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The total torsion element graph of semimodules over commutative semirings

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Abstract. We introduce and investigate the total torsion element graph of semimodules over a commutative semiring with non-zero identity. The main purpose of this paper is to extend the definition and results given in [2] to more general semimodule case.

1. Introduction

In [6], Beck associated to any commutative ring R its zero-divisor graph $G(R)$ whose vertices are the zero-divisors of R (including 0), with two vertices a, b joined by an edge in case $ab = 0$. The problem Beck studied was how to color the vertices of $G(R)$ with the smallest number of colors such that no two adjacent vertices in the graph had the same color. Beck conjectured this number is the clique number of $G(R)$, but Beck's question was settled in the negative in [1]. In [3], Anderson and Livingston introduced and studied the subgraph $\Gamma(R)$ whose vertices are the nonzero zero-divisors of R. This graph turns out to best exhibit the properties of the set of zero-divisors of R, and the ideas and problems introduced in [3] were further studied in [15], [9] and [10]. Let R be a commutative ring with $Z(R)$ its set of zero-divisors elements. The total graph of R, denoted by $T(\Gamma(R))$, is the (undirected) graph with all elements of R as vertices, and for distinct $x, y \in R$, the vertices x and y are adjacent if

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and only if $x + y \in Z(R)$. The total graph of a commutative ring have been introduced and studied by D. F. Anderson and A. Badawi in [2]. In [12], the notion of the total torsion element graph of a module over a commutative ring is introduced.

Subsemimodules of semimodules over semirings play a central role in the structure theory and are useful for many purposes [13, 14]. However, they do not in general coincide with the submodules over rings and, for this reason, their use is somewhat limited in trying to obtain analogues of ring theorems for semirings. Indeed, many results in rings apparently have no analogues in semirings using only subsemimodules. In order to overcome this deficiency, the authors defined a more restricted class of subsemimodules in semirings, which are called the class of "k-subsemimodules" and the class of "Q_M-subsemimodules" [5, 8, 7]. In the present paper we introduce a new class of graphs, called the total torsion element graph of a semimodule over a semiring, and we completely characterize the structure of this graph. The total torsion element graph of a module over a commutative ring and the total torsion element graph of a semimodule over a commutative semiring are different concepts. Some of our results are analogous to the results given in [2]. The corresponding results are obtained by modification and here we give a complete description of the total torsion element graph of a semimodule. The study of the total torsion element graph of a semimodule M breaks naturally into two cases depending on whether or not $T(M)$, the set of torsion elements in M, is a subsemimodule of M. In the third section, we handle the case when $T(M)$ is a subsemimodule (either k-subsemimodule or Q_M-subsemimodule) of M; in the fourth section, we do the case when $T(M)$ is not a subsemimodule of M (see Sections $3, 4$).

2. Preliminaries

For the sake of completeness, we state some definitions and notation used throughout. For a graph Γ, by $E(\Gamma)$ and $V(\Gamma)$, we denote the set of all edges and vertices, respectively. We recall that a graph is connected if there exists a path connecting any two distinct vertices. The distance between two distinct vertices a and b, denoted by $\mathrm{d}(a, b)$, is the length of a shortest path connecting them (if such a path does not exist, then $d(a, a) = 0$ and $d(a, b) = \infty$). The diameter of a graph Γ, denoted by $\mathrm{diam}(\Gamma)$, is equal to $\sup\{d(a, b) : a, b \in V(\Gamma)\}$. A graph is complete if it is connected with diameter less than or equal to one. The girth of a graph Γ, denoted $\mathrm{gr}(\Gamma)$, is the length of a shortest cycle in Γ, provided Γ contains a cycle;

otherwise; $\mathrm{gr}(\Gamma) = \infty$. We denote the complete graph on n vertices by K^n and the complete bipartite graph on m and n vertices by $K^{m,n}$ (we allow m and n to be infinite cardinals). We will sometimes call a $K^{1,m}$ a star graph. We say that two (induced) subgraphs Γ_1 and Γ_2 of Γ are disjoint if Γ_1 and Γ_2 have no common vertices and no vertex of Γ_1 (respectively, Γ_2) is adjacent (in Γ) to any vertex not in Γ_1 (respectively, Γ_2).

Throughout this paper R is a commutative semiring with identity. In order to make this paper easier to follow, we recall in this section various notions from semimodule theory which will be used in the sequel. For the definitions of monoid, semirings, semimodules and subsemimodules we refer [13, 14, 7, 8]. All semiring in this paper are commutative with non-zero identity. Let M be a semimodule over a semiring R.

(1) A semiring R is said to be semidomain whenever $a, b \in R$ with $ab = 0$ implies that either $a = 0$ or $b = 0$.

(2) A subtractive subsemimodule (= k-subsemimodule) N is a subsemimodule of M such that if $x, x + y \in N$, then $y \in N$ (so $\{0_M\}$ is a k-subsemimodule of M).

(3) An element x of M is called a zero-sum in M if $x + y = 0$ for some $y \in M$. We use $S(M)$ to denote the set of all zero-sum elements of M.

(4) A semimodule M over a semiring R is called a M-cancellative semimodule if whenever $rm = rn$ for elements $m, n \in M$ and $r \in R$, then $n = m$.

