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On one class of partition polynomials

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Abstract. We consider relations between one class of partition polynomials, parafunctions of triangular matrices (tables), and linear recurrence relations.

Introduced by Bell in [1] the concept of partition polynomials is widely used in discrete mathematics. They arise in number theory [2], algebra (the theory of symmetric polynomials), combinatorics [3] (e.g., an expression for the sum of divisors of the natural number through unordered partitions), differentiation of composite functions (Faa di Bruno's formula) [4] etc.

In this article, one class of partition polynomials is investigated using the calculus of triangular matrices (see [5], [6]). Their relations to some linear recurrence equations and parafunctions of triangular matrices are found. It should be noted that a general approach to partition polynomial is considered in [6], where it is proved that some parafunction of triangular matrices corresponds to every partition polynomial.

1. Subsidiary concepts and statements

From now on K is a number field.

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Definition 1. A triangular table of elements K

$$A = \begin{pmatrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}_n \tag{1}$$

is called a triangular matrix, and the number n the order of A.

Note that a triangular matrix in our definition is not a matrix in its usual sense because it is a triangular rather than rectangular table of numbers.

For every element a_{ij} of the matrix (1), the $(i - j + 1)$ elements $a_{ik},\ k = j, \ldots, i$ are called the *derived elements* of the matrix determined by the *key element* a_{ij}.

The product of all the derived elements determined by the element a_{ij} is denoted by $\{a_{ij}\}$ and called *the factorial product of the key element* a_{ij}, i.e.

$$\{a_{ij}\} = \prod_{k=j}^{i} a_{ik}.$$

Definition 2. If A is a triangular matrix (1), then the paradeterminant and the parapermanent of the triangular matrix are, accordingly, the numbers:

$$\operatorname{ddet}(\mathrm{A}) = \sum_{r=1}^{n} \sum_{\alpha_1+\ldots+\alpha_r=n} (-1)^{n-r} \prod_{s=1}^{r} \{a_{\alpha_1+\ldots+\alpha_s, \alpha_1+\ldots+\alpha_{s-1}+1}\},$$

$$\operatorname{pper}(\mathrm{A}) = \sum_{r=1}^{n} \sum_{\alpha_1+\ldots+\alpha_r=n} \prod_{s=1}^{r} \{a_{\alpha_1+\ldots+\alpha_s, \alpha_1+\ldots+\alpha_{s-1}+1}\},$$

where summation is performed over the set of natural solutions of the equation $\alpha_1 + \ldots + \alpha_r = n$.

Definition 3. For an element a_{ij} of a triangular matrix A, the **corner** $R_{ij}(A)$ is the triangular matrix that consists of the elements a_{rs} of A whose indices satisfy $j \leqslant s \leqslant r \leqslant i$.

Obviously, the corner is the triangular matrix of $(i - j + 1)$–th order, and a_{ij} is its bottom left corner.

We assume below that

$$\operatorname{ddet}(R_{01}(A)) = \operatorname{ddet}(R_{n,n+1}(A)) = \operatorname{pper}(R_{01}(A)) = \operatorname{pper}(R_{n,n+1}(A)) = 1.$$

Theorem 1. *The following identities are valid:*

$$\mathrm{ddet}\,(A) = \sum_{s=1}^{n} (-1)^{n+s} \{a_{ns}\} \cdot \mathrm{ddet}\,(R_{s-1,1}),$$

$$\mathrm{pper}(A) = \sum_{s=1}^{n} \{a_{ns}\} \cdot \mathrm{pper}(R_{s-1,1}).$$

Remark 1. Essentially Theorem 1 makes it possible to decompose (expand) parafunctions of a triangular matrices in the elements of the last row.

2. On one class of partition polynomials

In [6] it is shown that the parafuctions of the matrices of the form

$$A = \begin{pmatrix} k_{11} \cdot x_1 & & & \\ k_{21} \cdot \frac{x_2}{x_1} & k_{22} \cdot x_1 & & \\ \cdots\cdots & \cdots\cdots & \cdots\ \cdots\cdots & \\ k_{n1} \cdot \frac{x_n}{x_{n-1}} & k_{n2} \cdot \frac{x_{n-1}}{x_{n-2}} & \cdots & k_{nn} \cdot x_1 \end{pmatrix}_n = \left(k_{ij} \cdot \frac{x_{i-j+1}}{x_{i-j}} \right)_{1 \le j \le i \le n}, x_0 = 1,$$

where k_{ij} is some fractional rational function of arguments i and j, are related to partition polynomials.

