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All difference family structures arise from groups

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Abstract. Planar nearrings have been used to define classes of 2-designs since Ferrero's work in 1970. These 2-designs are a class of difference families. Recent work from Pianta has generalised Ferrero and Clay's work with planar nearrings to investigate planar nearrings with nonassociative additive structure. Thus we are led to the question of nonassociative difference families.

Difference families are traditionally built using groups as their basis. This paper looks at what sort of generalized difference family constructions could be made, using the standard basis of translation and difference.

We determine minimal axioms for a difference family structure to give a 2-design. Using these minimal axioms we show that we obtain quasigroups. These quasigroups are shown to be isotopic to groups and the derived 2-designs from the nonassociative difference family are identical to the 2-designs from the isotopic groups. Thus all difference families arise from groups.

This result will be of interest to those using nonstandard algebras as bases for defining 2-designs.

1. Introduction

An interesting application of nearring theory has been in the generation of designs. The theory of planar nearrings and Ferrero Pairs was introduced by G. Ferrero [7] and heavily investigated by a group including J. Clay and W–F. Ke [2, 6, 10, 11]. The core of this work is the use of orbits of fixed-point-free automorphisms on groups in order to define a difference family

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structure. Recently S. Pianta [14] has investigated the generalization of planar nearrings with nonassociative addition. A natural question then follows: can the design ideas from Ferrero and Clay be generalized to Pianta's expanded class? Rather than investigate this from the nearring side, we decided to see how to generalize difference family structures to nonassociative additions, as this should prove to be useful for other investigators looking at generalizing other algebraic structures from which difference families can be defined.

Difference families are used to construct many 2–designs. The class of 2–designs that arise from difference families can be easily characterized [1, 5] as designs with a group of automorphisms that is strictly one-transitive on points. There have been many uses of them in order to find 2–designs, in addition to the references above we note the work of Furino[8] and Buratti [3, 4]. Difference families are based upon groups, usually additively written. The essential operations are the difference operation (subtraction in the group) and translation (the mapping obtained by adding a fixed element). The main requirement for a difference family is that the difference between two arbitrary elements must remain invariant under translation. In this paper we look at minimal structures of translation and difference which allow a generalized form of difference family construction. We will see that although we can generalize considerably, all such structures arise in a clear fashion from groups.

Quasigroups and loops are generalizations of groups that do not require the operation to be associative. A *quasigroup* is a (2)–algebra $(S,+)$ such that for all $a,b \in S$ all equations

$$a+x=b \qquad y+a=b$$

have unique solutions for x and y. The Cayley tables of such algebras form Latin squares. There are many special cases of such algebras. In particular a *loop* is a quasigroup with a two–sided identity $e \in S$, i.e. $a+e=e+a=a \; \forall a \in S$. A group is an associative loop. See for instance [13] for details about loops and quasigroups.

Quasigroups can be obtained by twisting a group in some way. A simple example is to take an additive group $(G,+)$ and to use the subtraction operation to obtain a quasigroup $(G,-)$ that is in general not associative. For another example, given a field K, let $k \in K$, $k \neq 0,1$ be arbitrary but fixed. Define $a * b = ka + (1-k)b$. Then $(K,*)$ is a quasigroup, in general nonassociative.

There exists a more general form of equivalence between quasigroups, or more general algebras. Two groupoids $(S,+)$ and $(T,*)$ are *isotopic* if there exist bijections $\alpha,\beta,\gamma : S \to T$ such that for all $a,b \in S$ $\alpha(a+b) = \beta(a) * \gamma(b)$. An isomorphism is an isotopism with all bijections identical.

The isotope of a quasigroup is a quasigroup, so the class of quasigroups is closed under isotopism. We note that the class of groups is not closed under isotopism. In fact many proper quasigroups of interest are isotopic to groups, for instance those examples above. An $(2,2)$–algebra $(N,+,*)$ such that $(N,+)$ and $(N,*)$ are quasigroups is a *biquasigroup*.

In the following we will first look at difference families and determine what properties are needed in order to be useful for such a construction. We demonstrate that such a structure is equivalent to a class of biquasigroups with constants. We then look at this class and see that they are all simply obtained from groups. The difference family structures come directly from a group. Our main results are Proposition 16 which gives an explicit construction of all such biquasigroups and Proposition 17 demonstrating that the difference families are identical.

