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RESEARCH ARTICLE

The edge chromatic number of $\Gamma_I(R)$*

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Abstract. For a commutative ring R and an ideal I of R, the ideal-based zero-divisor graph is the undirected graph $\Gamma_I(R)$ with vertices $\{x \in R-I : xy \in I$ for some $y \in R-I\}$, where distinct vertices x and y are adjacent if and only if $xy \in I$. In this paper, we discuss the nature of the edges of $\Gamma_I(R)$. We also find the edge chromatic number for the graph $\Gamma_I(R)$.

Introduction

Algebraic combinatorics is an area of mathematics that employs methods of abstract algebra in various combinatorial contexts and vice versa. Associating a graph with an algebraic structure is a research study that has attracted considerable attention. In fact, the study aims at exposing the relationship between algebra and graph theory and advancing applications of one with the other. Let R be a commutative ring with identity and I be a proper ideal of R. Let $Z(R)$ be the set of zero-divisors of R. The zero-divisor graph of R, denoted by $\Gamma(R)$, is an undirected graph whose vertices are the nonzero zero-divisors of R with two distinct vertices x and y joined by an edge if and only if $xy = 0$. Beck [7] introduced the concept of a zero-divisor graph of a commutative ring, but this work was

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mostly concerned with colorings of rings. The zero-divisor graph helps us to study the algebraic properties of rings using graph-theoretical tools. We can translate some algebraic properties of a ring to graph theory language and then the geometric properties of graphs help us to explore some interesting results in the algebraic structures of rings. The above definition first appeared in Anderson and Livingston [4], which contains several fundamental results concerning $\Gamma(R)$. This definition, unlike the earlier work of Anderson and Naseer [5] and Beck [7], does not take zero to be a vertex of $\Gamma(R)$. The zero-divisor graph has been extended to other algebraic structures in DeMeyer et al.[8] and Redmond [13].

Let R be a commutative ring and let I be an ideal of R. The ideal-based zero-divisor graph is an undirected graph $\Gamma_I(R)$ with vertices $\{x \in R - I : xy \in I \text{ for some } y \in R - I\}$, where distinct vertices x and y are adjacent if and only if $xy \in I$. It was introduced by S. P. Redmond [14]. In [14], he found the values of parameters such as connectivity, clique, diameter, girth, etc. in relation with the zero-divisor graph. In this paper, we find the values of parameters such as vertex chromatic number, minimum and maximum degree of $\Gamma_I(R)$. In section 2, we give the definitions and theorems from [14] which are needed for subsequent sections. In section 3, we discusses the number and the nature of the edges of $\Gamma_I(R)$. In section 4, we determine the edge chromatic number of the graph $\Gamma_I(R)$.

A ring R is said to be decomposable if R can be written as $R_1 \times R_2$, where R_1 and R_2 are rings; otherwise, R is said to be indecomposable. If X is either an element or a subset of R, then $\mathrm{Ann}(X)$ denotes the annihilator of X in R. For any subset X of R, we define $X^* = X - \{0\}$, and $|X|$ denotes the number of elements in X.

For a graph G, the degree of a vertex v in G is the number of edges incident with v. Denote the degree of the vertex v in $\Gamma_I(R)$ by $\deg(v)$ and in $\Gamma(R/I)$ by $\deg_\Gamma(v)$. We denote the minimum and maximum degree of vertices of G by $\delta(G)$ and $\Delta(G)$, respectively. A graph G is regular if the degrees of all vertices of G are the same. We denote the complete graph with n vertices and complete bipartite graph with two parts of sizes m and n by K_n and $K_{m,n}$, respectively. The number of vertices in the set X is denoted by $|X|$.

A proper k-edge coloring of a graph G is an assignment of k colors $\{1, \ldots, k\}$ to the edges of G such that no two adjacent edges have the same color. The edge chromatic number $\chi^{'}(G)$ of a graph G is the minimum k for which G has a proper k-edge coloring. A graph G is said to be edge critical if G is connected, $\chi'(G) = \Delta(G) + 1$, and for any edge e of G, we have $\chi'(G - \{e\}) < \chi'(G)$.