(5) A subsemimodule N of a semimodule M over a semiring R is called a partitioning subsemimodule (= Q_M-subsemimodule) if there exists a subset Q_M of M such that $M = \cup\{q + N : q \in Q_M\}$ and if $q_1, q_2 \in Q_M$ then $(q_1 + N) \cap (q_2 + N) \neq \emptyset$ if and only if $q_1 = q_2$. Let N be a Q_M-subsemimodule of M and let $M/N = \{q + N : q \in Q_M\}$. Then M/N forms an R-semimodule under the operations $\oplus$ and $\odot$ defined as follows: $(q_1 + N) \oplus (q_2 + N) = q_3 + N$, where $q_3 \in Q_M$ is the unique element such that $q_1 + q_2 + N \subseteq q_3 + N$ and $r \odot (q_1 + N) = q_4 + I$, where $r \in R$ and $q_4 \in Q_M$ is the unique element such that $rq_1 + N \subseteq q_4 + N$. This R-semimodule M/N is called the quotient semimodule of M by N [7]. By [7, Lemma 2.3], there exists a unique element $q_0 \in Q_M$ such that $q_0 + N = N$. Thus $q_0 + N$ is the zero element of M/N.

(6) A torsion element is an element $m \in M$ for which there exists a non-zero element r of R such that $rm = 0$. The set of torsion elements in M will be denoted by $T(M)$. Also, we use $T(M)^*$ to denote the set of non-zero torsion elements of M.

(7) We define the total torsion element graph of a semimodule M, denoted by $T(\Gamma(M))$, as follows: $V(T(\Gamma(M))) = M$, $E(T(\Gamma(M))) =$

$\{\{x, y\} : x + y \in T(M)\}$. We will use $\mathrm{Tof}(M)$ to denote the set of elements of M that are not torsion elements. Let $\mathrm{Tof}(\Gamma(M))$ be the (induced) subgraph of $T(\Gamma(M))$ with vertices $\mathrm{Tof}(M)$, and let $\mathrm{Tor}(\Gamma(M))$ be the (induced) subgraph of $T(\Gamma(M))$ with vertices $T(M)$.

3. $T(M)$ is a subsemimodule of M

Let M be a semimodule over a semiring R. The structure of the torsion element graph $T(\Gamma(M))$ may be completely described in those cases when torsion elements form a subsemimodule. We begin this section with the extreme cases $T(M) = M$ and $T(M) = \{0\}$.

Theorem 3.1. Let M be a semimodule over a semiring R.

(i) $T(\Gamma(M))$ is complete if and only if $T(M) = M$.

(ii) $T(\Gamma(M))$ is totally disconnected if and only if $T(M) = S(M) = \{0\}$.

Proof. (i) If $T(M) = M$, then for any two vertices $x, y \in M$, one has $x + y \in T(M)$; hence they are adjacent in $T(\Gamma(M))$. Conversely, assume that $T(\Gamma(M))$ is complete and let $m \in M$. Then m is adjacent to 0. Thus $m = m + 0 \in T(M)$, and hence we have equality.

(ii) Let $T(\Gamma(M))$ be totally disconnected. Then 0 is not adjacent to any vertex; hence $x = x + 0 \notin T(M)$ for every non-zero element x of M. Thus $T(M) = \{0\}$. If there is a non-zero element m of $S(M)$, then there exists $0 \neq m' \in M$ such that $m + m' = 0 \in T(M)$, which is a contradiction. Thus $S(M) = \{0\}$. Conversely, assume that there exist distinct $a, b \in M$ such that $a + b \in T(M) = \{0\}$. Then $a, b \in S(M)$, a contradiction. Hence $T(\Gamma(M))$ is totally disconnected. $\square$

Proposition 3.2. Let M be a semimodule over a commutative semiring R.

(i) If $2 = 1_R + 1_R \in Z(R)$ and $x \in M$, then $2x \in T(M)$. In particular, if $T(M) = \{0\}$, then M is an R-module.

(ii) If $x \in Tof(M)$, then $2 \in Z(R)$ if and only if $2x \in T(M)$.

Proof. (i) By assumption, there exists $0 \neq r \in R$ such that $2r = 0$. Since $r(2x) = (2r)x = 0$, we have $2x \in T(M)$. Finally, by assumption, $2x = 0$ for every $x \in M$, as required.

(ii) By (i), it is enough to show that if $2x \in T(M)$, then $2 \in Z(R)$. There exists a non-zero element s of R such that $(2s)x = s(2x) = 0$; hence $2s = 0$ since $x \notin T(M)$. Thus $2 \in Z(R)$. $\square$

Example 3.3. (1) A subsemimodule of a semimodule over a semiring in general need not be a k-subsemimodule and Q_M-subsemimodule. Let $M = R$ be the set of all real numbers x satisfying $0 < x \leq 1$, and define $a+b = a.b = \min\{a, b\}$ for all $a, b \in R$. Then $(R, +, .)$ is easily checked to be a commutative semiring with 1 as identity. Each real number r such that $0 < r < 1$ defines a subsemimodule $N = \{y \in M : y \leq r\}$ of M. However, $r + 1 = 1$ together $r \in N$ and $1 \notin N$ show that N is not a k-subsemimodule of M. In particular, N is not a Q_M-subsemimodule of M since every Q_M-subsemimodule is a k-subsemimodule by [7, Theorem 3.2].

(2) Let $R = M$ denote the semiring of nonnegative integers with the usual operations of addition and multiplication. If $m \in M - \{0\}$, the subsemimodule

$$N = \{km : k \in R\}$$

is a Q_M-subsemimodule of M when $Q_M = \{0, 1, \cdots, m-1\}$. In particular, N is a k-subsemimodule.

