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RESEARCH ARTICLE

Free $(\ell r, rr)$-dibands

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Abstract. We prove that varieties of $(\ell r, rr)$-dibands and $(\ell n, rn)$-dibands coincide and describe the structure of free $(\ell r, rr)$-dibands. We also show that operations of an idempotent dimonoid with left (right) regular bands coincide, construct a new class of dimonoids and for such dimonoids give an example of a semiretraction.

1. Introduction

The notions of a dialgebra and a dimonoid were introduced by J.-L. Loday [1]. For further details and background see [1], [4].

Varieties of algebras are classes of algebras which can be given with the help of a set of identities. The special position of varieties in general algebra is defined by a circumstance that many structures in classical algebra such as groups, rings, lattices, Boolean algebras and etc. form a variety. In our time the most deep results and problems of the variety theory are connected with the investigation of concrete varieties and construction of relatively free algebras. A variety of dimonoids is a class of dimonoids which is characterized by some five identities. By the language of these identities we can express many properties of dimonoids and their classes. The first result about dimonoids is a Loday's description [1] of an absolute free dimonoid generated by a given set. Free commutative dimonoids and free rectangular dimonoids were construcred in [5] and [6] respectively. Free

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normal dibands and some other relatively free dimonoids were constructed in [7]. T. Pirashvili [2] considered sets with two associative operations (so-called duplexes) and constructed a free duplex. In Pirashvili's paper it is also considered duplexes with some additional conditions. For such duplexes the structure of free objects is known (see [2]).

In this paper we find necessary and sufficient conditions under which an arbitrary dimonoid is a $(\ell r, rr)$-diband and, as a consequence, establish that the variety of $(\ell r, rr)$-dibands coincides with the variety of $(\ell n, rn)$-dibands. In terms of dibands of subdimonoids we also describe the structure of free $(\ell r, rr)$-dibands. It turns out that operations of an idempotent dimonoid with left (right) regular bands coincide and it is a left (right) regular band. Moreover, we construct a new class of dimonoids and for such dimonoids give an example of a semiretraction.

We refer to [6] and [7] for the terminology and notations.

2. A new class of dimonoids

In this section we construct a new class of dimonoids via semiretractions [8] and for dimonoids from this class give an example of a semiretraction.

A transformation τ of a dimonoid $(D, \dashv, \vdash)$ is called a left semiretraction, if

$$(x \dashv y)\tau = (x\tau \dashv y)\tau, \tag{1}$$

$$(x \vdash y)\tau = (x\tau \vdash y)\tau \tag{2}$$

for all $x, y \in D$. If instead of (1), (2) the identities

$$(x \dashv y)\tau = (x \dashv y\tau)\tau, \tag{3}$$

$$(x \vdash y)\tau = (x \vdash y\tau)\tau \tag{4}$$

hold, then we say about a right semiretraction. If for τ the identities (1) – (4) hold, then τ is called a (symmetric) semiretraction of $(D, \dashv, \vdash)$.

If operations of a dimonoid coincide, then from the definition of a left (right, symmetric) semiretraction of a dimonoid we obtain the notion of a left (right, symmetric) semiretraction of a semigroup (see [3], [9]).

For any transformation π of a dimonoid $(D, \dashv, \vdash)$ let

$$\nabla_\pi = \{(x, y) \in D \times D \mid x\pi = y\pi\}.$$

The following statement gives general characteristic of (symmetric) semiretractions.

Proposition 1 ([8], Sect. 3.2, Proposition)**.** *For an idempotent transformation π of a dimonoid $(D, \dashv, \vdash)$ the following statements are equivalent:*

1. *π is a (symmetric) semiretraction;*
2. *π is a left semiretraction and ∇_π is a congruence on $(D, \dashv, \vdash)$;*
3. *π is a right semiretraction and ∇_π is a congruence on $(D, \dashv, \vdash)$;*
4. *for all $x, y \in D$ the identities*

$$(x \dashv y)\pi = (x\pi \dashv y\pi)\pi,\ (x \vdash y)\pi = (x\pi \vdash y\pi)\pi$$

hold.