The following two theorems are valid:

Theorem 2. *Let the polynomials* $y_n(x_1, x_2, \ldots, x_n), n = 0, 1, \ldots$ *be determined by the recurrence equation*

$$y_n = x_1 y_{n-1} - x_2 y_{n-2} + \ldots + (-1)^{n-2} x_{n-1} y_1 + (-1)^{n-1} a_n x_n y_0, \quad (2)$$

where $y_0 = 1$, *then the following equalities are valid:*

$$y_n = \mathrm{ddet} \begin{pmatrix} a_1 x_1 & & & \\ a_2 \frac{x_2}{x_1} & x_1 & & \\ \vdots & \cdots & \ddots & \\ a_n \frac{x_n}{x_{n-1}} & \cdots & \frac{x_2}{x_1} & x_1 \end{pmatrix}, \quad (3)$$

$$y_n = \sum_{\lambda_1 + 2\lambda_2 + \ldots + n\lambda_n = n} (-1)^{n-k} \left(\sum_{i=1}^{n} \lambda_i a_i \right) \frac{(k-1)!}{\lambda_1! \lambda_2! \cdot \ldots \cdot \lambda_n!} x_1^{\lambda_1} x_2^{\lambda_2} \cdot \ldots \cdot x_n^{\lambda_n}, \quad (4)$$

where $k = \lambda_1 + \lambda_2 + \ldots + \lambda_n$.

Proof. The fact that the paradeterminant (3) complies with the recurrence relation (2) follows from its expansion in the last row elements.

We prove the equality (4).

Let the theorem be valid for $n = 0, 1, \ldots, m$. We prove its validity for $n = m + 1$. Let us find the coefficient at some fixed monomial

$$x_1^{\lambda_1^*}, x_2^{\lambda_2^*}, \ldots, x_m^{\lambda_m^*}, x_{m+1}^{\lambda_{m+1}^*} \tag{5}$$

in the polynomial y_{m+1}.

The following two cases are possible:

1) $\lambda_{m+1}^* = 0$. In this case, according to the recurrence relation

$$y_{m+1} = x_1 y_m - x_2 y_{m-1} + \ldots + (-1)^{m-1} x_m y_1 + (-1)^m a_{m+1} x_{m+1} y_0,$$

the required coefficient of the monomial (5) can be obtained as the sum of the coefficients corresponding to the summands

$$a(i) = (-1)^{i-1} x_i y_{m-i+1},\ i = 1, 2, \ldots, m$$

of this relation. It is obvious that the summands $a(i)$ correlate with the partition

$$\lambda_1^* + 2\lambda_2^* + \ldots + i(\lambda_i^* - 1) + (i+1)\lambda_{i+1}^* + \ldots + (m-i+1)\lambda_{m-i+1}^* = m - i + 1$$

and the partition correlate with the coefficients

$$(-1)^{i-1}(-1)^{m-i+1-k-1}\left(\lambda_1^* a_1 + \ldots + (\lambda_i^* - 1)a_i + \ldots + \lambda_m^* a_m\right) \times$$
$$\times \frac{(k-2)!}{\lambda_1^*! \cdot \ldots \cdot (\lambda_i^* - 1)! \cdot \ldots \cdot \lambda_m^*!},$$

where $k = \lambda_1^* + \lambda_2^* + \ldots + \lambda_m^*$. Taking into account $\lambda_{m-i+2}^* = \ldots = \lambda_{m+1}^* = 0$, these coefficients can be written as

$$(-1)^{m-A+1}\left(B - a_i\right))\frac{(A-2)!}{\lambda_1^*! \cdot \ldots \cdot (\lambda_i^* - 1)! \cdot \ldots \cdot \lambda_m^*! \lambda_{m+1}^*!},$$

where

$$A = \sum_{i=1}^{m+1} \lambda_i^*, B = \sum_{i=1}^{m+1} \lambda_i^* a_i.$$

Therefore, the required coefficient, for the monomial

$$x_1^{\lambda_1^*} x_2^{\lambda_2^*} \cdot \ldots \cdot x_m^{\lambda_m^*} x_{m+1}^{\lambda_{m+1}^*}$$

equals

$$\sum_{i=1}^{m+1}(-1)^{m+1-A}(B-a_i)\lambda_i\frac{(A-2)!}{\lambda_1^*!\cdot\ldots\cdot\lambda_{m+1}^*!}=(-1)^{m+1-A}B\frac{(A-1)!}{\lambda_1^*!\cdot\ldots\cdot\lambda_{m+1}^*!}$$

2) $\lambda_{m+1}^*=1$. It is obvious that in this case

$$\lambda_1^*=\lambda_2^*=\ldots=\lambda_m^*=0,$$

and the recurrence relation implies that the required coefficient is

$$(-1)^m a_{m+1}.$$

But this coefficient can be represented as

$$(-1)^{m+1-A}B\frac{(A-1)!}{\lambda_1^*!\cdot\ldots\cdot\lambda_{m+1}^*!},$$

because $A=1, B=a_{m+1}$. □

The following theorem is proved by analogy.