(3) Assume that R is the semiring of nonnegative integers with the usual operations of addition and multiplication and let $M = (R, \gcd)$. It is easy to see that M is a commutative monoid in which every element is idempotent. Hence M is an R-semimodule in which $N = \{0, 2, 4, \cdots\}$ is a k-subsemomodule of M but not a Q_M-subsemimodule.

Proposition 3.4. Let M be a semimodule over a semidomain R.

(i) $T(M)$ is a k-subsemimodule of M.

(ii) If M is a M-cancellative semimodule, then $T(M)$ is a Q_M-subsemimodule of M.

Proof. (i) Clearly, $T(M)$ is a subsemimodule of M. Let $x + y, x \in T(M)$ for some $x, y \in M$. Then $rx = s(x + y) = 0$ for some $0 \neq r, s \in R$ (so $rs \neq 0$). Therefore, $(rs)y = 0$, and so $y \in T(M)$.

(ii) Set $Q_M = (M - T(M)) \cup \{0\}$; we show that $T(M)$ is a Q_M-subsemimodule of M. Let $m \in M$. If $m \in T(M)$, then $m \in 0_M + T(M)$. If $m \notin T(M)$, then $m \in m + T(M)$. So $M = \cup\{q + T(M) : q \in Q_M\}$. Let $(q_1 + T(M)) \cap (q_2 + T(M)) \neq \emptyset$. So $q_1 + a = q_2 + b$ for some $a, b \in T(M)$. It follows that $ra = sb = 0$ for some $0 \neq r, s \in R$. Therefore, $rsq_1 = rsq_2$ with $rs \neq 0$ since R is a semidomain. Thus $q_1 = q_2$ since M is a M-cancellative semimodule, as needed. □

Proposition 3.5. Let M be a semimodule over a commutative semiring R such that $T(M)$ is a subsemimodule of M.

(i) $Tor(\Gamma(M))$ is a complete (induced) subgraph of $T(\Gamma(M))$.

(ii) If N is a subsimimodule of M, then $T(\Gamma(N))$ is the induced subgraph of $T(\Gamma(M))$ if and only $T(N) = N \cap T(M)$.

(iii) If $(0 : M) \neq 0$, then $T(\Gamma(M))$ is a complete graph.

Proof. The proofs are straightforward. □

Example 3.6 shows that there are some semimodules over commutative semirings such that their torsion subsemimodules are not k-subsemimodues.

Example 3.6. Assume that E_+ be the set of all non-negative integers and let $M = R = \{(a, b) : a, b \in E_+\}$. Define $(a, b) + (c, d) = (\min\{a, c\}, \max\{b, d\})$ and $(a, b) * (c, d) = (ac, bd)$ for all $(a, b), (c, d) \in R$. Then $(R, +, *)$ is easily checked to be a commutative semiring. An inspection will show that $T(M) = \{(a, b) \in M : a = 0 \textit{ or } b = 0\}$ is a subsemimodule of M. However, $(0, 1) + (2, 5) = (0, 5) \in T(M)$ together with $(2, 5) \notin T(M)$ and $(0, 1) \in T(M)$ show that $T(M)$ is not a k-subsemimodule of M. Also, $T(\Gamma(M))$ is a connected graph since every element is adjacent to $(0, 0)$ in $T(\Gamma(M))$. Moreover, $gr(T(\Gamma(M))) = 3$ since there is a 3-cyclic $(0, 0) - (0, 1) - (1, 0) - (0, 0)$ in $T(\Gamma(M))$.

Example 3.7 shows that there are some semimodules over commutative semirings such that their torsion subsemimodules are k-subsemimodues but they are not Q_M-subsemimodules.

Example 3.7. Assume that $R = M$ is the set of all non-negative integers and let $a, b, k \in R$. Define $a + b = \gcd(a, b)$ and

$$a * b = \begin{cases} 0 & \text{if } \gcd(a, b) = 2k, \\ 1 & \text{if } \gcd(a, b) = 2k + 1, \\ 0 & a = 0 \text{ or } b = 0. \end{cases}$$

Then $(R, +, *)$ is easily checked to be a commutative semiring. An inspection will show that $T(M) = \{0, 2, 4, 6, \cdots\}$ is a k-subsemimodule of M but is not a Q_M-subsemimodule of M by Example 3.3 (3). Moreover, $Tor(\Gamma(M))$ is a complete graph and $Tof(\Gamma(M))$ is a totally disconnected graph.

The main goal of this section is a general structure theorem (Theorem 3.10) for $Tof(\Gamma(M))$ when either $T(M)$ is a k- subsemimodule of M or $T(M)$ is a Q_M-subsemimodule. But first, we record the basic observation that if $T(M)$ is a k-subsemimodule of (resp. $T(M)$ is not a k-subsemimodule), then the subgraph $Tor(\Gamma(M))$ is disjoint from $Tof(\Gamma(M))$ (resp. $Tor(\Gamma(M))$ is not disjoint from $Tof(\Gamma(M))$. Thus we will concentrate on the subgraph $Tof(\Gamma(M))$ throughout this section.

Theorem 3.8. Let M be a semimodule over a commutative semiring R such that $T(M)$ is a k-subsemimodule of M. If m and m' are distinct elements of $Tof(M)$ that are connected by a path with $m + m' \notin T(M)$ (i.e., if m and m' are not adjacent), then there is a path in $Tof(\Gamma(M))$ of length at most 2 between m and m'.

