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# Labelling matrices and index matrices of a graph structure

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Abstract. The concept of graph structure was introduced by E. Sampathkumar in 'Generalised Graph Structures', Bull. Kerala Math. Assoc., Vol 3, No.2, Dec 2006, 65-123. Based on the works of Brouwer, Doob and Stewart, R.H. Jeurissen has ('The Incidence Matrix and Labelings of a Graph', J. Combin. Theory, Ser. B30 (1981), 290-301) proved that the collection of all admissible index vectors and the collection of all labellings for 0 form free $F$-modules ($F$ is a commutative ring). We have obtained similar results on graph structures in a previous paper. In the present paper, we introduce labelling matrices and index matrices of graph structures and prove that the collection of all admissible index matrices and the collection of all labelling matrices for 0 form free $F$-modules. We also find their ranks in various cases of bipartition and char $F$ (equal to 2 and not equal to 2).

## Introduction

E. Sampathkumar introduced the concept of graph structure in [9]. It is in particular, a generalisation of the notions like graphs [5], signed graphs [2], [11], [12] and edge-coloured graphs [6] with the colourings. He defined a graph structure $G$ as $G = (V, R_1, R_2, ...R_k)$ where $V$ is a non-empty set and $R_1, R_2, ..., R_k$ are relations on V which are mutually disjoint such that each $R_i, i = 1, 2, ..., k$ is symmetric and irreflexive.

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R.H. Jeurissen [8], based on the works of Brouwer [1], Doob [4] and Stewart [10], has proved some results using incidence matrices of graphs. He has defined index vectors and labelings and also admissible index vectors. He has proved that if $F$ is a commutative ring and $G$ a graph, the collection of all admissible index vectors and the collection of all labellings for 0 are free $F$-modules. He has also found out the ranks of these modules in various cases, namely, $G$ is bipartite, non-bipartite, char $F = 2$, char $F \neq 2$ etc.

We have introduced similar concepts in graph structures in [3]. Instead of index vectors and labellings, we have introduced $R_i$-index vectors and $R_i$-labellings there and proved some results. Here we are introducing labelling matrices and index matrices of a graph structure. We are also proving that the collection of all admissible index matrices and the collection of all labelling matrices for 0 form free $F$-modules. We are also finding their ranks in various cases, namely, completely bipartite, not $R_i$-bipartite for some $i$ s, char $F = 2$, char $F \neq 2$ etc.

## 1. Preliminaries

We first go through some concepts introduced in [9].

**Definition 1** ([9])**.** In a graph structure $G = (V, R_1, R_2, ..., R_k)$, if $(u, v) \in R_i$, $(u, v)$ is an $R_i$-edge.

**Definition 2** ([9])**.** An $R_i$-path between two vertices $u$ and $v$ is an alternating sequence of vertices and edges consisting only of $R_i$-edges.

**Definition 3** ([9])**.** A set $S$ of vertices in a graph structure $G = (V, R_1, R_2, ..., R_k)$ is $R_i$-connected for any given $i$ if any two vertices in $S$ are connected by an $R_i$-path.

Now we recall the concept of $R_i$-distance [3].

**Definition 4** ([3])**.** The minimum number of $R_i$-edges from a vertex $u$ to a vertex $v$ in any $R_i$-path of a graph structure $G = (V, R_1, R_2, ..., R_k)$ is called the $R_i$-distance from $u$ to $v$. It is the number of $R_i$-edges from a vertex $u$ to a vertex $v$ in an $R_i$-tree.

The incidence matrix of a graph structure is defined as follows in [9].

**Definition 5** ([9])**.** The incidence matrix $B$ of a graph structure $G = (V, R_1, R_2, ..., R_k)$ is a $k \times p$ matrix $b = (b_{ij})$ where $b_{ij}$ is the number of $R_i$-edges incident to the vertex $v_j$.

We have defined the $R_i$-incidence matrix of a graph structure in [3] similar to the incidence matrix of a graph as follows.

**Definition 6** ([3])**.** Let $G = (V, R_1, R_2, ..., R_k)$ be a graph structure. The $R_i$-incidence matrix of $G$ is defined as $I_{R_i} = (b_{ij})$, where $b_{rs} = 1$ if $v_r$ is incident with an $R_i$-edge $e_s$ and 0 otherwise.

First we select a spanning $R_i$-tree and label the $R_i$-edges inside and outside the spanning $R_i$-tree as in [3]. We recall the procedure.

Let $G$ be a finite $R_i$-connected graph structure on $p$ vertices with $q_i$ number of $R_i$- edges, $i = 1, 2, ..., k$. Choose a spanning $R_i$-tree $T_i$ as follows.

Let $v_0$ be the root. Number the vertices as $v_1, v_2, ...$ successively. First those at $R_i$-distance 1 from $v_0$, then those at $R_i$-distance 2 and so on (Those at same $R_i$-distance are numbered arbitrarily). Label the $R_i$-edge $(v_r, v_s)$ with $r < s$ in the spanning $R_i$-tree $T_i$ as $e_i^s$. Number the $R_i$-edges outside the spanning $R_i$-tree $T_i$ as $e_i^{p_i}, e_i^{p_i+1}, ..., e_i^{q_i}$.

Then the $T_i$-part of the $R_i$-incidence matrix will be of the form

$$\begin{array}{c} v_0 \\ v_1 \\ v_2 \\ . \\ . \\ . \\ . \\ . \\ . \\ . \\ . \\ v_{p-1} \end{array}
\begin{bmatrix}
1 & . & . & . & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & . & . & . & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & . & & & 0 & 1 & 0 & 0 & 0 & 0 & 1 & . & . & . & 1 \\
0 & & . & & 0 & 0 & . & & & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & & & . & 0 & 0 & & . & & 0 & 0 & . & & & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & & & . & 0 & 0 & & . & & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & & & . & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

Consider a vertex $v_s$ and let $v_s, v_{k_1}, v_{k_2}, ..., v_{k_r}, v_0$ be the unique $R_i$-path in $T_i$ from $v_s$ to $v_0$. Denote it by $c_i^r$. Then $c_i^r - c_i^{k_1} + c_i^{k_2} - ... + (-1)^s c_i^t = [(-1)^t 00...0]'$.

