$$((1,2,5) \cdot (1,2,-1)) \cdot (3,-1,-6) = (1,2,5) \cdot (3,-1,-6) = (3,2,5).$$

But also $(\mathbb{Q} \times \mathbb{Q}) * \mathbb{Q} \notin \mathcal{A}_n$ for $n \geq 2$, since for $a, b, c \in \mathbb{Q}$ there exist $a', b', c' \in \mathbb{Z}$ with $na' = a, nb' = b$ and $nc' = c$, so that if the group were in $\mathcal{A}_n$ we'd have $(a \cdot b) \cdot c = (na' \cdot nb') \cdot nc' = na' \cdot (nb' \cdot nc') = a \cdot (b \cdot c)$. This gives us (taking account of **2.3**)

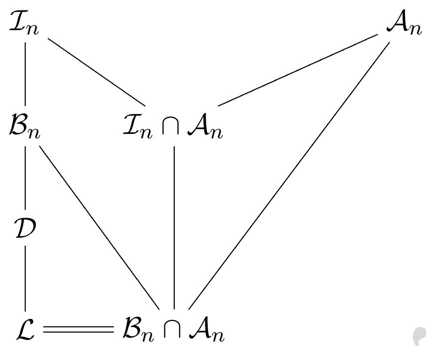
Theorem 2.5. *For each $n \geq 2$ we have*

$$\mathcal{L} \subset \mathcal{D} \subset \mathcal{B}_n; \mathcal{D} \| \mathcal{A}_n.$$

If $\mathcal{V}$ is a variety of directoid groups, $0 \neq a \in G \in \mathcal{V}$, then the cyclic group generated by $a \cdot 0$ is a directed group isomorphic to $\mathbb{Z}$ with its standard order. As noted above, this generates $\mathcal{L}$. We therefore have

Theorem 2.6. *$\mathcal{L}$ is the unique atom in the lattice of varieties of directoid groups.*

By **2.4** (proof) $J_n \in \mathcal{A}_n \cap \mathcal{I}_n$ and $J_n \notin \mathcal{B}_n$, so $J_n \notin \mathcal{D}$. Hence $\mathcal{D} \| \mathcal{I}_n \cap \mathcal{A}_n$. Thus the varieties we have discussed are related as shown in the following diagram, all inclusions being proper and all pairs of varieties for which no connection is indicated being incomparable.



A variety $\mathcal{V}$ of directoid groups has some claim to be regarded as a "variety of directed abelian groups" (2-torsion-free ones) if it satisfies

(e): If $A \in \mathcal{V}$ and B is a directoid group order isomorphic to A, then $B \in \mathcal{V}$.

At this point we should mention the concept of an $e-variety$ (see, e.g., [9],[13]). A regular semigroup can be regarded as a semigroup with an additional unary operation $'$ such that $a = aa'a$ and $a'aa' = a'$ for all a. The operation $'$ has to be *chosen* in the way we choose a directoid operation on a directed group. An $e-variety$ of regular semigroups is then a class of regular semigroups closed under homomorphic images, products and *regular* subsemigroups. Being an e-variety is equivalent to satisfying the regular semigroup analogue of our condition (e). We'll now look at the corresponding notions for directoid groups.

Let $\mathcal{I} = \bigcap_{n>1} \mathcal{I}_n = \{A : a \in A, n \in \mathbb{Z}^+ \& na \geq 0 \Rightarrow a \geq 0\}$. The directed groups in $\mathcal{I}$ are said to be *isolated* or to have *isolated order.*

Theorem 2.7. *The varieties $\mathcal{I}, \mathcal{I}_n, n = 2, 3, 4, \ldots$ satisfy condition (e) and $\mathcal{I}$ is the smallest variety of directoid groups which does so.*

Proof. The first assertion is clear. Let $\mathcal{W}$ be a variety satisfying condition (e). Then $\mathcal{L} \subset \mathcal{W}$ so every directoid group whose underlying directed group is an $l-$group must be in $\mathcal{W}$ also. Let A be in $\mathcal{I}$. Then A is an ordered subgroup of an $l-$group L. (This is deducible from a result of Lorenzen [15]; a direct proof was given by Dieudonné [4]. See also [1], §4.5.) By **2.9** of [8] the directoid group structure on A extends to one on L(which in general is a perturbation of the original $l-$group structure). Since the new directoid group on L is in $\mathcal{W}$, so is A. Thus $\mathcal{I} \subseteq \mathcal{W}$. □

Theorem 2.8. *Let $\mathcal{H}$ be a non-empty class of directed abelian groups closed under homomorphic images (for order homomorphisms), products and directed subgroups. Then $\mathcal{H} = \{0\}$ or $\mathcal{H}$ is the class of all directed abelian groups.*

Proof. We first note that if f is an order homomorphism then for $a, b \leq c$ we have $f(a), f(b) \leq f(c)$ so the class of all directed abelian groups is homomorphically closed. Also, if A_λ is directed for each $\lambda \in \Lambda$ then if $(a_\lambda)_\Lambda, (b_\lambda)_\Lambda \in \prod A_\lambda$ then for each λ there is a $c_\lambda \in A_\lambda$ such that $a_\lambda, b_\lambda \leq c_\lambda$, whence $(a_\lambda)_\Lambda, (b_\lambda)_\Lambda \leq (c_\lambda)_\Lambda$. Thus the class of all directed abelian groups satisfies the three properties we are considering.