Thus, the problem of the description of congruences on dimonoids from a given class is reduced to the description of semiretractons of dimonoids. That is, if we know the action of a semiretraction π on a dimonoid, then we can construct the unique congruence ∇_π which corresponds to π. Conversely, if we know the structure of a congruence on a dimonoid, then we can give the class of semiretractions π such that relations ∇_π coincide with a given congruence.

Examples of semiretractions of dimonoids can be found in [8].

Let S be an arbitrary semigroup and τ be an idempotent semiretraction of S. Define operations $\dashv$ and $\vdash$ on $S \times S$ by

$$(a, b) \dashv (c, d) = (a, (bcd)\tau),$$

$$(a, b) \vdash (c, d) = ((abc)\tau, d)$$

for all $(a, b), (c, d) \in S \times S$. The algebra $(S \times S, \dashv, \vdash)$ will be denoted by $S[\tau]$.

The following statement gives the opportunity to construct new dimonoids with the help of semiretractions.

Proposition 2. *For every idempotent semiretraction τ of a semigroup S the algebra $S[\tau]$ is a dimonoid.*

Proof. For all $(a, b), (c, d), (x, y) \in S \times S$ we obtain

$$((a, b) \dashv (c, d)) \dashv (x, y) = (a, (bcd)\tau) \dashv (x, y) =$$

$$= (a, ((bcd)\tau\, xy)\tau) = (a, (bcd\, xy)\tau),$$

$$(a, b) \dashv ((c, d) \dashv (x, y)) = (a, b) \dashv (c, (dxy)\tau) =$$

$$= (a, (bc(dxy)\tau)\tau) = (a, (bcd\,xy)\tau),$$

$$(a, b) \dashv ((c, d) \vdash (x, y)) = (a, b) \dashv ((cdx)\tau, y) =$$

$$= (a, (b(cdx)\tau\, y)\tau) = (a, ((b(cdx)\tau)\tau\, y)\tau) =$$

$$= (a, ((bcdx)\tau\, y)\tau) = (a, (bcd\,xy)\tau),$$

$$((a, b) \vdash (c, d)) \vdash (x, y) = ((abc)\tau, d) \vdash (x, y) =$$

$$= (((abc)\tau\, dx)\tau, y) = ((abcdx)\tau, y),$$

$$(a, b) \vdash ((c, d) \vdash (x, y)) = (a, b) \vdash ((cdx)\tau, y) =$$

$$= ((ab(cdx)\tau)\tau\, , y) = ((abcdx)\tau, y),$$

$$((a, b) \dashv (c, d)) \vdash (x, y) = (a, (bcd)\tau) \vdash (x, y) =$$

$$= ((a(bcd)\tau\, x)\tau, y) = (((a(bcd)\tau)\tau\, x)\tau, y) =$$

$$= (((abcd)\tau\, x)\tau, y) = ((abcdx)\tau, y),$$

$$((a, b) \vdash (c, d)) \dashv (x, y) =$$

$$= ((abc)\tau, d) \dashv (x, y) = ((abc)\tau, (dxy)\tau),$$

$$(a, b) \vdash ((c, d) \dashv (x, y)) =$$

$$= (a, b) \vdash (c, (dxy)\tau) = ((abc)\tau, (dxy)\tau)$$

according to Proposition 1. Comparing these expressions, we conclude that $S[\tau]$ is a dimonoid. □

Observe that a dimonoid which was constructed by J.-L. Loday in [1] is a particular case of the dimonoid $S[\tau]$. Indeed, if S is a monoid and τ is an identity transformation of S, then $S[\tau]$ coincides with a dimonoid from [1], see p.12.

We finish this section with the consideration of a semiretraction of the dimonoid $S\,[\tau]$.

Let $\pi : S\,[\tau] \to S\,[\tau] : (a, b) \mapsto (a\tau, b\tau)$.