1. Preliminaries

Definition 1 ([14])**.** Let R be a commutative ring and let I be an ideal of R. The ideal based zero-divisor graph is the undirected graph $\Gamma_I(R)$ with vertices $\{x \in R - I : xy \in I$ for some $y \in R - I\}$, where distinct vertices x and y are adjacent if and only if $xy \in I$.

Example 1. For $R \cong \mathbb{Z}_{16} \times \mathbb{Z}_2$ and $I = 0 \times \mathbb{Z}_2$, $\Gamma_I(R)$ is shown in Figure 1.

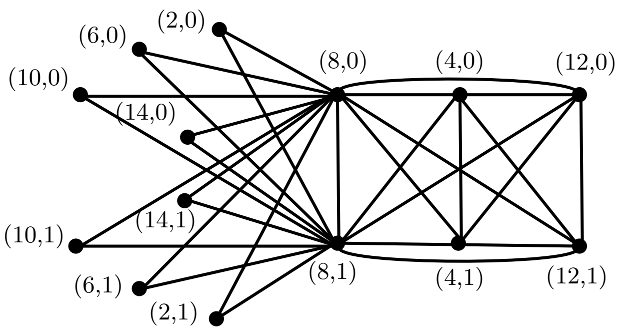

FIGURE 1.

Remark 1 ([14])**.** Let I be an ideal of a ring R. Then $\Gamma_I(R)$ is a graph on a finite number of vertices if and only if either R is finite or I is a prime ideal. Moreover, if $\Gamma(R/I)$ is a graph on N vertices, then $\Gamma_I(R)$ is a graph on $N|I|$ vertices.

Theorem 1 ([14])**.** *Let I be an ideal of a ring R, and let $x, y \in R - I$. Then*

(a) *if $x + I$ is adjacent to $y + I$ in $\Gamma(R/I)$, then x is adjacent to y in $\Gamma_I(R)$.*

(b) *if x is adjacent to y in $\Gamma_I(R)$ and $x + I \neq y + I$, then $x + I$ is adjacent to $y + I$ in $\Gamma(R/I)$.*

(c) *if x is adjacent to y in $\Gamma_I(R)$ and $x + I = y + I$, then $x^2, y^2 \in I$.*

Corollary 1 ([14])**.** *If x and y are (distinct) adjacent vertices in $\Gamma_I(R)$, then all (distinct) elements of $x + I$ and $y + I$ are adjacent in $\Gamma_I(R)$. If $x^2 \in I$, then all the distinct elements of $x + I$ are adjacent in $\Gamma_I(R)$.*

Remark 2 ([14]). Clearly there is a strong relationship between $\Gamma(R/I)$ and $\Gamma_I(R)$. Let I be an ideal of a ring R. One can verify that the following method can be used to construct a graph $\Gamma_I(R)$. Let $\{a_\lambda\}_{\lambda\in\Lambda} \subseteq R$ be a set of coset representatives of the vertices of $\Gamma(R/I)$. For each $i \in I$, define a graph G_i with vertices $\{a_\lambda + i : \lambda \in \Lambda\}$, where edges are defined by the relationship $a_\lambda + i$ is adjacent to $a_\beta + i$ in G_i if and only if $a_\lambda + I$ is adjacent to $a_\beta + I$ in $\Gamma(R/I)$ (i.e., $a_\lambda a_\beta \in I$).

Define the graph G to have as its vertex set $V = \bigcup\limits_{i\in I} G_i$. We define the edge set of G to be:

(1) all edges contained in G_i for each $i \in I$

(2) for distinct $\lambda, \beta \in \Lambda$ and for any $i, j \in I$, $a_\lambda + i$ is adjacent to $a_\beta + j$ if and only if $a_\lambda + I$ is adjacent to $a_\beta + I$ in $\Gamma(R/I)$ (i.e.,$a_\lambda a_\beta \in I$)

(3) for $\lambda \in \Lambda$ and distinct $i, j \in I, a_\lambda + i$ is adjacent to $a_\lambda + j$ if and only if $a_\lambda^2 \in I$.

Definition 2 ([14]). Using the notation as in the above construction, we call the subset $a_\lambda + I$ a column of $\Gamma_I(R)$. If $a_\lambda^2 \in I$, then we call $a_\lambda + I$ a connected column of $\Gamma_I(R)$.