In general we are interested in finite structures. However, the results here also apply for infinite structures. Note also that many of these results are folklore in the quasigroup community. However, because this article is aimed an combinatorialists investigating general algebraic structures for designs, we include proofs using elementary methods.

2. Difference families

A *(set) 2–design* is a pair $(V,\mathcal{B})$, where V is a set (of *points*), $\mathcal{B}$ is a set of subsets of V all of size k (called *blocks*) and for all pairs $a,b\in V$, $a\neq b$, $|\{B\in\mathcal{B} : a,b\in B\}|=\lambda$ for some constants k and λ. The number 2 in the name refers to the pairs of elements a,b. There are many variations on this definition, see e.g. [1, 5] for details.

Given a (2)–algebra $(N,+)$ and a set of subsets $\mathcal{B}$ of N, define the *development* of $\mathcal{B}$ in N, $dev(\mathcal{B})$ to be the collection $\{B+n : B\in\mathcal{B},\ n\in N\}$, possibly containing duplicates. The set development is the collection with no duplicates.

Given a group $(N,+)$, not necessarily abelian, and a set $\mathcal{B}=\{B_i : i=1,\dots,s\}$ of subsets of N, called *base blocks*, such that

- all B_i have the same size
- for all $B,C\in\mathcal{B}$, $n\in N$, $B+n=C \Leftrightarrow B=C$ and $n=0$
- there exists some λ such that for all nonzero $d\in N$, $|\{(B,a,b) : B\in\mathcal{B},\ a,b\in B,\ a-b=d\}|=\lambda$

Then $\mathcal{B}$ forms a *difference family* (DF) in the group $(N,+)$.

Theorem 1 (See e.g. [5, 1]). *Let $\mathcal{B}$ be a difference family on a group $(N,+)$. Then $(N, dev\mathcal{B})$ is a set 2-design.*

In the proof of this result, we can see that the requirement that $(N, +)$ be a group is too strong. We use only the translation and difference operations. Thus it would seem that this construction can be generalized to be based upon other structures. The following result does this.

Theorem 2. *Let N be a set with binary operation $-$ (difference) and unary operations $t_i \in T$ (translations) and $\mathcal{B}$ a set of subsets of N such that*

- *for all $a, b \in N$ there is a unique t_i such that $t_i a = b$.*
- *for all $a, b \in N$, the equation $a - x = b$ has a unique solution.*
- *$a - b = t_i a - t_i b$ for all $a, b \in N$, for all t_i.*
- *there exists some λ such that for all $\alpha, \beta \in N$, $\alpha \neq \beta$, $|\Delta(\alpha - \beta)| = \lambda$ where $\Delta(d) = \{(B, a, b) : B \in \mathcal{B}, a, b \in B, a \neq b, a - b = d\}$.*
- *there exists some integer k such that $|t_i B| = k$ for all t_i for all $B \in \mathcal{B}$.*
- *$t_i B = t_j C$ for $B, C \in \mathcal{B}$ implies $i = j$ and $B = C$.*

Then $(N, dev\mathcal{B})$ with $dev\mathcal{B} = \{t_i B : B \in \mathcal{B}\}$ is a set 2-design.

Proof. All the blocks in $dev\mathcal{B}$ have size k by construction. They are all distinct by the final requirement. We need only show that the number of blocks on a pair of points is constant.

Let $a \neq b \in N$. We show that $|\Delta(a - b)| = |\{t_i B : t_i \in T, B \in \mathcal{B}, a, b \in t_i B\}|$. There are exactly λ triples (B, α, β) in $\Delta(a-b)$ such that $\alpha, \beta \in B$, $\alpha - \beta = a - b$. For each (B, α, β) in $\Delta(a - b)$ there is a unique t_i such that $t_i \alpha = a$. We know

$$a - b = \alpha - \beta = t_i \alpha - t_i \beta = a - t_i \beta$$

so by the unique solution property of difference, $b = t_i \beta$. Thus $a, b \in t_i B$, so we have a mapping Θ from $\Delta(a - b)$ into $\{t_i B : t_i \in T, B \in \mathcal{B}, a, b \in t_i B\}$. This map Θ is injective by the final condition.