For each $I_{T_i}$, there exists an upper triangular $p-1 \times p-1$ matrix $D_i$ with elements $0, \pm 1$ and an all 1-diagonal such that

$$I_{T_i} D_i = \begin{bmatrix}
1 & \pm1 & \pm1 & . & . & . & \pm1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & & & & & 0 \\
0 & & . & & & & 0 \\
0 & & & . & & & 0 \\
0 & & & & . & & 0 \\
0 & & & & & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

The top row has +1 or -1 in $j$th position depending on whether $v_j$ is at odd or even $R_i$-distance from $v_0$.

Consider a column of $B_i$, the part of $I_{R_i}$ corresponding to the $R_i$-edges outside $T_i$. Suppose its 1's are in $j$th and $k$th rows with $j < k$. Subtracting $j$th and $k$th columns of $I_{T_i}D_i$ from it (if $j = 0$, only $k$), we get

$\begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$ if $R_i$-distance in $T_i$ from $v_j$ to $v_0$ is odd and that from $v_k$ to $v_0$ is even or if $R_i$ - distance in $T_i$ from $v_j$ to $v_0$ is even and that from $v_k$ to $v_0$ is odd.

$\begin{bmatrix} 2 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$ if $v_j$ and $v_k$ are at even $R_i$-distance from $v_0$

$\begin{bmatrix} -2 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$ if $v_j$ and $v_k$ are at odd $R_i$-distance from $v_0$.

Let $E_i$ be $(0, -1)$ matrix with one or two -1s in each column.

$(I_{T_i}D_i|B_i)\begin{bmatrix} I & E_i \\ 0 & I \end{bmatrix}$ gives

$$\left[\begin{array}{cccccccc|ccccccccccccc}
1 & \pm1 & \pm1 & . & . & . & \pm1 & & 0 & . & . & 0 & 2 & . & . & 2 & -2 & . & . & -2 \\
1 & 0 & 0 & . & . & . & 0 & & 0 & . & . & 0 & 0 & . & . & 0 & 0 & . & . & 0 \\
0 & 1 & 0 & . & . & . & 0 & & 0 & . & . & 0 & 0 & . & . & 0 & 0 & . & . & 0 \\
0 & 0 & 1 & . & . & . & 0 & & 0 & . & . & 0 & 0 & . & . & 0 & 0 & . & . & 0 \\
. & . & . & . & & & . & & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & & . & & . & & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & & & . & 0 & & 0 & . & . & 0 & 0 & . & . & 0 & 0 & . & . & 0 \\
0 & 0 & 0 & . & . & . & 1 & & 0 & . & . & 0 & 0 & . & . & 0 & 0 & . & . & 0
\end{array}\right]$$

after renumbering $R_i$-edges outside $T_i$.

The column operations on the $R_i$-incidence matrices will give the incidence matrix of the graph structure in the form

$$(I_G D|B)\begin{bmatrix} I & E \\ \Theta & I \end{bmatrix} = \begin{bmatrix} I_1 & \Theta & . & . & . & \Theta \\ \Theta & I_2 & \Theta & . & . & \Theta \\ . & \Theta & . & & & . \\ . & . & & . & & . \\ . & . & & & . & \Theta \\ \Theta & \Theta & . & . & \Theta & I_k \end{bmatrix}$$

where

$$I_G = \begin{bmatrix} I_{T_1}|B_1 & \Theta & . & . & . & \Theta \\ \Theta & I_{T_2}|B_2 & \Theta & . & . & \Theta \\ . & \Theta & . & & & . \\ . & . & & . & & . \\ . & . & & & . & \Theta \\ \Theta & \Theta & . & . & \Theta & I_{T_k}|B_k \end{bmatrix},$$

$$D = \begin{bmatrix} D_1 & \Theta & . & . & . & \Theta \\ \Theta & D_2 & \Theta & . & . & \Theta \\ . & \Theta & . & & & . \\ . & . & & . & & . \\ . & . & & & . & \Theta \\ \Theta & \Theta & . & . & \Theta & D_k \end{bmatrix}, \quad E = \begin{bmatrix} E_1 & \Theta & . & . & . & \Theta \\ \Theta & E_2 & \Theta & . & . & \Theta \\ . & \Theta & . & & & . \\ . & . & & . & & . \\ . & . & & & . & \Theta \\ \Theta & \Theta & . & . & \Theta & E_k \end{bmatrix}$$

and each $I_i$ has the form of

$$(I_{T_i} D_i | B_i) \begin{bmatrix} I & E_i \\ 0 & I \end{bmatrix}.$$

## 2. Labelling matrix and index matrix

We first recall the concepts of $R_i$-labelling and $R_i$-index vector introduced in [3].

**Definition 7** ([3]). Let $F$ be an abelian group or a ring and $G = (V, R_1, R_2, ..., R_k)$ be a graph structure with vertices $v_0, v_1, ..., v_{p-1}$ and $q_i$ number of $R_i$-edges. A mapping from $V$ to $F$ is an $R_i$-index vector and a mapping from $R_i$ to $F$ is an $R_i$-labelling.

**Definition 8** ([3]). An $R_i$-labelling $x_i$ is an $R_i$-labelling for the $R_i$-index vector $r_i$ if for each $j$, $r_i(v_j) = \sum_{e_r \in E_i^j} x_i(e_r)$, where $E_i^j$ is the set of all $R_i$-edges incident with $v_j$.