Theorem 3. *Let the polynomials $y_n(x_1,x_2,\ldots,x_n), n=0,1,\ldots$ be determined by the recurrence equation*

$$y_n=x_1y_{n-1}+x_2y_{n-2}+\ldots+x_{n-1}y_1+a_nx_ny_0,$$

where $y_0=1$, then the following equalities are valid

$$y_n=\operatorname{pper}\begin{pmatrix} a_1x_1 & & & \\ a_2\frac{x_2}{x_1} & x_1 & & \\ \vdots & \ldots & \ddots & \\ a_n\frac{x_n}{x_{n-1}} & \ldots & \frac{x_2}{x_1} & x_1 \end{pmatrix},$$

$$y_n=\sum_{\lambda_1+2\lambda_2+\ldots+n\lambda_n=n}\left(\sum_{i=1}^n\lambda_ia_i\right)\frac{(k-1)!}{\lambda_1!\lambda_2!\cdot\ldots\cdot\lambda_n!}x_1^{\lambda_1}x_2^{\lambda_2}\cdot\ldots\cdot x_n^{\lambda_n},$$

where $k=\lambda_1+\lambda_2+\ldots+\lambda_n$.

If $a_1=a_2=\ldots=a_n=1$ and $m=n$ in Theorem 2, then obtain Theorem 2.5.3 from [5]. The parapermanents of such matrices appear in an expression of homogeneous symmetric polynomials through power sums (cf. [5, pp. 174,338]).

If $a_i = i, i = 1, \ldots, n$, in Theorem 2, then we obtain the expression of Waring's formula as the paradeterminant of the triangular matrix

$$s_n = \left\langle \begin{array}{ccccc} \sigma_1 & & & & \\ 2\frac{\sigma_2}{\sigma_1} & \sigma_1 & & & \\ \vdots & \ldots & \ddots & & \\ (n-1)\frac{\sigma_{n-1}}{\sigma_{n-2}} & \frac{\sigma n-2}{\sigma n-1} & \ldots & \sigma_1 & \\ n\frac{\sigma_n}{\sigma_{n-1}} & \frac{\sigma_{n-1}}{\sigma_{n-2}} & \ldots & \frac{\sigma_2}{\sigma_1} & \sigma_1 \end{array} \right\rangle_n ,$$

here s_n are symmetric power sums and σ_i, $i = 1, \ldots, n$, are elementary symmetric polynomials.

The case $a_i = ri + s$, where r and s are some rational numbers such that $rs \neq 0$, is of particular interest. Then the following statement is valid.

Corollary 1. *The following equalities give the same polynomials:*

$$y_n = x_1 y_{n-1} - x_2 y_{n-2} + \ldots + (-1)^{n-2} x_{n-1} y_1 + (-1)^{n-1} (rn + s) x_n y_0,$$

$$y_n = \ ddet \begin{pmatrix} (r+s)x_1 & & & \\ (2r+s)\frac{x_2}{x_1} & x_1 & & \\ \vdots & \ldots & \ddots & \\ (rn+s)\frac{x_n}{x_{n-1}} & \ldots & \frac{x_2}{x_1} & x_1 \end{pmatrix},$$

$$y_n = \sum_{\lambda_1 + 2\lambda_2 + \ldots + n\lambda_n = n} (-1)^{n-k} (rn + sk) \frac{(k-1)!}{\lambda_1! \lambda_2! \cdot \ldots \cdot \lambda_n!} x_1^{\lambda_1} x_2^{\lambda_2} \cdot \ldots \cdot x_n^{\lambda_n},$$

where $k = \lambda_1 + \lambda_2 + \ldots + \lambda_n, \ y_0 = 1.$

A similar result is valid for the theorem 3.

Corollary 2. *The following equalities give the same polynomials:*

$$y_n = x_1 y_{n-1} + x_2 y_{n-2} + \ldots + x_{n-1} y_1 + (rn + s) x_n y_0,$$

$$y_n = \ pper \begin{pmatrix} (r+s)x_1 & & & \\ (2r+s)\frac{x_2}{x_1} & x_1 & & \\ \vdots & \ldots & \ddots & \\ (rn+s)\frac{x_n}{x_{n-1}} & \ldots & \frac{x_2}{x_1} & x_1 \end{pmatrix},$$

$$y_n = \sum_{\lambda_1 + 2\lambda_2 + \ldots + n\lambda_n = n} (rn + sk) \frac{(k-1)!}{\lambda_1! \lambda_2! \cdot \ldots \cdot \lambda_n!} x_1^{\lambda_1} x_2^{\lambda_2} \cdot \ldots \cdot x_n^{\lambda_n},$$

where $k = \lambda_1 + \lambda_2 + \ldots + \lambda_n, \ y_0 = 1.$

References

[1] *Bell E.T.* Partition polynomials. — Ann. Math. 29, 1927, p. 38 – 46.

[2] *Fine N.J.* Sums over partitions.— Report of the Institute in the Theory of Numbers, Boulder, 1959, p. 86-94.

[3] *Riordan J.* Kombinatornye tozhdestva. (Russian) [Combinatorial identities] Translated from the English by A. E. Zhukov. "Nauka", Moscow, 1982. 256 pp.

[4] *Riordan J.* An Introduction to Combinatorial Analysis, Wiley, New York, 1958.

[5] *Zatorsky R.A.* Triangular matrix calculus and its application. — Ivano-Frankivs'k: Simyk, 2010. – 508 p. (in Ukrainian).

[6] *Zatorsky R.A.* Theory of paradeterminants and its applications. Algebra and Discrete Mathematics Number 1. 2007. – pp. 109 – 138.

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