We now show that Θ is surjective. Let $a, b \in t_i B$. Then there exist some $\alpha, \beta \in B$ such that $a = t_i \alpha$, $b = t_i \beta$,

$$\alpha - \beta = t_i \alpha - t_i \beta = a - b$$

so $(B, \alpha, \beta) \in \Delta(a-b)$, and $t_i B$ is in the image of Θ. Thus Θ is a bijection and we are done. □

We now have a generalized form of difference family. In the next sections we will investigate the algebraic properties underlying this result.

3. Algebraic properties

Let us investigate the algebraic properties of the results above. For this section, let N, t_i and $-$ be as defined in Theorem 2 above.

Definition 3. *Fix $a_0 \in N$. For all $b \in N$, let $t_b \in T$ such that $t_b a_0 = b$. This is unique. Define $x + y := t_y x$.*

Note that the first condition of Theorem 2 above then states that the equation $a + x = b$ has a unique solution.

Lemma 4. *There exists a unique $o \in N$ that is a right additive zero on N.*

Proof. Fix $a \in N$. By the unique solution property of translations, there is some $o \in T$ such that $a + o = a$. Then for all $b \in N$, $a - b = (a + o) - (b + o) = a - (b + o)$, so by the unique solution property of differences, $b + o = b$ and o is a right additive zero on N. If some $p \in N$ were also a right additive zero, then $a + o = a + p = a$ so by the unique solution property $o = p$ and we see that o is unique. □

Theorem 5. *$(N, -)$ and $(N, +)$ are both quasigroups.*

Proof. We know $a - x = b$ has a unique solution. Suppose $x - a = b$ has two solutions, $x_1 \neq x_2$. There is some y such that $x_1 + y = x_2$. Then

$$x_1 - a = (x_1 + y) - (a + y) = x_2 - (a + y) = b = x_2 - a$$

Then $x_2 - x = b$ has solutions a and $a + y$ for x. Thus $a + y = a$ so $y = o$, $x_1 = x_2$ and $x - a = b$ has at most one solution.

Now we have to show that at least one solution to the equation $x - a = b$ exists. Let $c \in N$ be arbitrary but fixed, then $c - x = b$ has a unique solution x_1. By the above argument the equation $x - x_1 = b$ has the unique solution c. We also know that the equation $x_1 + x = a$ has a unique solution x_2. Then

$$b = c - x_1 = (c + x_2) - (x_1 + x_2) = (c + x_2) - a$$

shows that $c + x_2$ is a solution to $x - a = b$, so $(N, -)$ is a quasigroup.

We now turn our attention to the addition operation. As noted above, $a + x = b$ has a unique solution. Suppose $x + a = b$ has two solutions x_1, x_2. Then $x_1 - x_2 = (x_1 + a) - (x_2 + a) = b - b$. There is some unique k such that $b + k = x_1$. Then $x_1 - x_2 = b - b = (b + k) - (b + k) = x_1 - x_1$ so $x_1 = x_2$ by quasigroup property of $(N, -)$, so $x + a = b$ has at most one solution.

Let d be arbitrary but fixed. Then $x - d = b - (d + a)$ has a unique solution x_1 by the quasigroup property of $(N, -)$. Then

$$x_1 - d = (x_1 + a) - (d + a) = b - (d + a)$$

so due to the unique solution property of $x-(d+a) = x_1 - d$ we know that $x_1 + a = b$, so x_1 is a solution to $x + a = b$ and $(N, +)$ is a quasigroup. □

We have shown that the structure used in Theorem 2 can be seen as a set with two operations that form quasigroups. We formalize this, as it is clear that from such a pair of quasigroups we can form the translations used in Theorem 2.

Definition 6. *A* difference family biquasigroup (DFBQ) $(N, +, -)$ *is a* $(2, 2)$*–algebra that is a biquasigroup such that the identity* $a - b = (a + c) - (b + c)$ *is satisfied.*

Lemma 7. *There is a constant* $e \in N$ *such that* $e = a - a$ *for all* a.

Proof. Fix some $a \in N$. Define $e := a - a$. For all $b \in N$, there exists some c such that $a + c = b$. Thus $b - b = (a + c) - (a + c) = a - a = e$ and the second statement is proved. □

We may write a DFBQ as $(N, +, -, o, e)$ where o is the right additive identity and $e = a - a$ for all a.

4. In general

In this section we examine the structure of a general DFBQ. We will use these results in the next section to demonstrate that a general DFBQ is isotopic to a group and that the resulting designs are identical.