Remark 3. Denote the vertices of $\Gamma(R/I)$ by $V(\Gamma(R/I)) = \{a_i + I : i \in \Lambda\}$. From Remark 2, we can denote the vertex set of $\Gamma_I(R)$ as $V(\Gamma_I(R)) = \{a_i + h : i \in \Lambda, h \in I\}$.

Theorem 2 (Vizing's Theorem [16, p.16]). *If G is a simple graph, then either $\chi'(G) = \Delta(G)$ or $\chi'(G) = \Delta(G) + 1$.*

Theorem 3 (Vizing's Adjacency Lemma [16, p.24]). *If G is a critical graph, then G has at least $\Delta(G) - \delta(G) + 2$ vertices of maximum degree.*

Theorem 4 ([9]). *The sum of the degrees of the points of a graph G is twice the number of lines.*

Theorem 5 (Konig's Theorem [16, p. 11]). *For any bipartite graph G, we have $\chi'(G) = \Delta(G)$.*

Theorem 6 ([14, Theorem 5.7]). *Let I be a nonzero ideal of a ring R. Then $\Gamma_I(R)$ is bipartite if and only if either (a) $\mathrm{gr}(\Gamma_I(R)) = \infty$ or (b) $\mathrm{gr}(\Gamma_I(R)) = 4$ and $\Gamma(R/I))$ is bipartite.*

Theorem 7 ([4, Theorem 2.8]). *Let R be a commutative ring. Then $\Gamma(R)$ is a complete graph if and only if either $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $xy = 0$ for every $x, y \in Z(R)$. In particular, if R is a reduced commutative ring and not a field, then $\Gamma(R)$ is a complete graph if and only if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Remark 4. We note that if G is a graph and $\chi'(G) = \Delta(G) + 1$, then there exist a subgraph of G say G_1, such that $\chi'(G_1) = \Delta(G) + 1$ and for any edge e of G_1 we have $\chi'(G - \{e\}) = \Delta(G)$. Clearly G_1 has a connected subgraph, say H, such that $\chi'(H) = \Delta(G) + 1$. The graph H is a critical graph with maximum degree $\Delta(G)$. If x is a vertex of H with degree $\Delta(G)$, then by Vizing's Adjacency Lemma, H has at least $\Delta(G) - \deg_H(v) + 2$ vertices of degree $\Delta(G)$, for any vertex v which is adjacent to x. Therefore if G is a graph such that for every vertex u of maximum degree there exist an edge uv such that $\Delta(G) - \deg(v) + 2$ is more than the number of vertices with maximum degree in G, then by the above argument and Vizing's Theorem, we have $\chi'(G) = \Delta(G)$.

Remark 5 ([2])**.** Assume that $R = R_1 \times R_2 \times \cdots \times R_n$ is a finite decomposable ring. We note that if $x = (x_1, x_2, \ldots, x_n)$ has a maximum degree in $\Gamma(R)$, then x has exactly one non-zero component, say x_1. Now suppose that R_1 is a local ring. We consider two cases: If R_1 is a field, then $\Delta(\Gamma(R)) = \deg(x) = |R_2| \cdots |R_n| - 1$. If R_1 is not field, then we have $x_1 \in \mathrm{Ann}(Z(R_1))^*$ and $\Delta(\Gamma(R)) = \deg(x) = |Z(R_1)||R_2| \cdots |R_n| - 2$.

Theorem 8 ([10, Lemma 4.1])**.** *Let I be an ideal of a ring R. Then in $\Gamma_I(R)$,*

$$\deg(a) = \begin{cases} |I| \deg_\Gamma(a + I) & \text{if } a^2 \notin I. \\ |I| \deg_\Gamma(a + I) + |I| - 1 & \text{if } a^2 \in I. \end{cases}$$

Theorem 9 ([10, Lemma 4.3])**.** *Let $I \neq (0)$ be an ideal of a ring R which is not prime. Then*

$$\Delta(\Gamma_I(R)) = \begin{cases} |I| - 1 & \text{if } \Gamma(R/I) \text{ has a single vertex,} \\ |I|\Delta(\Gamma(R/I)) + |I| - 1 & \text{if } \Gamma_I(R) \text{ has a connected} \\ & \text{column } a + I \text{ with} \\ & \deg_\Gamma(a + I) = \Delta(\Gamma(R/I)), \\ |I|\Delta(\Gamma(R/I)) & \text{otherwise.} \end{cases}$$

2. Edge property of $\Gamma_I(R)$

In this section, we discuss the number and nature of the edges of $\Gamma_I(R)$.