Lemma 1. *π is a semiretraction of the dimonoid $S\,[\tau]$.*

Proof. For all $(a, b), (c, d) \in S[\tau]$ we have

$$((a, b) \dashv (c, d))\,\pi = (a, (bcd)\,\tau)\,\pi = (a\tau, (bcd)\,\tau) =$$
$$= (a\tau, ((bc)\,\tau d\tau)\,\tau) = (a\tau, ((b\tau c\tau)\,\tau d\tau)\,\tau) =$$
$$= (a\tau, (b\tau c\tau d\tau)\,\tau) = (a\tau, (b\tau c\tau d\tau)\,\tau)\,\pi =$$
$$= ((a\tau, b\tau) \dashv (c\tau, d\tau))\,\pi = ((a, b)\,\pi \dashv (c, d)\,\pi)\,\pi,$$
$$((a, b) \vdash (c, d))\,\pi = ((abc)\,\tau, d)\,\pi = ((abc)\,\tau, d\tau) =$$
$$= (((ab)\,\tau c\tau)\,\tau, d\tau) = (((a\tau b\tau)\,\tau c\tau)\,\tau, d\tau) =$$
$$= ((a\tau b\tau c\tau)\,\tau, d\tau) = ((a\tau b\tau c\tau)\,\tau, d\tau)\,\pi =$$
$$= ((a\tau, b\tau) \vdash (c\tau, d\tau))\,\pi = ((a, b)\,\pi \vdash (c, d)\,\pi)\,\pi.$$

Hence, by Proposition 1, π is a semiretraction. □

3. Dimonoids and left (right) regular bands

In this section we show that operations of an idempotent dimonoid $(D, \dashv, \vdash)$ with a left (respectively, right) regular band $(D, \vdash)$ (respectively, $(D, \dashv)$) coincide. We also find necessary and sufficient conditions under which an arbitrary dimonoid is a $(\ell r, rr)$-diband and, as a consequence, establish that the variety of $(\ell r, rr)$-dibands coincides with the variety of $(\ell n, rn)$-dibands. At the end of the section we describe the structure of free $(\ell r, rr)$-dibands.

Recall that an idempotent semigroup S is called a left regular band, if

$$aba = ab \tag{5}$$

for all $a, b \in S$. If instead of (5) the identity

$$aba = ba \tag{6}$$

holds, then S is a right regular band. An idempotent semigroup S is called a left normal band, if

$$axy = ayx \tag{7}$$

for all $a, x, y \in S$. If instead of (7) the identity

$$xya = yxa \tag{8}$$

holds, then S is a right normal band. A semigroup S is called left (respectively, right) commutative, if it satisfies the identity $xya = yxa$ (respectively, $axy = ayx$).

Lemma 2. *Operations of a dimonoid* $(D, \dashv, \vdash)$ *with an idempotent operation* $\dashv$ *(respectively,* $\vdash$*) coincide, if* $(D, \vdash)$ *(respectively,* $(D, \dashv)$*) is a left (respectively, right) regular band.*

Proof. For all $x, y \in D$ we have

$$x \dashv y = (x \dashv y) \vdash (x \dashv y) = x \vdash (y \vdash (x \dashv y)) =$$
$$= x \vdash ((y \vdash x) \dashv y) = (x \vdash (y \vdash x)) \dashv y =$$
$$= (x \vdash y) \dashv y = x \vdash (y \dashv y) = x \vdash y$$

according to the idempotent property of operations, the axioms of a dimonoid and the identity (5).

For all $x, y \in D$ we have

$$x \vdash y = (x \vdash y) \dashv (x \vdash y) = ((x \vdash y) \dashv x) \dashv y =$$
$$= (x \vdash (y \dashv x)) \dashv y = x \vdash ((y \dashv x) \dashv y) =$$
$$= x \vdash (x \dashv y) = (x \vdash x) \dashv y = x \dashv y$$

according to the idempotent property of operations, the axioms of a dimonoid and the identity (6). □

From Lemma 2 it follows that an idempotent dimonoid $(D, \dashv, \vdash)$ with left (respectively, right) regular bands $(D, \dashv)$ and $(D, \vdash)$ is a left (respectively, right) regular band.

If $(D, \dashv)$ is a semigroup, then D with an operation $\vdash$, defined by $x \vdash y = y \dashv x$ for all $x, y \in D$, is a semigroup. The semigroup $(D, \vdash)$ is called dual to the semigroup $(D, \dashv)$.

