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SURVEY ARTICLE

On Baer-Shemetkov’s decomposition in modules and related topics

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Dedicated to the memory of an outstanding mathematician Leonid A. Shemetkov

Abstract. This article is dedicated to the memory of an outstanding algebraist Leonid A. Shemetkov. His ideas and results not only shaped modern soluble finite group theory, but significantly influenced other branches of algebra. In this article, we traced the influence of L.A. Shemetkov’s ideas on some areas of modules theory and infinite groups theory.

On March 24, 2013, one of the greatest experts in the theory of finite groups Leonid A. Shemetkov passed away. He was one of those people who developed and shaped the modern theory of finite non-simple groups. His contribution to this field of algebra is hard to overestimate. L.A. Shemetkov productively and intensively investigated finite groups, and his ideas and influence on its development have been reflected in numerous works of his many students and followers. Led by him, the Gomel School is one of the world leading scientific schools in group theory. His outstanding results have been reflected in books and review articles. L.A. Shemetkov’s research affected significantly finite group theory development, and his ideas have been expanded to some important areas of infinite groups as well. Not only had the people directly related to his school felt L.A. Shemetkov’s influence. L.A. Shemetkov and his school had tight relations with Ukrainian algebraists. Along with his results,

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direct communications with him and his views on the problems have helped to shape some productive approaches to infinite groups study. In the current paper, we set out to do a review of the well known Shemetkov's results; it has been done already. We want to reflect something else. Here we would like to consider (and to show in the process of their development) some cross-related areas of group theory and module over group rings theory, the shaping of which L.A. Shemetkov had a great influence. The theory of infinite groups is significantly distinct from the theory of finite groups. The expanding of the results obtained for finite groups to infinite groups often possible only for sufficiently narrow classes of infinite groups, or instead of a single counterpart it leads to a series of results. And formation theory plays an important role here. In finite groups, this theory has been is rapidly developing for the last 40 years. L.A. Shemetkov was one of those who laid the foundation of formation theory (see the book [17]). The intensive development of finite group formation theory caused a natural interest among professionals working in infinite groups. A whole series of works devoted to the extension of the main results of formations of finite groups to locally finite groups have been published. Thus the concept of the formation was substantially narrowed, namely the condition (**F2**) (please see below) was replaced by the following:

if $\{H_\lambda | \lambda \in \Lambda\}$ is a family of normal subgroups, then $G/\bigcap_{\lambda\in\Lambda} H_\lambda \in \mathfrak{X}$.

There was quite a long time when the theme the purpose of which was almost a literal transfer of the main results of the theory of formations obtained for finite groups to certain classes of infinite groups has evolved. The transfer of the results of formation theory was possible only where there exists a very extensive Sylow theory (for example, in periodic almost locally nilpotent groups, periodic linear groups, and periodic FC-groups). Under the mentioned narrowed definition, many natural classes of groups, such as, finite groups, (locally) soluble groups, (locally) nilpotent groups, π-groups, etc., are not the formations. We will use here the classical definition of a formation, i.e. namely the definition of a formation that has arisen and works in finite groups.

A class of groups $\mathfrak{X}$ is said to be *a formation of groups* if the following conditions hold:

(**F1**) If $G \in \mathfrak{X}$ and H is a normal subgroup of G, then $G/H \in \mathfrak{X}$;

(**F2**) If H, K are the normal subgroups of G such that $G/H \in \mathfrak{X}$, $G/K \in \mathfrak{X}$, then $G/(H \cap K) \in \mathfrak{X}$.

For these formations, some of the commonly used in finite groups approaches work very well in infinite groups. They began to be implemented first for the particular formation, i.e. for the formation $\mathfrak{F}$ of all finite groups, and then for other formations which play an important role in the theory of infinite groups. Here we want to show some areas where it is effectively implemented. One of this areas, which are so remote from the area of finite groups, is artinian modules over group rings. An important aspect of the theory of Artinian modules is a problem of finding their natural direct decompositions. One of the first results in this area is the following well-known Fitting's Lemma. Here we present it in the following form.

Theorem 1 (Fitting)**.** *Let R be a ring, G a finite nilpotent group and A an RG-module. If A has a finite composition length, then $A = Z \oplus E$, where every RG-chief factor U/V of Z (respectively of E) satisfies the condition $G = C_G(U/V)$ (respectively, $G \neq C_G(U/V)$).*

Now we need some basic concepts. One of the first basic concepts of the formation theory is the concept of an $\mathfrak{X}$-central factor. We will formulate them in parallel for groups and modules.

Let G be a group, U, V be normal subgroups of G, such that $U \leq V$. The factor V/U is called $\mathfrak{X}$-*central* (respectively, $\mathfrak{X}$-*eccentric*), if $G/C_G(V/U) \in \mathfrak{X}$ (respectively, $G/C_G(V/U) \notin \mathfrak{X}$). Clearly, if $\mathfrak{X} = \mathfrak{I}$ is a class of all identity groups, then we come to the concept of central factor. Let G be a finite group and suppose that G has a series of normal subgroups $\langle 1 \rangle = H_0 \leq H_1 \leq ... \leq H_{n-1} \leq H_n = G$ whose factors are $\mathfrak{X}$-central. If $g \in H_1$, then $g^G = \{g^x | x \in G\} \subseteq H_1$. If $\mathfrak{X}$ is a formation, then $G/C_G(g^G) \in \mathfrak{X}$ and we come to the following important concept.