**Definition 9** ([3]). $R_i$-index vectors for which an $R_i$-labelling exists are called admissible $R_i$-index vectors.

Now we introduce the concepts of labelling matrix and index matrix of a graph structure.

**Definition 10.** Let $F$ be an abelian group or a ring. Let $r_i$ be an $R_i$-index vector and $x_i$ be an $R_i$-labelling for $i = 1, 2, ..., k$. Then

a) $r = \begin{bmatrix} r_1 & 0 & . & . & . & 0 \\ 0 & r_2 & 0 & . & . & 0 \\ . & 0 & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ 0 & . & & . & 0 & r_k \end{bmatrix}$ is defined to be an index matrix for $G$

b) $x = \begin{bmatrix} x_1 & 0 & . & . & . & 0 \\ 0 & x_2 & 0 & . & . & 0 \\ . & 0 & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ 0 & . & & . & 0 & x_k \end{bmatrix}$ is defined to be a labelling matrix for $G$.

**Definition 11.** The map

$$x : \begin{bmatrix} R_1 \\ R_2 \\ . \\ . \\ . \\ R_k \end{bmatrix} \to F^k$$

is a labelling for $r : V^k \to F^k$ if $\sum_{m \in E_s} x_i(m) = r_i(x_s)$ for $s = 0, 1, 2, ..., p - 1; i = 1, 2, ..., k$.

Now we prove a necessary and sufficient condition for $x$ to be a labelling matrix for $r$. For that first we recall the following lemma of [3].

**Lemma 1** ([3])**.** *Let $G = (V, R_1, R_2, ..., R_k)$ be a graph structure. $x_i$ is an $R_i$-labelling for $R_i$-index vector $r_i$ iff $I_{R_i} x_i = r_i$.*

Now we move on to prove the condition for $x$ to be a labelling matrix for $r$.

**Lemma 2.** *Let $G = (V, R_1, R_2, ..., R_k)$ be a graph structure. $x$ is a labelling for $r$ iff $I_G x = r$ where*

$$I_G = \begin{bmatrix} I_{R_1} & 0 & . & . & . & 0 \\ 0 & I_{R_2} & 0 & . & . & 0 \\ . & 0 & . & & & . \\ . & . & & . & & . \\ . & . & & & . & 0 \\ 0 & 0 & . & . & 0 & I_{R_k} \end{bmatrix}.$$

*Proof.* Let $x$ be a labelling for $r$. Then by Lemma 2,

$$I_G x = \begin{bmatrix} I_{R_1} & 0 & . & . & . & 0 \\ 0 & I_{R_2} & 0 & . & . & 0 \\ . & 0 & . & & & . \\ . & . & & . & & . \\ . & . & & & . & 0 \\ 0 & 0 & . & . & 0 & I_{R_k} \end{bmatrix} \begin{bmatrix} x_1 & 0 & . & . & . & 0 \\ 0 & x_2 & 0 & . & . & 0 \\ . & 0 & . & & & . \\ . & . & & . & & . \\ . & . & & & . & 0 \\ 0 & 0 & . & . & 0 & x_k \end{bmatrix}$$

$$= \begin{bmatrix} I_{R_1}x_1 & 0 & . & . & . & 0 \\ 0 & I_{R_2}x_2 & 0 & . & . & 0 \\ . & 0 & . & & & . \\ . & . & & . & & . \\ . & . & & & . & 0 \\ 0 & 0 & . & . & 0 & I_{R_k}x_k \end{bmatrix} = \begin{bmatrix} r_1 & 0 & . & . & . & 0 \\ 0 & r_2 & 0 & . & . & 0 \\ . & 0 & . & & & . \\ . & . & & . & & . \\ . & . & & & . & 0 \\ 0 & 0 & . & . & 0 & r_k \end{bmatrix} = r.$$

Conversely, let $I_G x = r$. Then $I_{R_i} x_i = r_i$ for $i = 1, 2, ..., k$. So $x_i$ is a labelling for $r_i, i = 1, 2, ..., k$. ie., $x$ is a labelling for $r$. □

**Definition 12.** Let $G = (V, R_1, R_2, ..., R_k)$ be a graph structure. If $r_i$ is an admissible $R_i$-index vector for $i = 1, 2, ..., k$, then

$$r = \begin{bmatrix} r_1 & 0 & . & . & . & 0 \\ 0 & r_2 & 0 & . & . & 0 \\ . & 0 & . & & & . \\ . & . & & . & & . \\ . & . & & & . & 0 \\ 0 & 0 & . & . & 0 & r_k \end{bmatrix}$$

is called an admissible index matrix for $G$.

Now we recall the definition of complete bipartition given in [9].

**Definition 13** ([9])**.** A graph structure $G$ is completely bipartite if for each $i, 1 \leq i \leq k$, $G$ is $R_i$-bipartite (with same sets of bipartition).

Now we go through some of the results proved in [3].

**Theorem 1** ([3])**.** *If $F$ is an abelian group and $v_1, v_2, ..., v_p$ are vertices of an $R_i$-bipartite graph structure $G$, then $r_i$ is an admissible $R_i$-index vector iff* $\sum_{j=1}^{s} r_i(v_j) = \sum_{j=s+1}^{p} r_i(v_j)$ *where* $S = \{v_1, v_2, ..., v_s\}$ *and* $U = \{v_{s+1}, v_{s+2}, ..., v_p\}$ *are the sets of $R_i$-bipartition.*

**Theorem 2** ([3]). *If $F$ is an abelian group and $v_1, v_2, ..., v_p$ are the vertices of a non-$R_i$-bipartite graph structure $G$, then $r_i$ is an admissible $R_i$-index vector iff $\sum_{j=1}^{p} r_i(v_j) = 2f_i$, for some $f_i \in F$.*

We now prove certain preliminary results.