Proof. Let $T(M)$ be a k-subsemimodule of M. It is enough to show that if m_1, m_2, m_3 and m_4 are distinct vertices of $Tof(M)$ and there is a path $m_1 - m_2 - m_3 - m_4$ from m_1 to m_4, then m_1 and m_4 are adjacent. Now we have $m_1 + m_2 + m_3 + m_4 \in T(M)$. Then $T(M)$ being k-subsemimodule of M gives $m_1 + m_4 \in T(M)$, and so m_1 and m_4 are adjacent, as required. □

Compare the next theorem with [12, Theorem 2.1 (1)].

Theorem 3.9. Let M be a semimodule over a commutative semiring R.

(i) If $T(M)$ is a k-subsemimodule of M, then $Tor(\Gamma(M))$ is disjoint from $Tof(\Gamma(M))$.

(ii) If $T(M)$ is not a k-subsemimodule of M, then $Tor(\Gamma(M))$ is not disjoint from $Tof(\Gamma(M))$.

Proof. (i) Let $T(M)$ is a k-subsemimodule of M. If $Tor(\Gamma(M))$ is not disjoint from $Tof(\Gamma(M))$, then there exist $a \in T(M)$ and $b \in Tof(M)$ such that $a + b \in T(M)$. Thus $b \in T(M)$ since $T(M)$ is a k-subsemimodule of M which is a contradiction. Thus $Tor(\Gamma(M))$ is disjoint from $Tof(\Gamma(M))$.

(ii) Assume that $T(M)$ is not a k-subsemimodule of M. So there exist $a \in T(M)$ and $b \in Tof(M)$ such that $a + b \in T(M)$. Let $x \in M$. We define the subset $P_T(x)$ as follows:

$$P_T(x) = \{m \in T(M) : \text{there is a path of finite length between } x \text{ and } m\}.$$

Clearly, if $x \in T(M)$, then $T(M) \subseteq P_T(x)$, and so $P_T(x) \neq \emptyset$. Set $N = \{x \in M : P_T(x) \neq \emptyset\}$. Therefore $T(M) \subset N$, since $b \in N$ and $b \notin T(M)$. Now, we show that N is a subsemimodule of M. Let $x_1, y_1 \in N$. Therefore, there exist $m_1, m'_1 \in T(M)$, $x_1, x_2, \cdots, x_n \in M$ and $y_1, y_2, \cdots, y_k \in M$ such that $x_1 - x_2 - \cdots - x_n - m_1$ and $y_1 - y_2 - \cdots - y_k - m'_1$ are paths of finite lengths between x_1, m_1 and y_1, m'_1, and so we have $x_i + x_{i+1}, y_j + y_{j+1}, x_n +$

$m_1, y_k + m'_1, m_1 + m'_1 \in T(M)$ for each $1 \leq i \leq n-1$ and $1 \leq j \leq k-1$. We may assume that $n \leq k$. So $(x_i + y_i) + (x_{i+1} + y_{i+1}) \in T(M)$ for each $1 \leq i \leq n-1$ and $1 \leq j \leq k-1$. Then

$$(x_1 + y_1) - (x_2 + y_2) - \cdots - (x_n + y_n) - \\ (m_1 + y_{n+1}) - (m'_1 + y_{n+2}) - (m_1 + y_{n+3}) - \cdots - m_1$$

is a path of finite length between $x_1 + y_1$ and m_1. Hence $P_T(x_1 + y_1) \neq \emptyset$, and so $x_1 + y_1 \in N$. Now, let $r \in R$. Therefore, $rx_1 - rx_2 - \cdots - rx_n - rm_1$ is a path between rx_1 and rm_1 of finite length, and so $P_T(rx) \neq \emptyset$. Thus N is a subsemimodule of M and $T(M) \subset N$. It is easy to see that $T(\Gamma(N))$ is a connected subgraph of $T(\Gamma(M))$ containing $Tor(\Gamma(M))$. Hence, $Tor(\Gamma(M))$ is not disjiont from $Tof(\Gamma(M))$. □

Compare the next theorem with [12, Theorem 2.5].

Theorem 3.10. Let M be a semimodule over a commutative semiring R, $|T(M)| = \alpha$ and $|Q_M - T(M)| = \beta$.

(i) If $T(M)$ is a k-subsemimodule of M and $2 \in Z(R)$, then $Tof(\Gamma(M))$ is the union of disjoint complete subgraphs.

(ii) If $T(M)$ is a k-subsemimodule of M and $2 \notin Z(R)$, then $Tof(\Gamma(M))$ is the union of totally disconnected subgraphs and some connected subgraphs.

(iii) If $T(M)$ is a Q_M-subsemimodule of M and $2 \in Z(R)$, then $Tof(\Gamma(M))$ is the union of β disjoint K^{λ}'s such that $\lambda \leq \alpha$.

(iv) If $T(M)$ is a Q_M-subsemimodule of M and $2 \notin Z(R)$, then $Tof(\Gamma(M))$ is the union of totally disconnected subgraphs and complete bipartite subgraphs.

Proof. (i) Let $2 \in Z(R)$. We set up an equivalence relation $\sim$ on $Tof(M)$ as follows: for $y, y' \in Tof(M)$, we write $y \sim y'$ if and only if $y + y' \in T(M)$. By Proposition 3.2 and $T(M)$ being a k-subsemimodule of M, it is straightforward to check that $\sim$ is an equivalence relation on $Tof(M)$: for $y \in Tof(M)$, we denote the equivalence class which contains y by $[y]$. Now let $y \in Tof(M)$. If $[y] = \{y\}$, then $(y+a) + (y+b) = 2y + (a+b) \in T(M)$ for every $a, b \in T(M)$ by Proposition 3.2. So $y + T(M)$ is a complete subgraph with at most α vertices. If $|[y]| = \gamma > 1$, then for every $y' \in [y]$ we have $(y+a) + (y'+b) = (y+y') + a + b \in T(M)$, where $a, b \in T(M)$. Thus $y + T(M)$ is a part of a complete graph K^{ν} with $\nu \leq \alpha\gamma$ vertices. Therefore, $Tof(\Gamma(M))$ is the union of disjoint complete subgraphs.