Definition 8. *A collection* $\mathcal{B}$ *and a DFBQ* $(N, +, -)$ *such that:*

- *there exists an integer* k *such that* $|B| = k$ *for all* $B \in \mathcal{B}$
- *there exists some* λ *such that for all* $\alpha, \beta \in N$, $\alpha \neq \beta$, $|\Delta(\alpha-\beta)| = \lambda$ *where* $\Delta(d) = \{(B, a, b) : B \in \mathcal{B},\ a, b \in B,\ a \neq b,\ a - b = d\}$.
- $B + b = C + c$ *for* $B, C \in \mathcal{B}$, $b, c \in N$ *implies* $B = C$ *and* $b = c$.

is called a quasigroup difference family (QDF)

It is clear that such a QDF will give a 2-design using the same methods as Theorem 2.

Proposition 9. *Let $(N,+,-,o,e)$ be a DFBQ. Let $\bar{e}$ be such that $e+\bar{e}=o$, then define $\phi: x \mapsto x+\bar{e}$ and $\alpha: x \mapsto x-e$. Define the operations*

$$a \oplus b = \phi^{-1}(\phi a + \phi b) \quad (1)$$
$$a \ominus b = \alpha^{-1}(a-b) \quad (2)$$

Then $(N,\oplus,\ominus,e,e)$ is a DFBQ with $a \ominus e = a$ for all $a \in N$.

Proof. $(N,\oplus)$ and $(N,\ominus)$ are quasigroups by isotopism. Note that $\phi a - \phi b = a-b$ by the DFBQ property, so $\phi^{-1}a - \phi^{-1}b = a-b$. Then

$$\begin{aligned}(a\oplus c)\ominus(b\oplus c) &= \alpha^{-1}((a\oplus c)-(b\oplus c)) && (3)\\ &= \alpha^{-1}((\phi a+\phi c)-(\phi b+\phi c)) && (4)\\ &= \alpha^{-1}(\phi a-\phi b) && (5)\\ &= \alpha^{-1}(a-b) && (6)\\ &= a\ominus b && (7)\end{aligned}$$

so we have the DFBQ property. The constants are both e since $a \oplus e = \phi^{-1}(\phi a+\phi e) = \phi^{-1}(\phi a + o) = a$ and $a \ominus a = \alpha^{-1}(a-a) = \alpha^{-1}(e) = e$.

The second claim is seen by $\alpha a = a-e$ thus $a \ominus e = \alpha^{-1}(a-e) = a$. □

Note that $a \ominus b = \phi a \ominus \phi b$. This is important for the next result, the converse.

Proposition 10. *Let $(N,\oplus,\ominus,e,e)$ be a DFBQ with $a \ominus e = a$ for all $a \in N$. Let ϕ be a permutation of N such that $a \ominus b = \phi a \ominus \phi b$, α a permutation of N such that $\alpha e = e$. Define*

$$a + b = \phi(\phi^{-1}a \oplus \phi^{-1}b) \quad (8)$$
$$a - b = \alpha(\phi^{-1}a \ominus \phi^{-1}b) \quad (9)$$

Let $o := \phi e$. Then $(N,+,-,o,e)$ is a DFBQ, $\alpha a = a-e$, $\phi a = a+\bar{e}$ where $e+\bar{e}=o$.

Proof. $(N,+)$ and $(N,-)$ are quasigroups by isotopism. The DFBQ property is seen by

$$\begin{aligned}(a+c)-(b+c) &= \alpha(\phi^{-1}\phi(\phi^{-1}a\oplus\phi^{-1}c)\ominus\phi^{-1}\phi(\phi^{-1}b\oplus\phi^{-1}c)) && (10)\\ &= \alpha((\phi^{-1}a\oplus\phi^{-1}c)\ominus(\phi^{-1}b\oplus\phi^{-1}c)) && (11)\\ &= \alpha(\phi^{-1}a\ominus\phi^{-1}b) && (12)\\ &= a-b && (13)\end{aligned}$$

The constants are given by $a+o = \phi(\phi^{-1}a \oplus \phi^{-1}o) = \phi(\phi^{-1}a \oplus e) = a$ and $a-a = \alpha(a\ominus a) = \alpha e = e$.