**Theorem 3.** *Let $G = (V, R_1, R_2, ..., R_k)$ be a complete bipartite graph structure. If $F$ is an abelian group and $v_1, v_2, ..., v_p$ are vertices of $G$, then $r$ is an admissible index matrix iff $\sum_{j=1}^{s_i} r_i(v_j) = \sum_{j=s_i+1}^{p} r_i(v_j)$, for $i = 1, 2, ..., k$ where $S = \{v_1, v_2, ..., v_{s_i}\}$ and $U = \{v_{s_i+1}, v_{s_i+2}, ..., v_p\}$ are sets of bipartition.*

*Proof.* Let $r$ be admissible. Then $r_i$ is an admissible $R_i$-index vector for $i = 1, 2, ..., k$. $G$ is $R_i$-bipartite for $i = 1, 2, ..., k$. Therefore by Theorem 1,

$$\sum_{j=1}^{s_i} r_i(v_j) = \sum_{j=s_i+1}^{p} r_i(v_j) = \sum_{j=s_i+1}^{p} r_i(v_j)$$

for $i = 1, 2, ..., k$.

Conversely, let

$$\sum_{j=1}^{s_i} r_i(v_j) = \sum_{j=s_i+1}^{p} r_i(v_j) = \sum_{j=s_i+1}^{p} r_i(v_j)$$

for $i = 1, 2, ..., k$.
Then by Theorem 1, $r_i$ is an $R_i$-admissible index vector for $i = 1, 2, ..., k$. So $r$ is admissible. □

**Theorem 4.** *Let $G = (V, R_1, R_2, ..., R_k)$ be a graph structure. If $F$ is an abelian group and $v_1, v_2, ..., v_p$ are the vertices of a graph structure $G$ which is not $R_i$-bipartite for $i = 1, 2, ..., k$, then $r$ is admissible iff $\sum_{j=1}^{p} r_i(v_j) = 2f_i$ for some $f_i \in F, i = 1, 2, ..., k$.*

*Proof.* Let $r$ be admissible. Then $r_i$ is an admissible $R_i$-index vector, $i = 1, 2, ..., k$. $G$ is not $R_i$-bipartite for $i = 1, 2, ..., k$. Therefore by Theorem 2, $\sum_{j=1}^{p} r_i(v_j) = 2f_i$ for some $f_i \in F, i = 1, 2, ..., k$.

Conversely, let $\sum_{j=1}^{p} r_i(v_j) = 2f_i$ for some $f_i \in F, i = 1, 2, ..., k$. Then by Theorem 2, $r_i$ is an admissible $R_i$-index vector for $i = 1, 2, ..., k$. Therefore $r$ is admissible. □

Now we move on to prove that the collection of all admissible index matrices for a graph structure form a free $F$-module and find its rank.

**Theorem 5.** *Let $G = (V, R_1, R_2, ..., R_k)$ be a graph structure. If $F$ is an integral domain, the admissible index matrices for $G$ form a free $F$-module. Its rank is*

(i) *$k(p-1)$ if $G$ is completely bipartite or char $F = 2$.*
(ii) *$kp$ if $G$ is not $R_i$-bipartite for $i = 1, 2, ..., k$ and char $F \neq 2$.*
(iii) *$kp$ if $G$ is not $R_i$-bipartite except for $i = i_1, i_2, ..., i_r$, char $F \neq 2$ and 2 is invertible.*
(iv) *$kp - r$ if $G$ is not $R_i$-bipartite except for $i = i_1, i_2, ..., i_r$, char $F \neq 2$ and 2 is not invertible.*

*Proof.* Index matrices of $G$ belong to $F^{kp\times k}$. Let $A =$ set of all admissible index matrices. It is a subset of $F^{kp\times k}$.
Let $r, s \in A$. Then $r_i, s_i$ are admissible $R_i$-index vectors for $i = 1, 2, ..., k$. Therefore there exists $R_i$-labellings $x_{r_i}, x_{s_i}$, for $r_i$ and $s_i$ for $i = 1, 2, ..., k$, and we have

$$r - s = \begin{bmatrix} r_1 & 0 & 0 & . & . & 0 \\ 0 & r_2 & 0 & . & . & 0 \\ 0 & 0 & . & & & . \\ . & . & & . & & . \\ . & . & & & . & 0 \\ 0 & 0 & . & . & 0 & r_k \end{bmatrix} - \begin{bmatrix} s_1 & 0 & 0 & . & . & 0 \\ 0 & s_2 & 0 & . & . & 0 \\ 0 & 0 & . & & & . \\ . & . & & . & & . \\ . & . & & & . & 0 \\ 0 & 0 & . & . & 0 & s_k \end{bmatrix}$$

$$= \begin{bmatrix} r_1 - s_1 & 0 & 0 & . & . & 0 \\ 0 & r_2 - s_2 & 0 & . & . & 0 \\ 0 & 0 & . & & & . \\ . & . & & . & & . \\ . & . & & & . & 0 \\ 0 & 0 & . & . & 0 & r_k - s_k \end{bmatrix}.$$

$$I_G(x_r - x_s) = \begin{bmatrix} I_{R_1} & 0 & 0 & . & . & 0 \\ 0 & I_{R_2} & 0 & . & . & 0 \\ 0 & 0 & . & & & . \\ . & . & & . & & . \\ . & . & & & . & 0 \\ 0 & 0 & . & . & 0 & I_{R_k} \end{bmatrix}$$