Now let $\mathcal{H} \neq 0$ satisfy the three conditions. Let $\mathcal{V}(\mathcal{H})$ denote the class of directoid groups whose underlying ordered groups are in $\mathcal{H}$. If $I \lhd G \in \mathcal{V}(\mathcal{H})$ then as a directed group G/I is in $\mathcal{H}$.Thus $G/I \in \mathcal{V}(\mathcal{H})$. If $\{G_\lambda : \lambda \in \Lambda\} \subseteq \mathcal{V}(\mathcal{H})$ then in $\prod G_\lambda$ (the directoid group product) the following are equivalent:

$$(x_\lambda)_\Lambda \leq (y_\lambda)_\Lambda; \quad (x_\lambda)_\Lambda (y_\lambda)_\Lambda = (y_\lambda)_\Lambda; \quad (x_\lambda y_\lambda)_\Lambda = (y_\lambda)_\Lambda;$$

$$x_\lambda y_\lambda = y_\lambda \forall \lambda; \quad x_\lambda \leq y_\lambda \forall \lambda.$$

Thus the order is the product order and so $\prod G_\lambda$ as a directed group is in $\mathcal{H}$, whence $\mathcal{V}(\mathcal{H})$ is closed under products. If $G \in \mathcal{V}(\mathcal{H})$ and H is a directoid subgroup of G, then H is a directed subgroup of the directed group $G \in \mathcal{H}$ so $H \in \mathcal{H}$, i.e. $H \in \mathcal{V}(\mathcal{H})$. Hence $\mathcal{V}(\mathcal{H})$ is a variety of directoid groups and clearly it satisfies condition (e) so by **2.7** $\mathcal{I} \subseteq \mathcal{V}(\mathcal{H})$.

Now let D be any directed abelian group, *not necessarily* $2-$*torsion-free.* Let $f : F \to D$ be a surjective *group* homomorphism for a suitably large free abelian group F. If we give F the discrete order, f becomes an *ordered* group homomorphism. Now the lexicographic product $\mathbb{Z} * F$ ($(n, u) \geq (m, v)$ if and only if $n > m$ or $(n, u) = (m, v)$) is directed and is in $\mathcal{I}$ with any directoid operation. Hence the directed group $\mathbb{Z} * F$ is in $\mathcal{H}$. Now take the lexicographic product $\mathbb{Z} * D$ ($(m, d) \geq (n, e)$ if and only if $n > m$ or $n = m$ and $d \geq e$). This is a directed group and the function

$$f^* : \mathbb{Z} * F \to \mathbb{Z} * D; f^*(n, u) = (n, f(u))$$

is a surjective ordered group homomorphism so that $\mathcal{H}$ contains $\mathbb{Z} * D$ and hence also D. □

Thus (albeit trivially) the 2-torsion-free members of the analogues of e-varieties of directed abelian groups form varieties of directoid groups satisfying (e), but unlike the case of regular semigroups, the converse is false, as can be seen from **2.7**. The key to this is **2.11** of [8].

Note that if $\mathcal{V}$ is a variety of directoid groups, we do not necessarily get another variety (in particular, a variety satisfying condition (e)) by taking all directoid groups order isomorphic to members of $\mathcal{V}$.

Theorem 2.9. *Let $\tilde{\mathcal{L}}$ denote the class of directoid groups whose underlying directed groups are $l-$groups. Then $\tilde{\mathcal{L}}$ is homomorphically closed, closed under direct products and convex directoid subgroups; it is not closed under directoid subgroups.*

Proof. If $A \in \tilde{\mathcal{L}}$, let $f : A \to B$ be a surjective homomorphism of directed groups. Then by Proposition 2.9 of [7], $Ker(f)$ is convex and if $x, y \in Ker(f)$,i.e. $x - 0, y - 0 \in Ker(f)$ then $x \cdot y = x \cdot y - 0 \cdot 0 \in Ker(f)$. Since $x, y \leq x \vee y \leq x \cdot y$ we have $x \vee y \in Ker(f)$. Thus $Ker(f)$ is an $l-$ideal. Hence B is an $l-$ group with respect to the quotient order defined by f. But by **2.10** of [8] this is the order of the directoid group B, so $B \in \tilde{\mathcal{L}}$. Product closure is clear. If $D \in \tilde{\mathcal{L}}$ and C is a convex directoid subgroup of D then for any $c_1, c_2 \in C$ there exists a $d \in C$ with $c_1, c_2 \leq d$ and hence $c_1, c_2 \leq c_1 \vee c_2 \leq d$, so $c_1 \vee c_2 \in C$ (where $c_1 \vee c_2$ is calculated in D). Clearly then $c_1 \vee c_2$ is a least upper bound for c_1 and c_2 in C, so $C \in \tilde{\mathcal{L}}$. For the final assertion, take the Jaffard group as a subgroup of $\mathbb{Z} \times \mathbb{Z}$, observe that $\mathbb{Z} \times \mathbb{Z}$ is in $\tilde{\mathcal{L}}$ and argue as in the proof of **2.7**. $\square$

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