(ii) Let $2 \notin Z(R)$ and $y \in Tof(M)$. Set

$$A_y = \{y' \in Tof(M) : y + y' \in T(M)\}.$$

If $A_y = \emptyset$, then $y + y' \notin T(M)$ for every $y' \in Tof(M)$. In this case, we show that $y + T(M)$ is a totally disconnected subgraph of $Tof(\Gamma(M))$. If $(y + a) + (y + b) \in T(M)$ for some $a, b \in T(M)$, then $2y + a + b = (y + a) + (y + b) \in T(M)$; so $2y \in T(M)$, which is a contradiction by Proposition 3.2. Therefore, $y + T(M)$ is a totally disconnected subgraph of $Tof(\Gamma(M))$. We may assume that $A_y \neq \emptyset$. Then $y + y' \in T(M)$ for some $y' \in Tof(M)$. Thus $(y+a)+(y'+b) = (y+y')+(a+b) \in T(M)$ for every $a, b \in T(M)$; hence each element of $y+T(M)$ is adjacent to each element of $y' + T(M)$. If $|A_y| = \nu$, then we have a connected subgraph of $Tof(\Gamma(M))$ with at most $\alpha\nu$ vertices. Hence, If $2 \notin Z(R)$, then $Tof(\Gamma(M))$ is the union of totally disconnected subgraphs and some connected subgraphs.

(iii) First, we show that $q+T(M) \subseteq Tof(M)$ for every $q \in Q_M - T(M)$. If $q + a \notin Tof(M)$ for some $a \in T(M)$, then $q + a \in T(M)$; hence $q \in T(M)$ since $T(M)$ is a k-subsemimodule which is a contradiction. Let $2 \in Z(R)$ and $q \in Q_M - T(M)$. Then each coset $q + T(M)$ is a complete subgraph of $Tof(M)$ with λ vertices such that $\lambda \leq \alpha$ (note that $(q_1+T(M)) \cap (q_2+T(M)) \neq \emptyset$ if and only if $q_1 = q_2$) since $(q+a)+(q+b) = 2q + (a + b) \in T(M)$ for all $a, b \in T(M)$ by Proposition 3.2 and $T(M)$ is a subsemimodule. Now we show that distinct cosets form disjoint subgraphs of $Tof(\Gamma(M))$. If $q_1 + a$ and $q_2 + b$ are adjacent for some $q_1, q_2 \in Q_M - T(M)$ and $a, b \in T(M)$, then $(q_1 + a) + (q_2 + b) \in T(M)$ gives $q_1+q_2 \in T(M)$ since $T(M)$ is a k-subsemimodule of M. So $q_2+2q_1 = q_1 + (q_1 + q_2) \in q_1 + T(M)$. Likewise, $q_2 + 2q_1 \in q_2 + T(M)$ by Proposition 3.2. So $q_2 + 2q_1 \in (q_1 + T(M)) \cap (q_2 + T(M))$; hence $q_1 = q_2$. Thus $Tof(\Gamma(M))$ is the union of β disjoint induced subgraphs $q + T(M)$, each of which is a K^λ such that $\lambda \leq \alpha$.

(iv) Next assume that $2 \notin Z(R)$ and let $q \in Q_M - T(M)$. If $q + q' \notin T(M)$ for every $q' \in Q_M - T(M)$, then $A_q = \emptyset$. Then by (ii), $q + T(M)$ is a totally disconnected subgraph of $Tof(\Gamma(M))$. So we may assume that $q + q' \in T(M)$ for some $q' \in Q_M - T(M)$. Then by (ii) each element of $q + T(M)$ is adjacent to each element of $q' + T(M)$; we show that q' is the unique element. Let $q + q'' \in T(M)$ for some $q'' \in Q_M - T(M)$. Therefore, $q + q' + q'' = q' + (q + q'') \in q' + T(M)$. Likewise, $q + q' + q'' = q''+(q+q') \in q''+T(M)$. Thus $(q'+T(M)) \cap (q''+T(M)) \neq \emptyset$ gives $q' = q''$. Therefore $(q + T(M)) \cup (q' + T(M))$ is a complete bipartite subgraph of $Tof(\Gamma(M))$. So $Tof(\Gamma(M))$ is the union of totally disconnected subgraphs and complete bipartite subgraphs. □

Proposition 3.11. Let M be a semimodule over a commutative semiring R.

(i) If $T(M)$ is a k-subsemimodule of M and $Tof(\Gamma(M))$ is complete, then $|Tof(M)| = 1$ or $|Tof(M)| = 2$.

(ii) If $T(M)$ is a Q_M-subsemimodule of M and $Tof(\Gamma(M))$ is complete, then $|M/T(M)| = 2$ or $|M/T(M)| = 3$.

(iii) If $T(M)$ is a Q_M-subsemimodule of M, $|M/T(M)| = 2$ and $2 \in Z(R)$, then $Tof(\Gamma(M))$ is complete.