$$\left\{\begin{bmatrix} x_{r_1} & 0 & 0 & . & . & 0 \\ 0 & x_{r_2} & 0 & . & . & 0 \\ 0 & 0 & . & & & . \\ . & . & & . & & . \\ . & . & & & . & 0 \\ 0 & 0 & . & . & 0 & x_{r_k} \end{bmatrix} - \begin{bmatrix} x_{s_1} & 0 & 0 & . & . & 0 \\ 0 & x_{s_2} & 0 & . & . & 0 \\ 0 & 0 & . & & & . \\ . & . & & . & & . \\ . & . & & & . & 0 \\ 0 & 0 & . & . & 0 & x_{s_k} \end{bmatrix}\right\}$$

$$= \begin{bmatrix} r_1 - s_1 & 0 & 0 & . & . & 0 \\ 0 & r_2 - s_2 & 0 & . & . & 0 \\ 0 & 0 & . & & & . \\ . & . & & . & & . \\ . & . & & & . & 0 \\ 0 & 0 & . & . & 0 & r_k - s_k \end{bmatrix},$$

that is, $x_r - x_s$ is a labelling for $r - s$ since $x_{r_i} - x_{s_i}$ is a labelling for $r_i - s_i$ for $i = 1, 2, ..., k$. So $r - s \in A$.

Let $f \in F, r \in A$. Then $r_i$ is $R_i$-admissible for $i = 1, 2, ..., k$. There exists an $R_i$-labelling $x_i$ for $r_i, i = 1, 2, ..., k$. $fx_i$ is a labelling for $fr_i, i = 1, 2, .., k$. So $fx$ is a labelling for $fr$ and hence $fr \in A$. Therefore $A$ is an $F$-module and it is a submodule of $F^{kp \times k}$.

**Case 1: $G$ is completely bipartite**

$r$ is admissible iff $\sum_{j=1}^{s} r_i(v_j) = \sum_{j=s+1}^{p} r_i(v_j)$ for $i = 1, 2, ..., k$ by Theorem 3.

$$r_i(v_p) = r_i(v_1) + ... + r_i(v_s) - r_i(v_{s+1}) - ... - r_i(v_{p-1}).$$

$$r = \begin{bmatrix} r_1(v_1) & 0 & . & . & . & 0 \\ r_1(v_2) & 0 & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ r_1(v_p) & 0 & . & . & . & . \\ 0 & r_2(v_1) & . & . & . & . \\ . & . & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & r_2(v_p) & . & . & . & . \\ . & 0 & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & 0 \\ . & . & . & . & . & r_k(v_1) \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & 0 & r_k(v_p) \end{bmatrix} = \begin{bmatrix} r_1(v_1) & 0 & . & . & . & 0 \\ 0 & 0 & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & . & . & . & . & . \\ r_1(v_1) & . & . & . & . & . \\ 0 & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & 0 \end{bmatrix}$$

$$
+\begin{bmatrix}
0 & 0 & . & . & . & 0\\
r_1(v_2) & 0 & . & . & . & 0\\
. & . & . & . & . & .\\
. & . & . & . & . & .\\
0 & . & . & . & . & .\\
r_1(v_2) & . & . & . & . & .\\
0 & . & . & . & . & .\\
. & . & . & . & . & .\\
. & . & . & . & . & .\\
. & . & . & . & . & .\\
. & . & . & . & . & .\\
. & . & . & . & . & .\\
. & . & . & . & . & .\\
. & . & . & . & . & .\\
. & . & . & . & . & .\\
0 & 0 & . & . & . & 0
\end{bmatrix}
+\ldots+
\begin{bmatrix}
0 & 0 & . & . & . & 0\\
0 & 0 & . & . & . & 0\\
. & . & . & . & . & .\\
0 & . & . & . & . & .\\
r_1(v_{p-1}) & . & . & . & . & .\\
-r_1(v_{p-1}) & . & . & . & . & .\\
0 & . & . & . & . & .\\
. & . & . & . & . & .\\
. & . & . & . & . & .\\
. & . & . & . & . & .\\
. & . & . & . & . & .\\
. & . & . & . & . & .\\
. & . & . & . & . & .\\
. & . & . & . & . & .\\
. & . & . & . & . & .\\
0 & 0 & . & . & . & 0
\end{bmatrix}
$$

$$
+\ldots+
\begin{bmatrix}
0 & 0 & . & . & . & 0\\
0 & 0 & . & . & . & 0\\
. & . & . & . & . & .\\
. & . & . & . & . & .\\
. & . & . & . & . & .\\
. & . & . & . & . & .\\
. & . & . & . & . & .\\
. & . & . & . & . & .\\
. & . & . & . & . & .\\
. & . & . & . & . & 0\\
. & . & . & . & . & r_k(v_1)\\
. & . & . & . & . & 0\\
. & . & . & . & . & .\\
. & . & . & . & . & .\\
. & . & . & . & . & 0\\
0 & 0 & . & . & 0 & -r_k(v_1)
\end{bmatrix}
+\ldots+
\begin{bmatrix}
0 & 0 & . & . & . & 0\\
0 & 0 & . & . & . & 0\\
. & . & . & . & . & .\\
. & . & . & . & . & .\\
. & . & . & . & . & .\\
. & . & . & . & . & .\\
. & . & . & . & . & .\\
. & . & . & . & . & .\\
. & . & . & . & . & .\\
. & . & . & . & . & .\\
. & . & . & . & . & .\\
. & . & . & . & . & .\\
. & . & . & . & . & .\\
. & . & . & . & . & 0\\
. & . & . & . & . & r_k(v_{p-1})\\
0 & 0 & . & . & 0 & -r_k(v_{p-1})
\end{bmatrix}
$$

Denote the above $k(p-1)$ matrices by

$$r_{11}, r_{12}, ..., r_{1(p-1)}, ..., r_{k1}, ..., r_{k(p-1)}.$$

Let $f_{11}, ..., f_{k(p-1)}$ be elements of $F$ and let $f_{11}r_{11} + ... + f_{k(p-1)}r_{k(p-1)} = 0$. Then $f_{11}r_{11} = ... = f_{k(p-1)}r_{k(p-1)} = 0$. Therefore $f_{11} = f_{12} = ... = f_{k(p-1)} = 0$ and so $r_{11}, r_{12}, ..., r_{k(p-1)}$ are linearly independent. Therefore $A$ is a free $F$-module of rank $k(p-1)$.

