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Ideals in $(\mathcal{Z}^+, \leq_D)$

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Abstract. A convolution is a mapping $\mathcal{C}$ of the set $\mathcal{Z}^+$ of positive integers into the set $\mathcal{P}(\mathcal{Z}^+)$ of all subsets of $\mathcal{Z}^+$ such that every member of $\mathcal{C}(n)$ is a divisor of n. If for any n, $D(n)$ is the set of all positive divisors of n , then D is called the Dirichlet's convolution. It is well known that $\mathcal{Z}^+$ has the structure of a distributive lattice with respect to the division order. Corresponding to any general convolution $\mathcal{C}$, one can define a binary relation $\leq_{\mathcal{C}}$ on $\mathcal{Z}^+$ by ' $m \leq_{\mathcal{C}} n$ if and only if $m \in \mathcal{C}(n)$ ' . A general convolution may not induce a lattice on $\mathcal{Z}^+$. However most of the convolutions induce a meet semi lattice structure on $\mathcal{Z}^+$ .In this paper we consider a general meet semi lattice and study it's ideals and extend these to $(\mathcal{Z}^+, \leq_D)$, where D is the Dirichlet's convolution.

Introduction

A convolution is a mapping $\mathcal{C} : \mathcal{Z}^+ \longrightarrow \mathcal{P}(\mathcal{Z}^+)$ such that $\mathcal{C}(n)$ is a set of positive divisors on n, $n \in \mathcal{C}(n)$ and $\mathcal{C}(n) = \bigcup\limits_{m \in \mathcal{C}(n)} \mathcal{C}(m)$, for any $n \in \mathcal{Z}^+$, where $\mathcal{Z}^+$ is the set of all positive integers and $\mathcal{P}(\mathcal{Z}^+)$ is the set of all subsets of $\mathcal{Z}^+$. Popular examples are the Dirichlet's convolution D and the Unitary convolution U defined respectively by

$$\mathrm{D}(n) = \text{The set of all positive divisors of } n$$

$$\text{and } \mathrm{U}(n) = \{d \ / \ d|n \ \text{ and } (d, \tfrac{n}{d}) = 1\}$$

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for any $n \in \mathcal{Z}^+$. If $\mathcal{C}$ is a convolution, then the binary relation $\leq_{\mathcal{C}}$ on $\mathcal{Z}^+$, defined by,

$$m \leq_{\mathcal{C}} n \text{ if and only if } m \in \mathcal{C}(n) \ ,$$

is a partial order on $\mathcal{Z}^+$ and is called the partial order induced by $\mathcal{C}$ [7]. It is well known that the Dirichlet's convolution induces the division order on $\mathcal{Z}^+$ with respect to which $\mathcal{Z}^+$ becomes a distributive lattice, where, for any $a, b \in \mathcal{Z}^+$, the greatest common divisor(GCD) and the least common multiple(LCM) of a and b are respectively the greatest lower bound(glb) and the least upper bound(lub) of a and b . In fact, with respect to the division order, the lattice $\mathcal{Z}^+$ satisfies the infinite join distributive law given by

$$a \vee (\bigwedge_{i \in I} b_i) = \bigwedge_{i \in I} (a \vee b_i)$$

for any $a \in \mathcal{Z}^+$ and $\{b_i\}_{i \in I} \subseteq \mathcal{Z}^+$. In this paper, we discuss various aspects of ideals in $(\mathcal{Z}^+, \leq_{\mathcal{C}})$. Actually a general convolution may not induce a lattice structure on $\mathcal{Z}^+$. However , most of the convolutions we are considering induce a meet semi lattice structure on $\mathcal{Z}^+$. For this reason, we first consider a general semi lattice and study it's ideals and later extend these to $(\mathcal{Z}^+, \leq_D)$.