Then $\alpha(a) = \alpha(a \ominus e) = \alpha(\phi^{-1}a \ominus \phi^{-1}e) = a - e$. Let $\bar{e}$ be such that $e + \bar{e} = o$. Then $e + \bar{e} = \phi(\phi^{-1}e \oplus \phi^{-1}\bar{e}) = o = \phi e$ so $\phi^{-1}e \oplus \phi^{-1}\bar{e} = e$. Then

$$\begin{aligned} a = a \ominus e &= \phi^{-1}a \ominus \phi^{-1}e && (14)\\ &= (\phi^{-1}a \oplus \phi^{-1}\bar{e}) \ominus (\phi^{-1}e \oplus \phi^{-1}\bar{e}) && (15)\\ &= (\phi^{-1}a \oplus \phi^{-1}\bar{e}) \ominus e && (16)\\ &= \phi^{-1}a \oplus \phi^{-1}\bar{e} && (17)\\ &= \phi^{-1}(a + \bar{e}) && (18) \end{aligned}$$

Thus $\phi a = a + \bar{e}$ and we are done. □

Definition 11. *A quasigroup $(Q, \circ)$ is a* Ward quasigroup *if $(a \circ c) \circ (b \circ c) = a \circ b$ for all $a, b, c \in Q$.*

Ward first investigated these structure in [17], Furstenberg referred to the equation above in [9]. Johnson and Vojtechovsky give further historical details in Section 2 of [16], where we also find the following.

Theorem 12 ([16] Theorem 2.1)**.** *Let $(Q, \circ)$ be a Ward quasigroup. Then there exists a unique element $e \in Q$ such that for all $x \in Q$, $x \circ x = e$. Define $\bar{x} = e \circ x$ and $x * y = x \circ \bar{y}$ for all $x, y \in Q$. Then $(Q, *, \bar{\ })$ is a group, and $x \circ y = x * \bar{y}$.*

Proposition 13. *Let $(N, +, -, e, e)$ be a DFBQ with $a - e = a$ for all a. Then it is isotopic to a group.*

Proof. Let I be the permutation of N such that $a + Ia = e$. Then

$$a - b = (a + Ib) - (b + Ib) = (a + Ib) - e = a + Ib. \qquad (19)$$

Thus $(a - b) - (c - b) = (a + Ib) - (c + Ib) = a - c$ so $(N, -)$ is a Ward quasigroup. Thus there is a group $(N, *, \cdot^{-1})$ with $a - b = a * b^{-1}$ and $a + b = a * (I^{-1}b)^{-1}$ by equation (19). □

The converse of this result holds too. The proof is simple calculation.

Lemma 14. *Let $(N, *, 1)$ be a group, I a permutation of N fixing 1. Define*

$$\begin{aligned} a + b &= a * (Ib)^{-1} && (20)\\ a - b &= a * b^{-1} && (21) \end{aligned}$$

Then $(N, +, -, 1, 1)$ is a DF biquasigroup with $a - 1 = a$ for all a.

Thus we obtain information on the form of the map ϕ in Proposition 10. We know the form of the operations from Proposition 13 so we can make some explicit statements about the structure.

Corollary 15. *Let* $(N, \oplus, \ominus, e, e)$ *and* ϕ *be as for Proposition 10. Let the operation* $*$ *be as from Proposition 13. Then there exists some* $k \in N$ *such that the map* ϕ *is of the form* $\phi(a) = a * k$.

Conversely, given $(N, \oplus, \ominus, e, e)$ *as in Proposition 10 and a binary operation* $*$ *such that* $(N, *)$ *is a group as in Proposition 13, select any element* $k \in N$*. Then* $\phi(a) := a * k$ *satisfies the requirements of Proposition 10.*

Proof. By Proposition 13 we know that $a \ominus b = a * b^{-1}$. Since $\phi a \ominus \phi b = a \ominus b$ we have $\phi a * (\phi b)^{-1} = a * b^{-1}$. Let $b = 1$ and we obtain $\phi a * (\phi 1)^{-1} = a$ so $\phi a = a * \phi 1$. Letting $k := \phi 1$ we are done.

The converse is seen by taking any element $k \in N$. Define $\phi a := a * k$. Then $\phi a \ominus \phi b = (a * k) * (b * k)^{-1} = a * b^{-1} = a \ominus b$ so we are done. □

5. General explicit descriptions

In this section, we will look at explicit descriptions of DFBQs and QDFs. Using the results above, we know the structure of all DFBQs.