**Case 2: char $F = 2$**

The case when $G$ is completely bipartite has already been discussed. So it is enough if we consider $G$ to be not completely bipartite.

Two subcases arise.

**Subcase 1: $G$ is not $R_i$-bipartite for $i = 1, 2, ..., k$.**

By theorem 4, $r$ is admissible iff $\sum_{j=1}^{p} r_i(v_j) = 2f_i$ for some $f_i \in F, i = 1, 2, ..., k$. Since char $F = 2$, $\sum_{j=1}^{p} r_i(v_j) = 0, i = 1, 2, ..., k$. Therefore $r_i(v_p) = -r_i(v_1) - r_i(v_2) - ... - r_i(v_{p-1})$, $i = 1, 2, ..., k$. So we can write

$$r = \begin{bmatrix} r_1(v_1) & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ -r_1(v_1) & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \cdot & \cdot & \cdot & 0 \\ r_1(v_2) & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ -r_1(v_2) & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & 0 \end{bmatrix} + ...$$

$$+ \begin{bmatrix} 0 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ r_1(v_{p-1}) & \cdot & \cdot & \cdot & \cdot & \cdot \\ -r_1(v_{p-1}) & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & 0 \end{bmatrix} + ... + \begin{bmatrix} 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & r_k(v_1) \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & 0 & -r_k(v_1) \end{bmatrix} + ...$$

$$+ \begin{bmatrix} 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & r_k(v_{p-1}) \\ 0 & 0 & \cdot & \cdot & 0 & -r_k(v_{p-1}) \end{bmatrix}$$

Let these elements be denoted by $r_{11}, r_{12}, ..., r_{k1}, ..., r_{k(p-1)}$ and let $f_{11}, f_{12}, ..., f_{1(p-1)}, ..., f_{k1}, ..., f_{k(p-1)} \in F$.

If

$$f_{11}r_{11} + f_{12}r_{12} + ... + f_{1(p-1)}r_{1(p-1)} + ... + f_{k1}r_{k1} + ... + f_{k(p-1)}r_{k(p-1)} = 0,$$

then $f_{11} = f_{12} = ... = f_{1(p-1)} = ... = f_{k1} = ... = f_{k(p-1)} = 0$ and hence $r_{11}, r_{12}, ..., r_{k1}, ..., r_{k(p-1)}$ are linearly independent. Therefore $A$ is a free $F$-module with rank $k(p-1)$.

**Subcase 2:** $G$ is $R_i$-bipartite for $i = i_1, i_2, ..., i_r$; $i_1, i_2, ..., i_r \in \{1, 2, ..., k\}$, $1 < r < k$. For $i = i_1, i_2, ..., i_r$, $r_i$ is admissible iff

$$\sum_{j=1}^{s} r_i(v_j) = \sum_{j=s+1}^{p} r_i(v_j)$$

by Theorem 1, that is,

$$r_i(v_p) = r_i(v_1) + r_i(v_2) + ... + r_i(v_s) - r_i(v_{s+1}) - ... - r_i(v_{p-1})$$

for $i = i_1, i_2, ..., i_r$. For $i \neq i_1, i_2, ..., i_r$, $r_i$ is admissible iff

$$\sum_{j=1}^{p} r_i(v_j) = 2f_i = 0$$

by Theorem 2, that is, for $i \neq i_1, i_2, ..., i_r, r_i(v_p) = -r_i(v_1) - r_i(v_2) - ... - r_i(v_{p-1})$. Therefore we can write $r$ as a linear combination of $k(p-1)$ matrices of which $kr$ matrices have the form

$$\begin{bmatrix}
0 & 0 & . & 0 & . & 0 \\
0 & 0 & . & . & . & 0 \\
. & . & . & . & . & . \\
. & . & . & . & . & . \\
. & . & . & . & . & . \\
. & . & . & 0 & . & . \\
. & . & . & r_i(v_j) & . & . \\
. & . & . & 0 & . & . \\
. & . & . & . & . & . \\
. & . & . & . & . & . \\
. & . & . & 0 & . & . \\
. & . & . & \pm r_i(v_j) & . & . \\
. & . & . & 0 & . & . \\
. & . & . & . & . & . \\
. & . & . & . & . & . \\
. & . & . & . & . & . \\
. & . & . & . & . & . \\
. & . & . & . & . & . \\
. & . & . & . & . & . \\
0 & 0 & . & 0 & . & 0
\end{bmatrix}$$

and the remaining have the form

$$\begin{bmatrix} 0 & 0 & . & 0 & . & 0 \\ 0 & 0 & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & 0 & . & . \\ . & . & . & r_i(v_j) & . & . \\ . & . & . & 0 & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & 0 & . & . \\ . & . & . & -r_i(v_j) & . & . \\ . & . & . & 0 & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & 0 & . & 0 \end{bmatrix}$$

Clearly they are linearly independent. Therefore $A$ is a free $F$-module of rank $k(p-1)$.

**Case 3: $G$ is not $R_i$-bipartite for $i = 1, 2, ..., k$ and char $F \neq 2$**

**Subcase 1: $G$ is not $R_i$-bipartite for $i = 1, 2, ..., k$ and char $F \neq 2$**

***a) 2 is invertible***

$$\sum_{j=1}^{p} r_i(v_j) = f_i = (2.2^{-1} f_i) = 2(2^{-1} f_i)$$

for some $f_i \in F, i = 1, 2, ..., k$, i.e., $\sum\limits_{j=1}^{p} r_i(v_j) = 2f_i'$, $f_i' = 2^{-1} f_i \in F$ for $i = 1, 2, ..., k$. Therefore $r_i$ is $R_i$-admissible for $i = 1, 2, ..., k$ by Theorem 2. So $r$ is admissible. Hence rank of $A$ is rank of $F^{kp \times k} = kp$.