1. Preliminaries

Let us recall that a partial order on a non-empty set X is defined as a binary relation $\leq$ on X which is reflexive ($a \leq a$), transitive ($a \leq b, b \leq c \Longrightarrow a \leq c$) and antisymmetric ($a \leq b, b \leq a \Longrightarrow a = b$) and that a pair $(X, \leq)$ is called a partially ordered set(poset) if X is a non-empty set and $\leq$ is a partial order on X. For any $A \subseteq X$ and $x \in X$, x is called a lower(upper) bound of A if $x \leq a$(respectively $a \leq x$) for all $a \in A$. We have the usual notations of the greatest lower bound(glb) and least upper bound(lub) of A in X. If A is a finite subset $\{a_1, a_2, \cdots, a_n\}$, the glb of A(lub of A) is denoted by $a_1 \wedge a_2 \wedge \cdots \wedge a_n$ or $\bigwedge_{i=1}^{n} a_i$ (respectively by $a_1 \vee a_2 \vee \cdots \vee a_n$ or $\bigvee_{i=1}^{n} a_i$). A partially ordered set $(X, \leq)$ is called a meet semi lattice if $a \wedge b$ (=glb$\{a, b\}$) exists for all a and $b \in X$. $(X, \leq)$ is called a join semi lattice if $a \vee b$ (=lub$\{a, b\}$) exists for all a and $b \in X$. A poset $(X, \leq)$ is called a lattice if it is both a meet and join semi lattice. Equivalently, lattice can also be defined as an algebraic system

$(X, \wedge, \vee)$, where $\wedge$ and $\vee$ are binary operations which are associative, commutative and idempotent and satisfying the absorption laws, namely $a \wedge (a \vee b) = a = a \vee (a \wedge b)$ for all $a, b \in X$; in this case the partial order $\leq$ on X is such that $a \wedge b$ and $a \vee b$ are respectively the glb and lub of $\{a, b\}$. The algebraic operations $\wedge$ and $\vee$ and the partial order $\leq$ are related by

$$a = a \wedge b \iff a \leq b \iff a \vee b = b.$$

Throughout the paper, $\mathcal{Z}^+$ and $\mathcal{N}$ denote the set of positive integers and the set of non-negative integers respectively.

Definition 1. A mapping $\mathcal{C} : \mathcal{Z}^+ \longrightarrow \mathcal{P}(\mathcal{Z}^+)$ is called a convolution if the following are satisfied for any $n \in \mathcal{Z}^+$.

(1) $\mathcal{C}(n)$ is a set of positive divisors of n

(2) $n \in \mathcal{C}(n)$

(3) $\mathcal{C}(n) = \bigcup_{m \in \mathcal{C}(n)} \mathcal{C}(m)$.

Definition 2. For any convolution $\mathcal{C}$ and m and $n \in \mathcal{Z}^+$, we define

$$m \leq n \text{ if and only if } m \in \mathcal{C}(n)$$

Then $\leq_{\mathcal{C}}$ is a partial order on $\mathcal{Z}^+$ and is called the partial order induced by $\mathcal{C}$ on $\mathcal{Z}^+$. In fact, for any mapping $\mathcal{C} : \mathcal{Z}^+ \longrightarrow \mathcal{P}(\mathcal{Z}^+)$ such that each member of $\mathcal{C}(n)$ is a divisor of n, $\leq_{\mathcal{C}}$ is a partial order on $\mathcal{Z}^+$ if and only if $\mathcal{C}$ is a convolution, as defined above [6],[8].

Definition 3. Let $\mathcal{C}$ be a convolution and p a prime number. Define a relation $\leq^p_{\mathcal{C}}$ on the set $\mathcal{N}$ of non-negative integers by

$$a \leq^p_{\mathcal{C}} b \text{ if and only if } p^a \in \mathcal{C}(p^b)$$

for any a and $b \in \mathcal{N}$.

It can be easily verified that $\leq^p_{\mathcal{C}}$ is a partial order on $\mathcal{N}$, for each prime p. The following is a direct verification.