Theorem 10. *Let I be an ideal of a ring R and suppose $\Gamma_I(R)$ has k connected columns. Then $|E(\Gamma_I(R))| = |I|^2 |E(\Gamma(R/I))| + k \binom{|I|}{2}$.*

Proof. Let $a+I$ be a vertex of $\Gamma(R/I)$. Then by Theorem 8, in $\Gamma_I(R)$, $\deg(a+i)=\deg(a+j),$ for all $i,j\in I$. Now

$$\begin{aligned}\sum_{a,i}\deg(a+i) &= \sum_{a^2\notin I,\ i\in I}\deg(a+i)+\sum_{a^2\in I,\ i\in I}\deg(a+i)\\ &= |I|[\sum_{a^2\notin I}\deg(a+i)+\sum_{a^2\in I}\deg(a+i)]\\ &= |I|[\sum_{a^2\notin I}|I|\deg_\Gamma(a+I)+\sum_{a^2\in I}(|I|\deg_\Gamma(a+I)+|I|-1)]\\ &= |I|^2\sum_{a^2\notin I}\deg_\Gamma(a+I)+|I|^2\sum_{a^2\in I}\deg_\Gamma(a+I)+|I|\sum_{a^2\in I}(|I|-1)\\ &= |I|^2\sum_{a+I\in V(\Gamma(R/I)}\deg_\Gamma(a+I)+k|I|(|I|-1)\end{aligned}$$

By Theorem 4,

$$\begin{aligned}|E(\Gamma_I(R))| &= \frac{\sum\limits_{a\in V(\Gamma_I(R))}\deg(a)}{2}=\frac{\sum\limits_{a,i}\deg(a+i)}{2}\\ &= \frac{|I|^2\sum\limits_{a+I\in V(\Gamma(R/I))}\deg_\Gamma(a+I)+k|I|(|I|-1)}{2}.\end{aligned}$$

Thus $|E(\Gamma_I(R))|=|I|^2|E(\Gamma(R/I))|+k\binom{|I|}{2}$. □

Theorem 11. *Let I be an ideal of a ring R such that R/I is a finite ring which is not a field. Suppose $\Gamma_I(R)$ has k connected columns. Then $\Gamma_I(R)$ has an even number of edges if and only if one of the following holds:*

(i) $|I|\in 4\mathbb{Z}$.

(ii) $|I|=4m+1, m\in\mathbb{Z}$ *and* $\Gamma(R/I)$ *has even number of edges.*

(iii) $|I|\in 2\mathbb{Z}-4\mathbb{Z}$ *and* k *is even.*

(iv) $|I|=2m+1$ *($m\in\mathbb{Z}$ and m is odd),* $\Gamma(R/I)$ *has even number of edges and k is even.*

(v) $|I|=2m+1$ *($m\in\mathbb{Z}$ and m is odd),* $\Gamma(R/I)$ *has odd number of edges and k is odd.*

Proof. Assume that $\Gamma_I(R)$ is a graph with even number of edges. Then $|E(\Gamma_I(R))|$ is even. By Theorem 10, $|I|^2|E(\Gamma(R/I))|+k\binom{|I|}{2}$ is even.

Case 1: $k\binom{|I|}{2}$ is even. In this case $|I|^2|E(\Gamma(R/I))|$ is even. If k is odd, then $|I|=4m$ or $4m+1$. Suppose $|I|=4m$. Then (i) holds. Suppose not. Then $|I|=4m+1$. Since $|I|^2|E(\Gamma(R/I))|$ is also even, $|E(\Gamma(R/I))|$ is even and so (ii) holds.