Lemma 3. *Let* $(D, \dashv)$ *be an arbitrary semigroup and* $(D, \vdash)$ *be a dual semigroup to* $(D, \dashv)$*. Then an algebra* $(D, \dashv, \vdash)$ *is a dimonoid if and only if* $(D, \dashv)$ *is a right commutative semigroup.*

Proof. Let $(D, \dashv, \vdash)$ be a dimonoid and $(D, \vdash)$ be dual to $(D, \dashv)$. Then for all $x, y, z \in D$,

$$x \dashv y \dashv z = x \dashv (y \vdash z) = x \dashv (z \dashv y)$$

according to the axiom of a dimonoid and the definition of the operation $\vdash$. Hence $(D, \dashv)$ is a right commutative semigroup.

Conversely, let $(D, \dashv)$ be a right commutative semigroup and $x, y, z \in D$. Then

$$(x \dashv y) \dashv z = x \dashv (z \dashv y) = x \dashv (y \vdash z),$$

$$(x \vdash y) \dashv z = (y \dashv x) \dashv z = y \dashv (x \dashv z) =$$
$$= y \dashv (z \dashv x) = (y \dashv z) \dashv x = x \vdash (y \dashv z),$$
$$x \vdash (y \vdash z) = (y \vdash z) \dashv x = z \dashv (y \dashv x) =$$
$$= z \dashv (x \dashv y) = (x \dashv y) \vdash z$$

according to the definition of the operation $\vdash$ and the right commutativity of $(D, \dashv)$. So, $(D, \dashv, \vdash)$ is a dimonoid. □

Dually, the following lemma can be proved.

Lemma 4. *Let $(D, \vdash)$ be an arbitrary semigroup and $(D, \dashv)$ be a dual semigroup to $(D, \vdash)$. Then an algebra $(D, \dashv, \vdash)$ is a dimonoid if and only if $(D, \vdash)$ is a left commutative semigroup.*

A dimonoid $(D, \dashv, \vdash)$ will be called a $(\ell r, rr)$-diband, if $(D, \dashv)$ is a left regular band and $(D, \vdash)$ is a right regular band. Recall that a dimonoid $(D, \dashv, \vdash)$ is called a $(\ell n, rn)$-diband, if $(D, \dashv)$ is a left normal band and $(D, \vdash)$ is a right normal band [7].

Note that every left (right) normal band is left (right) regular. The converse statement is not true. It is natural to consider the similar question for $(\ell r, rr)$-dibands and $(\ell n, rn)$-dibands.

The following theorem gives necessary and sufficient conditions under which an arbitrary dimonoid is a $(\ell r, rr)$-diband.

Theorem 1. *A dimonoid $(D, \dashv, \vdash)$ is a $(\ell r, rr)$-diband if and only if $(D, \dashv, \vdash)$ is a $(\ell n, rn)$-diband.*

Proof. Let $(D, \dashv, \vdash)$ be an arbitrary $(\ell r, rr)$-diband. For all $x, y \in D$ we have

$$x \vdash y \dashv x = (y \vdash (x \vdash y)) \dashv x = y \vdash ((x \vdash y) \dashv x) =$$
$$= y \vdash (x \vdash (y \dashv x)) = (y \dashv x) \vdash (y \dashv x) = y \dashv x,$$
$$x \vdash y \dashv x = x \vdash ((y \dashv x) \dashv y) = (x \vdash (y \dashv x)) \dashv y =$$
$$= ((x \vdash y) \dashv x) \dashv y = (x \vdash y) \dashv (x \vdash y) = x \vdash y$$

according to the idempotent property of operations, the axioms of a dimonoid and the identities (6) and (5). Hence

$$x \vdash y = y \dashv x \tag{9}$$

for all $x, y \in D$.

Further by Lemma 3 $(D, \dashv)$ is a right commutative semigroup. As $(D, \dashv)$ is also idempotent, then $(D, \dashv)$ is a left normal band.

Let $x, y, z \in D$. From

$$x \dashv y \dashv z = x \dashv z \dashv y$$

we have

$$z \vdash y \vdash x = y \vdash z \vdash x$$

by (9). So, $(D, \vdash)$ is a left commutative semigroup. As $(D, \vdash)$ is also idempotent, then $(D, \vdash)$ is a right normal band.

Thus, $(D, \dashv, \vdash)$ is a $(\ell n, rn)$-diband.