Theorem 1. *Let $\mathcal{C}$ be a convolution.*

(1) If $(\mathcal{Z}^+, \leq_{\mathcal{C}})$ is a meet(join) semilattice, then so is $(\mathcal{N}, \leq^p_{\mathcal{C}})$ for each prime p.

(2) If $(\mathcal{Z}^+, \leq_{\mathcal{C}})$ is a lattice, then so is $(\mathcal{N}, \leq^p_{\mathcal{C}})$ for each prime p.

Now, we have the following examples from [9] in which the convolutions induce meet semi lattice structures.

Example 1. Let D be the Dirichlet's convolution defined by

$$D(n) = \text{The set of all positive divisors of } n.$$

Then $\leq_D$ is precisely the division order on $\mathcal{Z}^+$ and, for each prime p, $\leq_D^p$ is the usual order on $\mathcal{N}$. $(\mathcal{Z}^+, \leq_D)$ is known to be distributive lattice.

Example 2. Let $U(n)$ be the Unitary convolution defined by

$$U(n) \;=\; \{d \in D(n) \;\mid\; d \;\text{ and }\; \tfrac{n}{d} \;\text{ are relatively prime}\}.$$

Then $(\mathcal{Z}^+, \leq_U)$ is a meet semilattice, but not a join semilattice.

Note that

$$U(p^a) = \{1, p^a\} \text{ for any } 0 < a \in \mathcal{N}.$$

Example 3. Let F_2 be the square-free convolution defined by

$$F_2(n) = \{n\} \cup \{d \in D(n) \;\mid\; p^2 \;\text{ does not divide } n \text{ for any prime } p\}.$$

Then $(\mathcal{Z}^+, \leq_{F_2})$ is a meet semilattice but not a join semilattice. Note that, for any prime p and $a \in \mathcal{N}$,

$$F_2(p^a) = \begin{cases} \{1\} & \text{if} \quad a = 0 \\ \{1, p\} & \text{if} \quad a = 1 \\ \{1, p, p^a\} & \text{if} \quad a > 1 \end{cases}$$

Example 4. For any $k \in \mathcal{Z}^+$, a positive integer d is said to be k-free if p^k does not divide d for any prime p. Let $F_k(n)$ be the set of all k-free divisors of n together with n. Then $(\mathcal{Z}^+, \leq_{F_k})$ is a meet semilattice but not a join semi lattice.

2. Ideals in Semi lattices

Recall that most of the convolutions like Dirichlet's convolution, Unitary convolution and k-free convolution induce meet semi lattice structure on $\mathcal{Z}^+$[9]. For this reason we consider a general meet semi lattice and study it's ideals. Throughout this section, unless otherwise stated, by a semi lattice we mean a meet semi lattice only.

Definition 4. Let $(S, \wedge)$ be a semi lattice. A non-empty subset I of S is called an ideal of S if the following are satisfied

(1) $x \in S$ and $x \leq a \in I \implies x \in I$

(2) For any a and $b \in I$, there exists $c \in I$ such that $a \leq c$ and $b \leq c$

Theorem 2. *Let a and b be elements of a meet semi lattice $(S, \wedge)$. Then the following are equivalent to each other.*

(1) There exists smallest ideal of S containing a and b.

(2) The intersection of all ideals of S containing a and b is again an ideal of S.

(3) a and b have least upper bound in S.

Proof. (1) $\iff$ (2) is trivial.

(1) $\implies$ (3) : Let I be the smallest ideal of S containing a and b. Then, there exists $x \in I$ such that

$$a \leq x \quad \text{and} \quad b \leq x$$

Therefore x is an upper bound of a and b. If y is any other upper bound of a and b, then $(y]$ is an ideal of S containing a and b and hence $I \subseteq (y]$. Since $x \in I$, we get that $x \in (y]$ and therefore $x \leq y$. Thus x is the least upper bound of a and b.