If k is even and $\binom{|I|}{2}$ is odd, then $|I| = 2m$ or $2m+1$. Suppose $|I| = 2m$. Then (iii) holds. Suppose not. Then $|I| = 2m + 1$. Since $|I|^2|E(\Gamma(R/I))|$ is also even, $|E(\Gamma(R/I))|$ is even and so (iv) holds.

If k is even and $\binom{|I|}{2}$ is even, then $|I| = 4m$ or $4m + 1$. If $|I| = 4m$, then (i) holds. If not, then $|E(\Gamma(R/I))|$ is even, since $|I|^2|E(\Gamma(R/I))|$ is also even. So (ii) holds.

Case 2: $k\binom{|I|}{2}$ is odd. In this case, $|I|^2|E(\Gamma(R/I))|$ is odd. Now k and $\binom{|I|}{2}$ are odd. Since $|I|^2|E(\Gamma(R/I))|$ is odd, $|I| = 2m + 1$ and $|E(\Gamma(R/I))|$ is odd. Thus (vii) is true.

Converse is obvious. □

3. Edge chromatic number

In this section, we determine the edge chromatic number of $\Gamma_I(R)$ when R/I is a finite local ring or R/I is a finite decomposable ring.

Theorem 12. *Let I be an ideal of a ring R. If $\Gamma_I(R)$ is a graph with $V(\Gamma(R/I)) = \mathrm{Ann}(Z(R/I)^*)$, then*

$$\chi'(\Gamma_I(R)) = \begin{cases} \Delta(\Gamma_I(R)) + 1 & \text{if } |V(\Gamma(R/I))| \text{ is odd and } |I| \text{ is odd,} \\ \Delta(\Gamma_I(R)) & \text{otherwise.} \end{cases}$$

Proof. Since $V(\Gamma(R/I)) = \mathrm{Ann}(Z(R/I)^*)$, $\Gamma(R/I)$ is a complete graph and $a^2 \in I$ for all $a + I \in V(\Gamma_I(R))$. Also, $|V(\Gamma_I(R))| = |I||\Gamma(R/I)|$. Hence the result follows. □

If R/I is an Artinian local ring which is not a field, then the Jacobson radical of R/I equals $Z(R/I)$. Thus $Z(R/I)$ is a nilpotent ideal and this implies that if R/I is not a field, then $\mathrm{Ann}(Z(R/I)) \neq \{0\}$. Also, every element of $\mathrm{Ann}(Z(R/I))^*$ is adjacent to every other vertex of $\Gamma(R/I)$.

Theorem 13. *Let I be an ideal of a ring R such that R/I is a finite local ring which is not a field and $\Gamma(R/I)$ is a graph on at least two vertices. Then $\chi'(\Gamma_I(R)) = \Delta(\Gamma_I(R))$, unless $\Gamma(R/I)$ is a complete graph.*

Proof. Let k be the number of vertices of degree $\Delta(\Gamma_I(R))$. Then $\Gamma(R/I)$ has $\frac{k}{|I|}$ vertices of maximum degree $\Delta(\Gamma(R/I))$.

Suppose $\chi'(\Gamma_I(R)) = \Delta(\Gamma_I(R)) + 1$. Then by Remark 4, $\Gamma_I(R)$ has a critical subgraph H such that $\Delta(\Gamma_I(R)) - \deg_H(v) + 2 \leqslant k$, where $v \in N(u)$ such that $\deg(v) < \deg(u) = \Delta(\Gamma_I(R))$. So $\Delta(\Gamma_I(R)) - \deg(v) + 2 \leqslant k$. We have $\Delta(\Gamma_I(R)) - \deg(v) + 2 = |I|(\Delta(\Gamma(R/I)) - \deg_\Gamma(v + I) + 2) - |I| + 1$.