***b) 2 is not invertible***

Consider

$$s_{11} = \begin{bmatrix} 2 & 0 & . & . & . & 0 \\ 0 & 0 & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & 0 \end{bmatrix}, s_{12} = \begin{bmatrix} 0 & 0 & . & . & . & 0 \\ 2 & 0 & . & . & . & 0 \\ 0 & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & 0 \end{bmatrix}, ...,$$

$$s_{1p} = \begin{bmatrix} 0 & 0 & . & . & . & 0 \\ 0 & 0 & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & . & . & . & . & . \\ 2 & . & . & . & . & . \\ 0 & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & 0 \end{bmatrix}, ..., s_{k1} = \begin{bmatrix} 0 & 0 & . & . & . & 0 \\ 0 & 0 & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & 0 \\ . & . & . & . & . & 2 \\ . & . & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & 0 \end{bmatrix}, ...,$$

$$s_{kp} = \begin{bmatrix} 0 & 0 & . & . & . & 0 \\ 0 & 0 & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & 0 \\ 0 & 0 & . & . & . & 2 \end{bmatrix}$$

Each $s_{ij}$ is admissible by Theorem 2. Also $s_{ij} \in A, i = 1, 2, ..., k; j = 1, 2, ..., p$. Let $f_{11}, f_{12}, ..., f_{1p}, ..., f_{k1}, ..., f_{kp} \in F$ and $f_{11}s_{11} + f_{12}s_{12} + ... + f_{kp}s_{kp} = 0$. Then $f_{11} = f_{12} = ... = f_{kp} = 0$ since char $F \neq 2$. Therefore

$s_{11}, s_{12}, ..., s_{kp}$ are linearly independent and hence $A$ is a free $F$-module with rank equal to the rank of $F^{kp \times k} = kp$.

**Subcase 2: $G$ is $R_i$-bipartite for $i = i_1, i_2, ..., i_r, 1 < r < k$ and not $R_i$-bipartite for $i = i_{r+1}, i_{r+2}, ..., i_k$ where $i_1, i_2, ..., i_k \in \{1, 2, ..., k\}$**

***a) 2 is invertible***

Then $\sum_{j=1}^{p} r_i(v_j) = f_i = 2f_i'$ where $f_i' = 2^{-1} f_i \in F$.

So $r_i$ is an admissible $R_i$-index vector for $i = 1, 2, ..., k$ and hence $r$ is admissible by Theorem 2. Therefore rank of $A$ is equal to rank of $F^{kp \times k} = kp$.

***b) 2 is not invertible***

As in subcase 1 b, define $s_{11}, ..., s_{kp}$. Then $s_{ij} \in A, i = 1, 2, ..., k; j = 1, 2, ..., p$. But for $i = i_1, i_2, ..., i_r$, $s_{ip} = s_{i1} + s_{i2} + ... + s_{is} - s_{i(s+1)} - ... - s_{i(p-1)}$. Therefore rank of $A$ is equal to rank of $F^{kp \times k} - r = kp - r$. □

Now we prove that the collection of all labelling matrices for 0 form a free $F$-module and find its rank. For that, first we recall a theorem from [3].

**Theorem 6** ([3])**.** *Let $G = (V, R_1, R_2, ..., R_k)$ be a graph structure. If $F$ is an integral domain, the $R_i$-labelling of $G$ for the $R_i$-index vector 0 form a free $F$-module. Its rank is $q_i - p + 1$ if $G$ is $R_i$-bipartite or char $F = 2$ and $q_i - p$ if $G$ is not $R_i$-bipartite and char $F \neq 2$.*

**Theorem 7.** *Let $G = (V, R_1, R_2, ..., R_k)$ be a graph structure. If $F$ is an integral domain, the labellings of $G$ for the index matrix 0 form a free $F$-module. Its rank is*

(i) $q_1 + q_2 + ... + q_k - k(p-1)$ *if $G$ is completely bipartite or char $F = 2$,*
(ii) $q_1 + q_2 + ... + q_k - kp$ *if $G$ is not $R_i$-bipartite for $i = 1, 2, ..., k$ and char $F \neq 2$,*
(iii) $q_1 + q_2 + ... + q_k - kp + r$ *if $G$ is $R_i$-bipartite for $i = i_1, i_2, ..., i_r$ and char $F \neq 2$.*

*Proof.* Define $\phi : F^{(q_1+q_2+...+q_k) \times k} \to \{I_G x\}$ as $\phi(x) = I_G x$. Let $x, y \in F^{(q_1+q_2+...+q_k) \times k}$.