(3) $\implies$ (1) : Let $a \vee b$ be the least upper bound of a and b. Then $a \leq a \vee b$ and $b \leq a \vee b$ and hence $(a \vee b]$ is an ideal containing a and b. If I is any ideal containing a and b, then there exists $x \in I$ such that

$$a \leq x \quad \text{and} \quad b \leq x \quad \text{and hence} \quad a \vee b \leq x$$

so that $a \vee b \in I$ and $(a \vee b] \subseteq I$. Thus $(a \vee b]$ is the smallest ideal of S containing a and b. □

Although the intersection of an arbitrary class of ideals need not be an ideal, a finite intersection is always an ideal.

Theorem 3. *Let $(S, \wedge)$ be a semi lattice and $\mathcal{I}(S)$ the set of all ideals of S. Then $(\mathcal{I}(S), \cap)$ is a semilattice and $a \mapsto (a]$ is an embedding of $(S, \wedge)$ onto $(\mathcal{I}(S), \cap)$.*

Proof. By the above theorem, it follows that $(\mathcal{I}(S), \cap)$ is a semi lattice.Also, for any a and b in S, we have

$$(a] \cap (b] \;=\; (a \wedge b]$$

$$\text{and} \quad (a] \subseteq (b] \iff a \in (b] \iff a \leq b$$

Therefore $a \mapsto (a]$ is an embedding of S into $\mathcal{I}(S)$. □

Theorem 4. *A semi lattice $(S, \wedge)$ is a lattice if and only if $\mathcal{I}(S)$ is a lattice and, in this case, $a \mapsto (a]$ is an embedding of the lattice S into the lattice $\mathcal{I}(S)$.*

Proof. It is well known that the set $\mathcal{I}(S)$ of ideals of a lattice $(S, \wedge, \vee)$ is again a lattice in which,

$$\begin{aligned} I \wedge J &= I \cap J \\ \text{and } I \vee J &= \{ \ x \in S \ \mid \ x \leq a \wedge b, \text{ for some } a \in I \text{ and } b \in J \ \} \end{aligned}$$

for any ideals I and J, in this case,

$$(a] \vee (b] \ = \ (a \vee b]$$

for any a and b in S, so that $a \mapsto (a]$ is an embedding of lattices.

Conversely, suppose that $\mathcal{I}(S)$ is a lattice. Let a and $b \in S$ and I be the least upper bound of $(a]$ and $(b]$ in $\mathcal{I}(S)$. Then I is the smallest ideal containing a and b and hence by Theorem 3.3, $a \vee b$ exists in S. Therefore S is a lattice. □

For a lattice $(L, \wedge, \vee)$, any ideal of the semi lattice $(L, \wedge)$ turns out to be the usual ideal of the lattice $(L, \wedge, \vee)$.

3. Ideals in $(\mathcal{Z}^+, \leq_D)$

Now we shall turn our attention to the particular case of the lattice structure on $\mathcal{Z}^+$ induced by the division ordering / and study the ideals of $\mathcal{Z}^+$. The division ordering is precisely the partial ordering $\leq_D$ induced by the Dirichlet's convolution D.

First we observe that $\theta : (\mathcal{Z}^+, /) \longrightarrow (\sum\limits_{P} \mathcal{N}, \leq)$ is an order isomorphism where θ is defined by
$\theta(a)(p)$ =The largest n in $\mathcal{N}$ such that p^n divides a,for any $a \in \mathcal{Z}^+$ and $p \in \mathcal{P}$ and $\sum\limits_{P} \mathcal{N} = \{ \ f : \mathcal{P} \longrightarrow \ \mathcal{N} \ \mid \ f(p) = 0$ for all but finite $p\,'s \ \}$. Here $\mathcal{P}$ stands for the set of primes and $\mathcal{N}$ stands for the set of non-negative integers.

Definition 5. Adjoin an external element ∞ to $\mathcal{N}$ and extend the usual ordering $\leq$ on $\mathcal{N}$ to $\mathcal{N} \cup \{\infty\}$ by defining $a < \infty$ for all $a \in \mathcal{N}$. We shall denote $\mathcal{N} \cup \{\infty\}$ together with this extended usual order by $\mathcal{N}^{\infty}$.