This implies that $\Delta(\Gamma(R/I)) - \deg_\Gamma(v+I) + 2 < \frac{k}{|I|}$. Since R/I is a finite local ring, $\operatorname{Ann}(Z(R/I)) \neq (0)$. Clearly $\Delta(\Gamma(R/I)) = |Z(R/I)^*| - 1$ and $|\operatorname{Ann}(Z(R/I))| - 1$ is the number of vertices of degree $\Delta(\Gamma(R/I))$. Hence

$$|Z(R/I)^*| - 1 - \deg_\Gamma(v+I) + 2 \leqslant |\operatorname{Ann}(Z(R/I))| - 1$$

and so $\deg_\Gamma(v+I) > |Z(R/I)^*| - |\operatorname{Ann}(Z(R/I))|$, which is a contradiction. Hence by Vizing's Theorem, $\chi'(\Gamma_I(R)) = \Delta(\Gamma_I(R))$. □

Theorem 14. *Let I be an ideal of a ring R. If $\Gamma(R/I)$ is a complete graph and $\Gamma_I(R)$ is a non-complete graph having no connected columns, then $\chi'(\Gamma_I(R)) = \Delta(\Gamma_I(R))$.*

Proof. Since $\Gamma(R/I)$ is complete graph, by Theorem 7, $R/I \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $(x+I)(y+I) = 0 + I$, for all $x + I \in Z(R/I)$. Since $\Gamma_I(R)$ has no connected column, $\Gamma(R/I) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and R/I is complete bipartite graph. Also $\operatorname{gr}(\Gamma_I(R)) = 4$. By Theorem 6, $\Gamma_I(R)$ is complete bipartite graph and by Theorem 5, $\chi'(\Gamma_I(R)) = \Delta(\Gamma_I(R))$. □

Note that if R is a finite commutative ring with identity, then $R = R_1 \times \cdots \times R_n$, where $n \geqslant 2$ and each (R_i, m_i) is a local ring, where $1 \leqslant i \leqslant n$.

Theorem 15. *Let I be an ideal of a ring R such that R/I is a finite decomposable ring. Then $\chi'(\Gamma_I(R)) = \Delta(\Gamma_I(R))$.*

Proof. Since R/I is finite $R/I = R_1 \times \cdots \times R_n$, where $n \geqslant 2$ and each R_i is a local ring. By Remark 5, without loss of generality suppose that the non-zero components of the vertices with maximum degree in $\Gamma(R/I)$ occur in $R_1, \ldots, R_k$.

Claim. Either $R_1, \ldots, R_k$ are fields or none of them are fields.

Suppose that R_1 is a field and R_2 is not a field. Now a vertex with maximum degree in $R_1 \times \{0\} \times \cdots \times \{0\}$ has degree $(|R_2| \cdots |R_n|) - 1$ and each vertex with maximum degree in $\{0\} \times R_2 \times \{0\} \times \cdots \times \{0\}$ has degree $(|R_1||Z(R_2)||R_3| \cdots |R_n|) - 2$. Thus we have $|Z(R_2)||R_3| \cdots |R_n|(|R_1| - |R_2/Z(R_2)|) = 1$, a contradiction.

By Remark 5, for any i, $1 \leqslant i \leqslant k$, $\Delta(\Gamma(R/I)) = (|R_1| \cdots |R_{i-1}||Z(R_i)| |R_{i+1}| \cdots |R_n) - \epsilon$, where $\epsilon = 1$ or 2. Hence $|R_1/Z(R_1)| = \cdots = |R_k/Z(R_k)|$. Since for each j, $k + 1 \leqslant j \leqslant n$, the degree of any vertex in $\{0\} \times \cdots \times \{0\} \times R_j \times \{0\} \times \cdots \times \{0\}$ is less than $\Delta(\Gamma(R/I))$. So we have

$$|R_j/Z(R_j)| \geqslant |R_1/Z(R_1)|. \tag{1}$$

For any t, $1 \leqslant t \leqslant n$, suppose that $e_t + I$ is the element whose t^{th} component is one and other components are zero.