Proof. (i) By Proposition 3.2, $2y \in T(M)$ for every $y \in Tof(M)$. Then $y + T(M)$ is a complete subgraph of $Tof(\Gamma(M))$; hence $|Tof(M)| = 1$ since $Tof(\Gamma(M))$ is complete. If $2 \notin Z(R)$, then for each $y \in Tof(M)$, there exists $y' \in Tof(M)$ such that $y + y' \in T(M)$. So $|Tof(M)| = 2$ since $Tof(\Gamma(M))$ is complete. In this case, $Tof(\Gamma(M))$ is a complete bipartite graph (see Theorem 3.10).

(ii) Since every Q_M-subsemimodule is a k-subsemimodule, the part (i) gives $|Tof(M)| = 1$ or $|Tof(M)| = 2$. If $|Tof(M)| = 1$, then $M = T(M) \cup (q + T(M))$ for $q \in Tof(M)$ and hence $|M/T(M)| = 2$. SimilarIy, if $|Tof(M)| = 2$, then $M = T(M) \cup (q + T(M)) \cup (q' + T(M))$ for $q, q' \in Tof(M)$ with $q \neq q'$, and hence $|M/T(M)| = 3$.

(iii) Let $|M/T(M)| = 2$ and $2 \in Z(R)$. Then $M = T(M) \cup (q + T(M))$ for some $q \in Q_M - T(M)$; so $2q \in T(M)$ by Proposition 3.2. Let $m, m' \in Tof(M)$. Then $m, m' \in q + T(M)$. So $m + m' = (q + a) + (q + b) = 2q + (a + b) \in T(M)$ for some $a, b \in T(M)$. Thus $Tof(\Gamma(M))$ is complete. □

Proposition 3.12. Let M be a semimodule over a commutative semiring R such that $T(M)$ is a Q_M-subsemimodule of M. Then:

(i) If $Tof(\Gamma(M))$ is connected, then $|M/T(M)| = 2$ or $|M/T(M)| = 3$.

(ii) If $|M/T(M)| = 2$ and $2 \in Z(R)$, then $Tof(\Gamma(M))$ is connected.

Proof. (i) Let $Tof(\Gamma(M))$ be a connected graph. Then $Tof(\Gamma(M))$ is a single complete graph K^λ or a bipartite graph by Theorem 3.10. Hence $Tof(\Gamma(M))$ is a complete graph. Now the assertion follows from Proposition 3.11.

(ii) This follows directly from Proposition 3.11. □

Theorem 3.13. Let M be a semimodule over a commutative semiring R.

(i) If $T(M)$ is a k-subsemimodule of M, then $\mathrm{diam}(Tof(\Gamma(M))) = 0$ if and only if $T(M) = \{0\}$ and $|M| = 2$.

(ii) Let $T(M)$ be a Q_M-subsemimodule of M. Then the following hold:

(a) $\mathrm{diam}(Tof(\Gamma(M))) = 1$ if and only if $2 \in Z(R)$ and $|M/T(M)| = 2$.

(b) $\mathrm{diam}(Tof(\Gamma(M))) = 2$ if and only if $|M/T(M)| = 3$, $2 \notin Z(R)$ and $q + q' \in T(M)$ for every $q, q' \in Q_M - T(M)$.

(c) Otherwise $\mathrm{diam}(Tof(\Gamma(M))) = \infty$.

Proof. (i) If $\mathrm{diam}(Tof(\Gamma(M))) = 0$, then $Tof(\Gamma(M))$ is a complete graph K^1, and so $|T(M)| = |Tof(M)| = 1$ by Theorem 3.10. Hence $T(M) = \{0\}$ and $|M| = 2$. The other implication is clear.

(ii) (a) If $\mathrm{diam}(Tof(\Gamma(M))) = 1$, then $Tof(\Gamma(M))$ is a complete graph K^λ with $\lambda \leq |T(M)|$ by Theorem 3.10. Therefore, $2 \in Z(R)$ and $|Q_M - T(M)| = 1$. Thus $M = T(M) \cup (q + T(M))$ for some $q \in Q_M - T(M)$; hence $|M/T(M)| = 2$. The converse follows from Theorem 3.10.

(ii) (b) If $\mathrm{diam}(Tof(\Gamma(M))) = 2$, then $Tof(\Gamma(M))$ is a complete bipartite graph $K^{1,2}$ or $K^{2,2}$; thus $2 \notin Z(R)$ and $|Q_M - T(M)| = 2$ by Theorem 3.10. Since $Tof(\Gamma(M))$ has not any totally disconnected subgraph, we must have $q + q' \in T(M)$ for every $q, q' \in Q_M - T(M)$. $\square$

Remark 3.14. Let R and M be as described in Example 3.7. So $T(M) = \{0, 2, 4, 6, \cdots\}$ is a k-subsemimodule of M but is not a Q-subsemimodule of M. Also, $Tor(\Gamma(M))$ is a complete graph and $Tof(\Gamma(M))$ is a totally disconnected graph. Since $\gcd(2, 4) = 2$, we have $2 * 4 = 0$; hence $2 \in Z(R)$. Moreover, $M = T(M) \cup (1 + T(M))$ and $\mathrm{diam}(Tof(\Gamma(M))) = \infty$. Hence Theorem 3.13 (ii) is not true when $T(M)$ is not a Q_M-subsemimodule of M.

Proposition 3.15. Let M be a semimodule over a commutative semiring R such that $T(M)$ is a k-subsemimodule of M. Then $\mathrm{gr}(Tof(\Gamma(M))) = 3, 4$ or ∞. In particular, if $Tof(\Gamma(M))$ contains a cycle, $\mathrm{gr}(Tof(\Gamma(M))) \leq 4$.