Theorem 5. *Let* $\alpha : \mathcal{P} \longrightarrow \mathcal{N}^{\infty}$ *be a mapping and define*

$$I_\alpha = \{ n \in \mathcal{Z}^+ \mid \theta(n)(p) \leq \alpha(p) \ \textit{for all} \ p \in \mathcal{P}\}$$

Then I_α *is an ideal of* $(\mathcal{Z}^+, /)$ *and every ideal of* $(\mathcal{Z}^+, /)$ *is of the form* I_α *for some mapping* $\alpha : \mathcal{P} \longrightarrow \mathcal{N}^{\infty}$

Proof. Since no prime divides the integer 1, we get that $\theta(1)(p) = 0 \leq \alpha(p)$ for all $p \in \mathcal{P}$ and hence $1 \in I_\alpha$. Therefore I_α is a non-empty subset of $\mathcal{Z}^+$.

$$\begin{aligned} m \text{ and } n \in I_\alpha &\Longrightarrow \theta(m)(p) \leq \alpha(p) \text{ and } \theta(n)(p) \leq \alpha(p) \text{ for all } p \in \mathcal{P} \\ &\Longrightarrow \theta(m \vee n)(p) = \text{Max}\,\{ \theta(m)(p), \theta(n)(p) \} \leq \alpha(p) \\ &\qquad \text{for all } p \in \mathcal{P} \\ &\Longrightarrow m \vee n \in I_\alpha \end{aligned}$$

and

$$\begin{aligned} m \leq_D n \in I_\alpha &\Longrightarrow \theta(m)(p) \leq \theta(n)(p) \leq \alpha(p) \text{ for all } p \in \mathcal{P} \\ &\Longrightarrow \theta(m)(p) \leq \alpha(p) \text{ for all } p \in \mathcal{P} \\ &\Longrightarrow m \in I_\alpha. \end{aligned}$$

Thus I_α is an ideal of $(\mathcal{Z}^+, /)$.

Conversely suppose that I is any ideal of $(\mathcal{Z}^+, /)$. Define $\alpha : \mathcal{P} \longrightarrow \mathcal{N}^{\infty}$ by

$$\alpha(p) = \text{Sup}\{ \theta(n)(p) \mid n \in I \} \ \text{ for any } \ p \in \mathcal{P}$$

Note that $\alpha(p)$ is either a non-negative integer or ∞, for any $p \in \mathcal{P}$. Therefore α is a mapping of $\mathcal{P}$ into $\mathcal{N}^{\infty}$.

$$\begin{aligned} n \in I &\Longrightarrow \theta(n)(p) \leq \alpha(p) \text{ for all } p \in \mathcal{P} \\ &\Longrightarrow n \in I_\alpha \end{aligned}$$

Therefore $I \subseteq I_\alpha$.

On the other hand, suppose $n \in I_\alpha$. Then $\theta(n)(p) \leq \alpha(p)$ for all $p \in \mathcal{P}$. Since $\theta(n) \in \sum_P \mathcal{N}$, $|\theta(n)|$ is finite. If $|\theta(n)| = \phi$, then $n = 1 \in I$.

Suppose $|\theta(n)|$ is non-empty. Let $|\theta(n)| = \{ p_1, p_2 \cdots, p_r \}$. Then $\theta(n)(p) = 0$ for all $p \neq p_i$, $1 \leq i \leq r$ and $\theta(n)(p_i) \in \mathcal{N}$. Now, for each $1 \leq i \leq r$, $\theta(n)(p_i) \leq \alpha(p_i) = \text{Sup}\{ \theta(m)(p_i) \mid m \in I \}$ and hence there exists $m_i \in I$ such that $\theta(n)(p_i) \leq \theta(m)(p_i)$. Now, put $m = m_1 \vee m_2 \vee \cdots \vee m_r$, then $m \in I$ and
$\theta(n)(p_i) \leq \text{Max.}\{ \theta(m_1)(p_i), \cdots, \theta(m_i)(p_i) \} = \theta(m)(p_i)$ for all $1 \leq i \leq r$. Also, since $\theta(n)(p) = 0$ for all $p \neq p_i$, we get that $\theta(n)(p) \leq \theta(m)(p)$ for