$\phi(x + y) = I_G(x + y) = I_G x + I_G y = \phi(x) + \phi(y)$,
For $f \in F, x \in F^{(q_1+q_2+...+q_k) \times k}$, $\phi(fx) = I_G(fx) = f(I_G x) = f\phi(x)$.
Therefore $\phi$ is a homomorphism. It is onto also. Let
$K = \{x \in F^{(q_1+q_2+...+q_k) \times k} : I_G x = 0\}$. It is the kernel of $\phi$. Clearly it is

a submodule of $F^{(q_1+q_2+...+q_k)\times k}$. Hence $K$ is an $F$-module.
$r$ is admissible iff $I_G x = r$ by Lemma 2. ie., iff $(I_G C)(C^{-1}x) = r$. So 0 is admissible iff $(I_G C)(C^{-1}x) = 0$. ie., iff $(I_G C)y = 0$ where $y = C^{-1}x$.
Therefore a general solution for $y$ pre-multiplied by $C$ is a labelling for 0.
A general solution for $y$ for $r = 0$ has the form

$$S = \begin{bmatrix} \Theta & A_1 & 0 & . & . & . & 0 & \\ \Theta & 0 & A_2 & 0 & . & . & . & 0 \\ . & . & & . & & & & . \\ . & . & & & . & & & . \\ . & . & & & & . & & . \\ \Theta & . & . & . & . & . & . & A_k \end{bmatrix}$$

where $\Theta$ is a zero matrix of suitable order,

$$A_1 = \begin{bmatrix} \alpha_p & . & . & . & \alpha_{s_1} & . & . & . & \alpha_{t_1} & . & . & . & \alpha_{q_1} \end{bmatrix},$$
$$..., A_k = \begin{bmatrix} \alpha_p & . & . & . & \alpha_{s_k} & . & . & . & \alpha_{t_k} & . & . & . & \alpha_{q_k}. \end{bmatrix}$$

Also

$$2(\alpha_{t_i+1} + \alpha_{t_i+2} + ... + \alpha_{s_i+1} - \alpha_{s_i} - ... - \alpha_{q_i}) = 0 \qquad (**)$$

for $i = 1, 2, ..., k$. So the number of zeroes in the top row of each $I_{R_i} C_i$ is $t_i - p + 1$.
Any labelling has the form $CS$. Therefore if $x$ is a labelling,

$$x = C\{ \begin{bmatrix} \Theta & A_{1p} & 0 & . & . & . & . & \Theta \\ \Theta & \Theta & . & . & . & . & . & \Theta \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Theta & . & . & . & . & . & . & \Theta \end{bmatrix} +$$

$$\begin{bmatrix} \Theta & A_{1(p+1)} & \Theta & . & . & . & . & \Theta \\ \Theta & \Theta & . & . & . & . & . & \Theta \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ \Theta & . & . & . & . & . & . & \Theta \end{bmatrix} + ...$$

$$+ \begin{bmatrix} \Theta & A_{1q_1} & \Theta & . & . & . & . & \Theta \\ \Theta & \Theta & . & . & . & . & . & \Theta \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ \Theta & . & . & . & . & . & . & \Theta \end{bmatrix} + ... + \begin{bmatrix} \Theta & \Theta & . & . & . & . & . & \Theta \\ \Theta & \Theta & . & . & . & . & . & \Theta \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ \Theta & . & . & . & . & . & . & A_{kp} \end{bmatrix}$$

$$+ \ldots + \begin{bmatrix} \Theta & \Theta & . & . & . & . & . & \Theta \\ \Theta & \Theta & . & . & . & . & . & \Theta \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ \Theta & . & . & . & . & . & . & A_{kq_k} \end{bmatrix} \}$$

where

$$A_{1p} = \begin{bmatrix} \alpha_p & 0 & . & . & . & 0 \end{bmatrix},$$
$$A_{1(p+1)} = \begin{bmatrix} 0 & \alpha_{p+1} & 0 & . & . & 0 \end{bmatrix}, \ldots, A_{1q_1} = \begin{bmatrix} 0 & . & . & . & 0 & \alpha_{q_1} \end{bmatrix},$$
$$\ldots,$$
$$A_{kp} = \begin{bmatrix} \alpha_p & 0 & . & . & 0 & 0 \end{bmatrix}, \ldots, A_{kq_k} = \begin{bmatrix} 0 & . & . & . & 0 & \alpha_{q_k} \end{bmatrix}.$$

**Case 1: char** $F = 2$

**(**)** is automatically satisfied for each $i$. Therefore the elements of the set of labellings for 0 are linearly independent. So the rank of the free $F$-module of labellings for 0 is $q_1 - p + 1 + q_2 - p + 1 + \ldots + q_k - p + 1 = (q_1 + q_2 + \ldots + q_k) - kp + k = q_1 + q_2 + \ldots + q_k - (p-1)k$ by Theorem 6.

**Case 2:** $G$ **is completely bipartite**

Since $G$ is $R_i$-bipartite for each $i$, the top elements of $I_{R_i}C_i$ corresponding to the $R_i$-edges outside the spanning $R_i$-tree will be 0 for $i = 1, 2, \ldots, k$. So **(**)** is irrelevant and hence the elements of the set of labellings for 0 are linearly independent. Therefore the rank of the free $F$-module of labellings for 0 is $q_1 - p + 1 + q_2 - p + 1 + \ldots + q_k - p + 1 = (q_1 + q_2 + \ldots + q_k) - kp + k = q_1 + q_2 + \ldots + q_k - (p-1)k$ by Theorem 6.

**Case 3:** $G$ **is not completely bipartite and char** $F \neq 2$

**Subcase 1:** $G$ **is not** $R_i$**-bipartite for** $i = 1, 2, \ldots, k$

In this case, due to **(**)**, one of the matrices can be expressed as a linear combination of others for each $i$. So the rank of the free $F$-module of labellings for 0 is $q_1 - p + q_2 - p + \ldots + q_k - p = (q_1 + q_2 + \ldots + q_k) - kp$ by theorem 6.

**Subcase 2: $G$ is $R_i$-bipartite for $i = i_1, i_2, ..., i_r, 1 < r < k$ and not $R_i$-bipartite for $i = i_{r+1}, ..., i_k; i_1, i_2, ..., i_k \in \{1, 2, ..., k\}$**

In this case, due to **(**)**, one of the matrices can be represented as a linear combination of others for each $i$ except for $i = i_1, i_2, ..., i_r$. so the rank of the free $F$-module of labellings for 0 is $(q_1 + q_2 + ... + q_k) - kp + r$ by Theorem 6. □

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