Case 1: $R_1, \ldots, R_k$ are not fields. Now $\Gamma(R/I)$ has $\sum_{t=1}^{k} |\operatorname{Ann}(Z(R_t))^*|$ vertices of maximum degree. Every vertex $a+I$ of maximum degree in $\Gamma(R/I)$ is adjacent to at least one of the $e_t + I$'s. Since $\Gamma_I(R)$ has a connected column, $\Gamma_I(R)$ has $|I| \sum_{t=1}^{k} |\operatorname{Ann}(Z(R_t))^*|$ vertices of maximum degree and every vertex a of maximum degree is adjacent to at least one e_t. Now for any $i, 1 \leqslant i \leqslant n$, we have

$$\begin{aligned}
\Delta(\Gamma(R/I)) - \deg_\Gamma(e_i + I) + 2 &\geqslant (|R_1|\cdots|R_{i-1}||Z(R_i)||R_{i+1}|\cdots|R_n| - 2)\\
&\quad - (|R_1|\cdots|R_{i-1}||R_{i+1}|\cdots||R_n| - 1) + 2\\
&= |R_1|\cdots|R_{i-1}|(|Z(R_i)| - 1)|R_{i+1}|\cdots|R_n| + 1\\
&= |R_1|\cdots|R_{i-1}||Z(R_i)^*||R_{i+1}|\cdots|R_n| + 1\\
&\geqslant |R_1| + \cdots + |R_{i-1}| + |Z(R_i)^*| + |R_{i+1}| + \cdots + |R_n| + 1\\
&\geqslant |\operatorname{Ann}(Z(R_1))| + \cdots + |\operatorname{Ann}(Z(R_{i-1}))| + \cdots + |\operatorname{Ann}(Z(R_n))| + 1\\
&\geqslant \sum_{i=1}^{k} |\operatorname{Ann}(Z(R_i)^*)| + k + 1 \geqslant \sum_{i=1}^{k} |\operatorname{Ann}(Z(R_i)^*)| + 2.
\end{aligned}$$

Let $m = \sum_{i=1}^{k} |\operatorname{Ann}(Z(R_i)^*)|$. Then $\Delta(\Gamma(R/I)) - \deg_\Gamma(e_i + I) + 2 \geqslant m + 2$. Now

$$\begin{aligned}
\Delta(\Gamma_I(R)) - \deg(e_i) + 2 &= |I|\Delta(\Gamma(R/I)) + |I| - 1 - |I|\deg_\Gamma(e_i + I) + 2\\
&= |I|(\Delta(\Gamma(R/I)) - \deg_\Gamma(e_i + I) + 2) - 2|I| + |I| + 1\\
&\geqslant (m+2)|I| - |I| + 1 > m|I| > |I| \sum_{i=1}^{k} |\operatorname{Ann}(Z(R_i)^*)|.
\end{aligned}$$

Hence by Remark 4, we conclude that $\chi'(\Gamma_I(R)) = \Delta(\Gamma_I(R))$.

Case 2: $R_1, \ldots, R_k$ are fields. Then $\Gamma(R/I)$ has $\sum_{t=1}^{k} |R_t^*|$ vertices of maximum degree.

Subcase 1: $n > 2$. Then every vertex of maximum degree in $\Gamma(R/I)$ is adjacent to $1-(e_t+I)$, for some t, $1 \leqslant t \leqslant k$. In this case $|R_1| = \cdots = |R_k|$ and if we set $|R_1| = a$, then by (1) we have $|R_j| \geqslant a$ for any j, $j > k$. Now since $a^{n-1} - a + 2 > n(a-1)$, for any $i, 1 \leqslant i \leqslant k$, we have

$$\begin{aligned}
&\Delta(\Gamma(R/I)) - \deg_\Gamma(1 - e_i + I) + 2 \\
&\qquad = (|R_1| \cdots |R_{i-1}||R_{i+1}| \cdots |R_n| - 1) - (|R_i| - 1) + 2 \\
&\qquad \geqslant a^{n-1} - a + 2 > \sum_{t=1}^{k} |R_t^*|. \qquad (2)
\end{aligned}$$

In this case, $\Gamma_I(R)$ has no connected columns. Also, $|I| \sum_{t=1}^{k} |R_t^*|$ vertices have maximum degree in $\Gamma_I(R)$ and every vertex a of maximum degree is adjacent to $1 - e_i$, for some i, $1 \leqslant i \leqslant k$. Note that

$$\Delta(\Gamma_I(R)) - \deg(1 - e_i) + 2 = |I|\Delta(\Gamma(R/I)) - |I| \deg_\Gamma(e_i + I) + 2.$$