all $p \in \mathcal{P}$ so that $n \leq_D m \in I$ and therefore $n \in I$. Therefore $I_\alpha \subseteq I$. Thus $I = I_\alpha$. $\square$

Note that, if α is the constant map $\overline{0}$ defined by $\alpha(p) = 0$ for all $p \in \mathcal{P}$, then $I_\alpha = \{1\}$ and that , if α is the constant map $\overline{\infty}$, then $I_\alpha = \mathcal{Z}^+$.

Definition 6. For any mappings α and β from $\mathcal{P}$ into $\mathcal{N}^\infty$, define

$$\alpha \leq \beta \quad \text{if and only if} \quad \alpha(p) \leq \beta(p) \text{ for all } p \in \mathcal{P}.$$

Thus $\leq$ is a partial order on $(\mathcal{N}^\infty)^\mathcal{P}$.

Theorem 6. *The map $\alpha \mapsto I_\alpha$ is an order isomorphism of the poset $((\mathcal{N}^\infty)^\mathcal{P}, \leq)$, onto the poset $(\mathcal{I}(\mathcal{Z}^+), \subseteq)$ of all ideals of $(\mathcal{Z}^+, /)$.*

Proof.
Let α and $\beta : \mathcal{P} \mapsto \mathcal{N}^\infty$ be any mappings. Clearly, $\alpha \leq \beta \Rightarrow I_\alpha \subseteq I_\beta$.

On the other hand, suppose that $I_\alpha \subseteq I_\beta$. We shall prove that $\alpha(p) \leq \beta(p)$ for all $p \in \mathcal{P}$ so that $\alpha \leq \beta$. To prove this, let us fix $p \in \mathcal{P}$. If $\beta(p) = \infty$ or $\alpha(p) = 0$, trivially $\alpha(p) \leq \beta(p)$. Therefore, we can assume that $\beta(p) < \infty$ and $\alpha(p) > 0$.

Consider $n = p^{\beta(p)+1}$. Then

$$\theta(n)(p) = \beta(p) + 1 \;\not\leq\; \beta(p).$$

and hence $n \notin I_\beta$. Since $I_\alpha \subseteq I_\beta$, $n \notin I_\alpha$ and therefore $\theta(n)(q) \not\leq \alpha(q)$ for some $q \in \mathcal{P}$. But $\theta(n)(q) = 0$ for all $q \neq p$. Thus

$$\beta(p) + 1 = \theta(n)(p) \not\leq \alpha(p)$$

$$\alpha(p) < \beta(p) + 1.$$

Therefore $\alpha(p) \leq \beta(p)$. This is true for all $p \in \mathcal{P}$. Thus $\alpha \leq \beta$. Also $\alpha \mapsto I_\alpha$ is a surjection. Thus $\alpha \mapsto I_\alpha$ is an order isomorphism of $((\mathcal{N}^\infty)^\mathcal{P}, \leq)$, onto $(\mathcal{I}(\mathcal{Z}^+), \subseteq)$. $\square$

Corollary 1. *For any α and $\beta : \mathcal{P} \to \mathcal{N}^\infty$,*

$$I_\alpha \cap I_\beta = I_{\alpha \,\wedge\, \beta}.$$

$$\textit{and } I_\alpha \cup I_\beta = I_{\alpha \,\vee\, \beta}.$$

where $\alpha \wedge \beta$ and $\alpha \vee \beta$ are point-wise g.l.b and l.u.b of α and β.

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