Proof. Let $Tof(\Gamma(M))$ contains a cycle. Then $Tof(\Gamma(M))$ is not a totally disconnected graph, so by the proof of Theorem 3.10, $Tof(\Gamma(M))$ has either a complete or a complete bipartite subgraph. Therefore, it must contain either a 3-cycle or a 4-cycle. Thus $\mathrm{gr}(Tof(\Gamma(M))) \leq 4$. $\square$

Theorem 3.16. Let M be a semimodule over a commutative semiring R such that $T(M)$ be a k-subsemimodule of M.

(i) $\mathrm{gr}(Tof(\Gamma(M))) = 3$ if and only if $2 \in Z(R)$ and $|y + T(M)| \geq 3$ for some $y \in Tof(M)$.

(ii) $\mathrm{gr}(Tof(\Gamma(M))) = 4$ if and only if $2 \notin Z(R)$ and $y + y' \in T(M)$ for some $y, y' \in Tof(M)$.

Proof. (i) Assume that $\mathrm{gr}(Tof(\Gamma(M))) = 3$. Then by Theorem 3.10, $Tof(\Gamma(M))$ is a complete graph K^{λ} with $3 \leq \lambda$. Therefore, $2 \in Z(R)$ and $|y + T(M)| \geq 3$ for some $y \in Tof(M)$.

(ii) If $\mathrm{gr}(Tof(\Gamma(M))) = 4$, then by Theorem 3.10, $Tof(\Gamma(M))$ has a complete bipartite subgraph; hence $2 \notin Z(R)$ and $y + y' \in T(M)$ for some $y, y' \in Tof(M)$ by Theorem 3.10. The other implications of (i) and (ii) follows directly from Theorem 3.10. □

Theorem 3.17. Let M be a semimodule over a commutative semiring R such that $T(M)$ be a k-subsemimodule of M.

(i) $\mathrm{gr}(T(\Gamma(M))) = 3$ if and only if $|T(M)| \geq 3$.

(ii) $\mathrm{gr}(T(\Gamma(M))) = 4$ if and only if $2 \notin Z(R)$, $|T(M)| < 3$ and $y + y' \in T(M)$ for some $y, y' \in Tof(M)$.

(iii) Otherwise, $\mathrm{gr}(T(\Gamma(M))) = \infty$.

Proof. (i) This follows from Proposition 3.5.

(ii) Since $\mathrm{gr}(Tor(\Gamma(M)) = 3$ or ∞, then $\mathrm{gr}(Tof(\Gamma(M))) = 4$. Therefore, $2 \notin Z(R)$ and $y + y' \in T(M)$ for some $y, y' \in Tof(M)$ by Theorem 3.16. On the other hand, $\mathrm{gr}(T(\Gamma(M)) \neq 3$; so $|T(M)| < 3$. The converse implication follows from Theorem 3.10. □

4. $T(M)$ is not a subsemimodule of M

We continue to use the notation already established, so M is a semimodule over a commutative semiring R. In this section , we study the torsion element graph $T(\Gamma(M))$ when $T(M)$ is not a subsemimodule of M.

Lemma 4.1. Let M be a semimodule over a semiring R such that $T(M)$ is not a subsemimodule of M. Then there are distinct $m, m' \in T(M)^*$ such that $m + m' \in \mathrm{Tof}(M)$.

Proof. It is enough to show that $T(M)$ is always closed under scalar multiplication of its elements by elements of R. Let $m \in T(M)$ and $r \in R$. There is a non-zero element $s \in R$ with $sm = 0$; hence $s(rm)) = r(sm) = 0$. Thus $rm \in T(M)$. This completes the proof. □

Theorem 4.2. Let M be a semimodule over a semiring R such that $T(M)$ is not a subsemimodule of M. Then $\mathrm{Tor}(\Gamma(M))$ is connected with $\mathrm{diam}(\mathrm{Tor}(\Gamma(M))) = 2$.

Proof. Let $x \in T(M)^*$. Then x is adjacent to 0. Thus $x - 0 - y$ is a path in $\mathrm{Tor}(\Gamma(M))$ of length two between any two distinct $x, y \in T(M)^*$.

Moreover, there exist nonadjacent $x, y \in T(M)^*$ by Lemma 4.1; thus $\mathrm{diam}(\mathrm{Tor}(\Gamma(M))) = 2$. □

Example 4.3. Let $R = \{0, 1, a\}$ be the idempotent semiring in which $1 + a = a + 1 = a$ and let $M = R \oplus R$. Then M is a semimodule over R with 9 elements. An inspection will show that $T(M) = \{(0,0), (1,0), (0,1), (a,0), (0,a)\}$ is not a subsemimodule of M and $M = \langle T(M)\rangle$. Moreover, $Tor(\Gamma(M))$ is disjoint from $Tof(\Gamma(M))$ and $Tof(\Gamma(M))$ is a totally disconnected subgraph of $T(\Gamma(M))$. Hence $T(\Gamma(M))$ is disconnected. So Theorem 3.1 (ii), (iii) and Theorem 3.2 in [12], in general, are not true when M is a semimodule over a semiring R.

Definition 4.4. A semimodule M over a semiring R is called a subtractive semimodule if every cyclic subsemimodule of M is a k-subsemimodule.