Take $m = |I| \sum_{t=1}^{k} |R_t^*|$. Suppose $\Delta(\Gamma_I(R)) - \deg(1 - e_i) + 2 \leqslant m$, where $1 \leqslant i \leqslant n$. Then $|I|(\Delta(\Gamma(R/I)) - \deg_\Gamma(1 - e_i + I) + 2) - 2|I| + 2 \leqslant m$ and so $\Delta(\Gamma(R/I)) - \deg_\Gamma(1 - e_i + I) + 2 \leqslant \frac{m}{|I|} + 1$. Hence

$$\Delta(\Gamma(R/I) - \deg_\Gamma(1 - e_i + I) + 2 \leqslant \sum_{t=1}^{k} |R_t^*| + 1. \qquad (3)$$

From (2),

$$\Delta(\Gamma(R/I)) - \deg_\Gamma(1 - e_i + I) + 2 = \sum_{t=1}^{k} |R_t^*| + 1.$$

This implies that

$$\begin{aligned}
\deg_\Gamma(1 - e_i + I) &= \Delta(\Gamma(R/I)) - \sum_{t=1}^{k} |R_t^*| + 1 \\
&= |R_1| \cdots |R_{i-1}||R_{i+1}| \cdots |R_n| - 1 - \sum_{t=1}^{k} |R_t^*| + 1 \\
&= |R_1| \cdots |R_{i-1}||R_{i+1}| \cdots |R_n| - \sum_{t=1}^{k} |R_t| + k \\
&\geqslant |R_1| + \cdots + |R_{i-1}| + |R_{i+1}| + \cdots + |R_n| - \sum_{t=1}^{k} |R_t| + k \\
&\geqslant \sum_{t=k+1}^{n} |R_t| - |R_i| + k \geqslant 2a - a + k > a
\end{aligned}$$

which contradicts that $\deg_\Gamma(1-e_t+I)=|R_i|-1=a-1$. Therefore,

$$\Delta(\Gamma(R/I))-\deg_\Gamma(1-e_i+I)+2>m=|I|\sum_{t=1}^{k}|R_t^*|.$$

By Remark 4, we conclude that $\chi^{'}(\Gamma_I(R))=\Delta(\Gamma_I(R))$.

Subcase 2: $n=2$. If $k=1$ and R_2 is not a field, then by (1) we have $|R_2|\geqslant 2|R_1|$. Also $\Gamma_I(R)$ has a connected column. Since in this case any vertex of maximum degree in $\Gamma(R/I)$ is adjacent to e_2+I, $\Delta(\Gamma(R/I))-\deg_\Gamma(e_2+I)+2=(|R_2|-1)-(|R_1|-1)+2>|R_1^*|$. Thus

$$\Delta(\Gamma(R/I))-\deg_\Gamma(e_2+I)+2\geqslant|R_1^*|+1.$$

If $\Delta(\Gamma(R/I))-\deg_\Gamma(e_2+I)+2=|R_1^*|+1$, then $\deg_\Gamma(e_2+I)=\Delta(\Gamma(R/I))-|R_1^*|+1\geqslant|R_1|+1$, which contradicts that $\deg_\Gamma(e_2+I)=|R_1|-1$. Thus

$$\Delta(\Gamma(R/I))-\deg_\Gamma(e_2+I)+2>|R_1^*|+1.$$

$$\begin{aligned}\Delta(\Gamma_I(R))-\deg(e_2)+2&=|I|(\Delta(\Gamma(R/I))-\deg_\Gamma(e_2+I)+2)-|I|+1\\&\geqslant|I|(|R_1^*|+2)-|I|+1>|I||R_1^*|.\end{aligned}$$

By Remark 4, $\chi^{'}(\Gamma_I(R))=\Delta(\Gamma_I(R))$. If either $k=1$ and R_2 is a field or $k=2$, then $\Gamma(R/I)$ is a complete bipartite graph and $\mathrm{gr}(\Gamma_I(R))=4$. By Theorem 6, $\Gamma_I(R)$ is a complete bipartite graph. Hence, by Theorem 5, we have $\chi^{'}(\Gamma_I(R))=\Delta(\Gamma_I(R))$ and the proof is complete. □

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