Algebra and Discrete Mathematics
Volume 16 **(2013)**. Number 1. pp. 16 – 19

RESEARCH ARTICLE

On modular representations of semigroups $S_p \times T_p$

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Communicated by V. V. Kirichenko

Abstract. Let p be simple, and let S_p and T_p be the symmetric group and the symmetric semigroup of degree p, respectively. The theorem of this paper says that the direct product $S_p \times T_p$ are of wild representation type over any field of characteristic p. The main case is $p = 2$.

Let k be a field. A semigroup is called *of tame representation type* (resp. *of wild representation type*) over k if so is the problem of classifying its representations (see precise general definitions in [1]).

We give the precise definition of semigroups of wild representation type in matrix language.

For a semigroup S and a k-algebra Λ, we denote by $R_\Lambda(S)$ the set of all matrix representations of S over Λ; $R_k(\Lambda)$ denotes the category of matrix representations of Λ over k.

A semigroup S is called of wild representation type (or simply wild) over k if there exists a matrix representation M of S over $\Lambda = K_2 = k < x, y >$ such that the following conditions hold:

1) the matrix representation $M \otimes X$ (of S over k) with $X \in R_k(\Lambda)$ is indecomposable if so is X;

2) the matrix representations $M \otimes X$ and $M \otimes X'$ are nonequivalent if so are X and X'.

2010 MSC: 16G, 20M30.

Key words and phrases: matrix, wild, transformation, symmetric semigroup, modular representations.

Here $K_2 = k < x, y >$ denotes the free associative k-algebra in two noncommuting variables x and y.

We call such an M a *perfect representation of S over* Λ.

In practice, to simplify the proofs of wildness (not only semigroup but also other objects) one can replace K_2 by any wild k-algebra.

The main result of this paper is the following theorem.

Theorem. *Let k be a field of characteristic $p \neq 0$ and let S_p and T_p be the symmetric group and the symmetric semigroup of degree p, respectively. Then the semigroup $S_p \times T_p$ is wild over k.*

Here $\times$ denotes, as usual, the sign of the direct product.

Note that T_p and $S_p \times T_p$ are monoids.

Since the factor semigroup of T_p by its only maximal two-sided ideal (generated by all the non-invertible elements) is isomorphic to S_p, the semigroup $S_p \times T_p$ is wild for $p \neq 2$ by the criterion of tameness and wildness of finite groups [2]. In case $p = 2$ we will indicate a perfect representation of $S_p \times T_p$ over the k-algebra $\Lambda = k\Gamma$ of paths of the quiver Γ with two vertices p_1, p_2 and two arrows $x : p_1 \to p_1$, $y : p_1 \to p_2$ (this quiver is wild [3, 4]).

The monoid T_2 of transformations of the set $\{1, 2\}$ is generated by the elements a, b, where $a(1) = 2$, $a(2) = 1$, $b(1) = 2$, $b(2) = 2$, with defining relations $a^2 = 1$, $b^2 = b$, $ab = b$ [5]. Obviously that the monoid $S_2 \times T_2$ is generated by the elements g, a, b with the additional relations $g^2 = 1$, $ga = ag$, $gb = bg$ (g denotes the non-identity element of S_2).

Consider the next matrix representation γ of $S_2 \times T_2$ over the algebra $\Lambda = k\Gamma$:

$$\gamma(g) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & y & 1 & x \\ 0 & 0 & 0 & 1 \end{pmatrix}, \gamma(a) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \gamma(b) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

($\gamma(1)$ is equal to the identity matrix).

We will prove that γ is a perfect representation.

Let φ, φ' be representations of Λ over k having the same dimension s and let $G = (\gamma \otimes \varphi)(g)$, $A = (\gamma \otimes \varphi)(a)$, $B = (\gamma \otimes \varphi)(b)$, $G' = (\gamma \otimes \varphi')(g)$, $A' = (\gamma \otimes \varphi')(a)$, $B' = (\gamma \otimes \varphi')(b)$. Consider the matrix equalities (in the variable X)

$$GX = XG', \qquad AX = XA', \qquad BX = XB', \tag{$*$}$$

viewing all their matrices as $s \times s$ block ones.

The equalities (of the $s \times s$ ij-blocks)

$$(GX)_{ij} = (XG')_{ij}, \qquad (AX)_{ij} = (XA')_{ij}, \qquad (BX)_{ij} = (XB')_{ij},$$

$i, j \in \{1, 2, 3, 4\}$ are denoted by $(1; ij)$, $(2; ij)$, $(3; ij)$, respectively.

We first write down all equalities of the forms $(2; ij)$ and $(3; ij)$ besides the trivial identities $0 = 0$ and $X_{ii} = X_{ii}$:

$$\begin{array}{lll}
(2;1,1) : X_{21} = 0, & (2;1,2) : X_{22} = X_{11}, & (2;1,3) : X_{23} = 0, \\
(2;1,4) : X_{24} = X_{13}, & (2;2,2) : 0 = X_{21}, & (2;2,4) : 0 = X_{23}, \\
(2;3,1) : X_{41} = 0, & (2;3,2) : X_{42} = X_{31}, & (2;3,3) : X_{43} = 0, \\
(2;3,4) : X_{44} = X_{33}, & (2;4,2) : 0 = X_{41}, & (2;4,4) : 0 = X_{43}, \\
(3;1,2) : X_{12} = 0, & (3;1,3) : X_{13} = 0, & (3;1,4) : X_{14} = 0, \\
(3;2,1) : 0 = X_{21}, & (3;3,1) : 0 = X_{31}, & (3;4,1) : 0 = X_{41}.
\end{array}$$

From these equalities it follows that

$$X = \begin{pmatrix} X_{11} & 0 & 0 & 0 \\ 0 & X_{11} & 0 & 0 \\ 0 & X_{32} & X_{33} & X_{34} \\ 0 & 0 & 0 & X_{33} \end{pmatrix}.$$

Then from the equalities

$$(1;3,2) : \varphi(y)X_{11} = X_{33}\varphi'(y), \quad (1;3,4) : \varphi(x)X_{33} = X_{33}\varphi'(x) \quad (**)$$

(the only two nontrivial equalities of the form $(1; ij)$ modulo the equalities $(2; ij)$ and $(3; ij)$) we have that the matrix k-representations φ and φ' of $\Lambda = k\Gamma$ are equivalent if so are the matrix k-representations $\gamma \otimes \varphi$ and $\gamma \otimes \varphi'$ of $S_2 \times T_2$ (because X_{11} and X_{33} are invertible if so is X).

Thus, for the representation γ condition 2) of the definition of wild semigroups holds.

From the form of the matrix X it follows that the endomorphism algebra of $\gamma \otimes \varphi$ is local if and only if so is the endomorphism algebra of φ (these algebras are defined, respectively, by $(*)$ and $(**)$ with $\varphi = \varphi'$). Therefore $\gamma \otimes \varphi$ is indecomposable if φ is indecomposable, and consequently γ satisfies condition 1) of the mentioned definition too.

The theorem is proved.

Because as a perfect matrix representation of the quiver Γ over the algebra $K_2' = k < x', y' >$ one can take the representation

$$x \to \begin{pmatrix} 0 & x' \\ 1 & y' \end{pmatrix}, \quad y \to \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

it follows from the proof of our theorem that the following representation λ of the semigroup $S_2 \times T_2$ over K'_2 is perfect:

$$\lambda(g) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & x' \\ 0 & 0 & 0 & 1 & 1 & y' \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \lambda(a) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\lambda(b) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

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Received by the editors: 17.07.2013
and in final form 17.07.2013.