Example 4.5. Assume that E_+ be the set of all non-negative integers and let $M = R = E_+ \cup \{\infty\}$. Define $a + b = \max\{a, b\}$ and $ab = \min\{a, b\}$ for all $a, b \in R$. Then R is a commutative semiring with $1_R = \infty$ and $0_R = 0$. An inspection will show that the list of subsemimodules of M are: M, E_+ and for every non-negative integer n

$$N_n = \{0, 1, ..., n\}.$$

It is clear that every proper subsemimodule of M is a k-subsemimodule. So M is a subtractive semimodule.

Lemma 4.6. Let R be a semiring which is not a ring, and let M be a subtractive R-semimodule. Then $S(M) \subseteq T(M)$.

Proof. If $S(M) = \{0\}$, we are done. Suppose that $0 \neq x \in S(M)$. Then there is a $y \in S(M)$ such that $x + y = 0$. Thus $y \in Rx$ since Rx is a k-subsemimodule. Then there exists $r \in R$ such that $(1 + r)x = 0$. It then follows from [11, Lemma 2.1] that $1 + r \neq 0$. Thus $x \in T(M)$, as required. □

Theorem 4.7. Let R be a semiring which is not a ring, and let M be a subtractive R-semimodule. If $|S(M)| \geq 3$, then $gr(Tor(\Gamma(M))) = 3$.

Proof. Let $0 \neq x, y \in S(M)$. Then $x, y \in T(M)$ by Lemma 4.6 and $x + y = 0 \in T(M)$. Thus $0 - x - y - 0$ is a 3-cycle in $Tor(\Gamma(M))$. □

Theorem 4.8. Let M be a semimodule over a commutative semiring R such that $T(M)$ is not a subsemimodule of M.

(i) If $T(\Gamma(R))$ is connected, then $T(\Gamma(M))$ is connected.

(ii) Either $gr(Tor(\Gamma(M))) = 3$ or $gr(Tor(\Gamma(M))) = \infty$.

Proof. (i) It is clear that if $r \in Z(R)$ and $x \in M$, then $rx \in T(M)$. Assume that $y \in M$ and let $0 - a_1 - a_2 - \cdots - a_k - 1$ be a path from 0 to 1 in $T(\Gamma(R))$. Then $0 - a_1y - a_2y - \cdots - a_ky - y$ is a path from 0 to y in $T(\Gamma(M))$. Since all vertices may be connected via 0, $T(\Gamma(M))$ is connected.

(ii) If $x+y \in T(M)$ for some distinct $x, y \in T(M)^*$, then $0-x-y-0$ is a 3-cycle in $Tor(\Gamma(M))$; so $\mathrm{gr}(Tor(\Gamma(M))) = 3$. Otherwise, $x+y \in Tof(M)$ for all distinct $x, y \in T(M)$ by Lemma 4.1. Therefore, in this case, each $x \in T(M)^*$ is adjacent to 0, and no two distinct $x, y \in T(M)^*$ are adjacent. Thus $Tor(\Gamma(M))$ is a star graph with center 0; hence $\mathrm{gr}(Tor(\Gamma(M))) = \infty$. $\square$

Lemma 4.9. Let M be a semimodule over a commutative semiring R such that $T(M)$ is not a subsemimodule of M. Then $|Z(R)| \geq 3$.

Proof. By Lemma 4.1, there are distinct $x, y \in T(M)$ such that $x+y \in Tof(M)$. Thus there exist $0 \neq r, s \in R$ such that $rx = sy = 0$. Since $rs(x+y) = 0$ and $x+y \notin T(M)$, we have $rs = 0$, as needed. $\square$

Theorem 4.10. Let M be a semimodule over a commutative semiring R such that $T(M)$ is not a subsemimodule of M.

(i) If $Z(R)$ is an ideal of R, then $gr(Tor(\Gamma(M))) = 3$

(ii) If $Z(R)$ is not an ideal of R, then $gr(Tof(\Gamma(M))) = 3$ or ∞.

Proof. (i) Let $Z(R)$ is an ideal of R. By Lemma 4.9, there are non-zero elements r, s of $Z(R)$ with $r+s \in Z(R)$ and $rs = 0$. Therefore, $t(r+s) = 0$ for some non-zero element t of R. Now let $m, m' \in Tof(M)$. Since $t(r+s)m = t(r+s)m' = 0$, we have $rm + sm, rm' + sm' \in T(M)$. On the other hand, $rm + rm', sm + sm' \in T(M)$ since $s(rm + rm') = r(sm+sm') = 0$. Hence $rm - rm' - sm' - sm - rm$ is a 4-cycle in $T(\Gamma(M))$. Then $gr(Tor(\Gamma(M))) = 3$ by Theorem 4.8 (ii).

(ii) We may assume that $Tof(\Gamma(M))$ contains a cycle. So there is a path $x - y - z$ in $Tof(M)$. If $x+z \in T(M)$, then we have a 3-cycle in $Tof(\Gamma(M))$. So we may assume that $x+z \notin T(M)$. There exist $r_1, r_2 \in Z(R)$ such that $r_1 + r_2 \notin Z(R)$ since $Z(R)$ is not an ideal of R. So there are $0 \neq t_1, t_2 \in R$ such that $r_1t_1 = r_2t_2 = 0$ and then $t_1t_2 = 0$ since $t_1t_2(r_1+r_2) = 0$. Therefore $t_1x + t_1z \in T(M)$ since $t_2(t_1x + t_1z) = 0$. Thus $t_1x - t_1y - t_1z - t_1x$ is a 3-cycle in $Tof(\Gamma(M))$ and the proof is complete. $\square$

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