Proposition 16. *Let* $(N, *, 1)$ *be a group. Let* α, β *be permutations of* N*,* $\alpha 1 = 1$*. Define*

$$a + b = a * \beta b \tag{22}$$

$$o = \beta^{-1}(1) \tag{23}$$

$$a - b = \alpha(a * b^{-1}) \tag{24}$$

Then $(N, +, -, o, 1)$ *is a DFBQ and all DFBQs are of this form.*

Proof. The forward direction is a calculation and is clear. Let $(N, +, -, o, e)$ be a DFBQ. We demonstrate that there exists a group structure $(N, *, ^{-1}, 1)$ and permutations α, β of N as above.

By Proposition 9 there exist ϕ and α such that defining

$$a \oplus b := \phi^{-1}(\phi a + \phi b) \tag{25}$$

$$a \ominus b := \alpha^{-1}(a - b) \tag{26}$$

we obtain $(N, \oplus, \ominus, e, e)$ is a DFBQ with $a \ominus e = a$. By Proposition 13 there exists some group $(N, *, ^{-1}, 1)$ such that $e = 1$, $a \ominus b = a * b^{-1}$ and $a \oplus b = a * (I^{-1}(b))^{-1}$. Thus

$$a + b = \phi(\phi^{-1} a * (I^{-1}(\phi^{-1} b))^{-1}) \tag{27}$$

$$a - b = \alpha(a * b^{-1}) \tag{28}$$

By Corollary 15 we know that $\phi x = x * k$, $\phi^{-1}x = x * k^{-1}$. Thus

$$\begin{aligned} a+b &= ((a*k^{-1})*(I^{-1}(b*k^{-1}))^{-1})*k & (29)\\ &= a*k^{-1}*(I^{-1}(b*k^{-1}))^{-1}*k & (30)\\ &= a*\beta(b) & (31)\end{aligned}$$

where $\beta(x) = k^{-1} * (I^{-1}(x * k^{-1}))^{-1} * k$ is a permutation of N. Since $a + \beta^{-1}(1) = a * \beta(\beta^{-1}(1)) = a * 1 = a$ we know $\beta^{-1}(1)$ is the unique right identity, so $o = \beta^{-1}(1)$. The permutation α fixes e which is seen to be 1 and we are done. □

This final result shows that all difference family structures are in fact group structures.

Proposition 17. *The quasigroup development and the group development of a difference family are identical.*

Proof. Suppose we have a QDF $\mathcal{B}$ on a DFBQ $(N, +, -, o, e)$. By Prop 16 above, we know that there is a group operation $*$ and some permutation of N such that $a + b = a * \beta(b)$. Thus if B is a subset of N,

$$dev_+B = \{B+n : n \in N\} = \{B*\beta(n) : n \in N\} = \{B*n : n \in N\} = dev_*B$$

so we obtain exactly the same set of sets. Thus $dev_+\mathcal{B} = dev_*\mathcal{B}$ and we are done. □

6. Conclusion

It would be desirable to generalize the definition of a difference family so as to use more general structures to derive designs using this formalism. With simple and reasonable requirements for our difference family structures, we have shown that we obtain a biquasigroup algebra and that such algebraic structures must be isotopic to groups. It is also seen that the resulting designs are identical, the central result in this paper.

Thus we see that there is no need to bring in complex algebraic structures in order to obtain new designs through the difference family method. We also see that should such a construction work in some algebraic structure, we can make some simple manipulations and obtain a group structure, enhancing our knowledge of the algebraic structure.

Questions remain open as to whether the requirements that we posit are all necessary. It may be reasonable to use a simpler structure for the difference and translation operations, however the postulates we use seem to be minimal.

Several applications can be seen here. For instance, planar nearrings have been shown to possess a difference family structure. Questions about nonassociative planar nearrings have been raised [14], and it might be appropriate to use these results to deduce structure about the nonassociative nearrings that could be so defined. The investigation of neardomains and K–loops [12] suggests that there are some strange and interesting properties when we drop the finiteness and associativity restriction. In particular there may be connections between the generalization of nearfields to neardomains and the generalization to planar nearrings and Ferrero pairs [6, 15], which may be connected to the construction of nonassociative difference families.

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