Conversely, let $(D, \dashv, \vdash)$ be a $(\ell n, rn)$-diband. Then

$$x \dashv y \dashv z = x \dashv z \dashv y,$$

$$z \vdash y \vdash x = y \vdash z \vdash x$$

for all $x, y, z \in D$. Substituting $z = x$ in the last two equalities and using the idempotent property of operations, we obtain

$$x \dashv y \dashv x = x \dashv y, \; x \vdash y \vdash x = y \vdash x.$$

Thus, $(D, \dashv, \vdash)$ is a $(\ell r, rr)$-diband. □

From Theorem 1 we obtain

Corollary 1. *The variety of $(\ell r, rr)$-dibands coincides with the variety of $(\ell n, rn)$-dibands.*

We call a dimonoid which is free in the variety of $(\ell r, rr)$-dibands (respectively, $(\ell n, rn)$-dibands) a free $(\ell r, rr)$-diband (respectively, free $(\ell n, rn)$-diband).

Let X be an arbitrary nonempty set. Assume $(X, \dashv)$ and $(X, \vdash)$ be a left zero semigroup and a right zero semigroup respectively. Recall that we call the dimonoid $X_{\ell z, rz} = (X, \dashv, \vdash)$ as a left and right diband (see [6]).

Let $B(X)$ be the semilattice of all nonempty finite subsets of X with respect to the operation of the set theoretical union, $X_{\ell z, rz}$ be a left and right diband and

$$B_{\ell z, rz}(X) = \{(x, A) \in X_{\ell z, rz} \times B(X) \,|\, x \in A\}.$$

By [7] $B_{\ell z, rz}(X)$ is the free $(\ell n, rn)$-diband.

From Theorem 1 we obtain

Corollary 2. *$B_{\ell z,rz}(X)$ is the free $(\ell r, rr)$-diband.*

For all $Y \in B(X)$, $i \in X$ put

$$T^Y = \{(x, A) \in B_{\ell z,rz}(X) \,|\, A = Y\},$$

$$T_{(i)} = \{(x, A) \in B_{\ell z,rz}(X) \,|\, x = i\},$$

$$B_i(X) = \{A \in B(X) \,|\, i \in A\}.$$

The notion of a diband of subdimonoids (see [10], [11]) is effective to describe structural properties of dimonoids. In terms of dibands of subdimonoids (see also [12]), similar to Theorems 4 (vii) and 3 (iv) from [7], the following theorem can be proved.

Theorem 2. *Let $B_{\ell z,rz}(X)$ be the free $(\ell r, rr)$-diband. Then*

(i) *$B_{\ell z,rz}(X)$ is the free semilattice $B(X)$ of subdimonoids T^Y, $Y \in B(X)$, such that $T^Y \cong Y_{\ell z,rz}$ for every $Y \in B(X)$;*

(ii) *$B_{\ell z,rz}(X)$ is a left and right diband $X_{\ell z,rz}$ of subsemigroups $T_{(i)}$, $i \in X_{\ell z,rz}$, such that $T_{(i)} \cong B_i(X)$ for every $i \in X_{\ell z,rz}$.*

If $\varphi : D_1 \to D_2$ is a homomorphism of dimonoids, then the corresponding congruence on D_1 will be denoted by Δ_φ.

Let

$$\mu : B_{\ell z,rz}(X) \to B(X) : (x, A) \mapsto (x, A)\mu = A,$$

$$q : B_{\ell z,rz}(X) \to X_{\ell z,rz} : (x, A) \mapsto (x, A)q = x.$$

It is evident that μ and q are homomorphisms.

If ρ is a congruence on a dimonoid $(D, \dashv, \vdash)$ such that operations of $(D, \dashv, \vdash)/\rho$ coincide and it is a semilattice, then we say that ρ is a semilattice congruence. If ρ is a congruence on a dimonoid $(D, \dashv, \vdash)$ such that $(D, \dashv, \vdash)/\rho$ is a left and right diband, then we say that ρ is a left zero and right zero congruence.

From Theorem 2 we obtain

Corollary 3. *Let $B_{\ell z,rz}(X)$ be the free $(\ell r, rr)$-diband. Then*

(i) *Δ_μ is the least semilattice congruence on $B_{\ell z,rz}(X)$;*

(ii) *Δ_q is the least left zero and right zero congruence on $B_{\ell z,rz}(X)$.*

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