Let G be a group and $\mathfrak{X}$ be a class of groups. Put

$$\mathfrak{X}C(G) = \{x \in G | G/C_G(g^G) \in \mathfrak{X}\}.$$

If $\mathfrak{X}$ is a formation, then it is not hard to prove that $\mathfrak{X}C(G)$ is a characteristic subgroup of G. A subgroup $\mathfrak{X}C(G)$ is said to be the $\mathfrak{X}C$-*center* of a group G. If $\mathfrak{X} = \mathfrak{I}$ is a class of all identity groups, then $\mathfrak{X}C(G) = \zeta(G)$ is an ordinary center of G. A group G is called an $\mathfrak{X}C$-*group* if $G = \mathfrak{X}C(G)$. Thus we see that the class of $\mathfrak{X}C$-groups is a natural extension of the class of abelian groups. If $\mathfrak{X} = \mathfrak{F}$ then $\mathfrak{X}C$-group is called an FC-*group.*

Starting from the $\mathfrak{X}C$-center of a group G, we construct *the upper $\mathfrak{X}C$-central series of G* as

$$\langle 1 \rangle = B_0 \leq B_1 \leq ... \leq B_\alpha \leq B_{\alpha+1} \leq ... \leq B_\gamma,$$

where $B_1 = \mathfrak{X}C(A)$, $B_{\alpha+1}/B_\alpha = \mathfrak{X}C(A/B_\alpha)$ for all ordinals $\alpha < \gamma$ and $\langle 0\rangle = \mathfrak{X}C(A/B_\gamma)$. The last term B_γ of this series is called the *upper* $\mathfrak{X}C$-*hypercenter of* A and is denoted by $\mathfrak{X}C^\infty(A)$. If $A = \mathfrak{X}C^\infty(A)$, then A is said to be $\mathfrak{X}C$-*hypercentral*; if γ is finite, then A is called $\mathfrak{X}C$-*nilpotent*.

Let $\mathfrak{X}$ be a class of groups, G a group and

$$\mathfrak{H}(\mathfrak{X}) = \{H | H \text{ is a normal subgroup of } G \text{ such that } G/H \in \mathfrak{X}\},$$
$$G^{\mathfrak{X}} = \bigcap \mathfrak{H}(\mathfrak{X}) = \bigcap\nolimits_{H\in\mathfrak{H}(\mathfrak{X})} H.$$

A subgroup $G^{\mathfrak{X}}$ is called the $\mathfrak{X}$-*residual of a group* G. If $\mathfrak{X}$ is a formation and a family $\mathfrak{H}(\mathfrak{X})$ is finite (in particular, if G is finite), then $G/G^{\mathfrak{X}} \in \mathfrak{X}$.

There are some analogs of these concepts for modules over group rings.

Let R be a ring, G a group and A an RG-module. If B, C are RG-submodules of A such that $B \le C$, the factor C/B is said to be $\mathfrak{X}$-*central* (respectively, $\mathfrak{X}$-*eccentric*) if $G/C_G(C/B) \in \mathfrak{X}$ (respectively, $G/C_G(C/B) \notin \mathfrak{X}$).

To rule out these factors, we define

$$\mathfrak{X}C_{RG}(A) = \{a \in A | G/C_G(aRG) \in \mathfrak{X}\}.$$

Observed at once, that if $\mathfrak{X}$ is a formation of groups, then $\mathfrak{X}C_{RG}(A)$ is an RG-submodule of A. The submodule $\mathfrak{X}C_{RG}(A)$ is called the $\mathfrak{X}C - RG$-*center of* A (shorter, the $\mathfrak{X}C$-*center of* A). Started from $\mathfrak{X}C$-center, we construct *the upper* $\mathfrak{X}C - RG$-*central series of the module* A as

$$\langle 0\rangle = A_0 \le A_1 \le \dots \le A_\alpha \le A_{\alpha+1} \le \dots \le A_\gamma,$$

where $A_1 = \mathfrak{X}C_{RG}(A)$, $A_{\alpha+1}/A_\alpha = \mathfrak{X}C_{RG}(A/A_\alpha)$ for all ordinals $\alpha < \gamma$ and $\mathfrak{X}C_{RG}(A/A_\gamma) = \langle 0\rangle$. The last term A_γ of this series is called the *upper* $\mathfrak{X}C - RG$-*hypercenter of* A (in short, the $\mathfrak{X}C$-*hypercenter of* A) and is denoted by $\mathfrak{X}C^\infty_{RG}(A)$. If $A = \mathfrak{X}C^\infty_{RG}(A)$, then A is said to be $\mathfrak{X}C - RG$-*hypercentral*; if γ is finite, then A is called $\mathfrak{X}C - RG$-*nilpotent*.

We note that, if $\mathfrak{X} = \mathfrak{I}$ is the class of all identity groups, then $\mathfrak{X}C_{RG}(A) = \zeta_{RG}(A)$ is called the RG-*center of* A and $\mathfrak{X}C^\infty_{RG}(A) = \zeta^\infty_{RG}(A)$ is called *the upper* RG-*hypercenter of* A. If $\mathfrak{X} = \mathfrak{F}$ is the class of all finite groups, then $\mathfrak{X}C_{RG}(A) = FC_{RG}(A)$ is called *the* FC-*center of* A and $\mathfrak{X}C^\infty_{RG}(A) = FC^\infty_{RG}(A)$ is called *the upper* FC-*hypercenter of* A.

An RG-submodule C of A is said to be $\mathfrak{X} - RG$-hypereccentric if it has an ascending series

$$\langle 0\rangle = C_0 \le C_1 \le \dots \le C_\alpha \le C_{\alpha+1} \le \dots \le C_\gamma = C$$

of RG-submodules of A such that each factor $C_{\alpha+1}/C_{\alpha}$ is an $\mathfrak{X}$-eccentric simple RG-module for every ordinal $\alpha < \gamma$.

If A is an artinian RG-module, then it has an ascending series of RG-submodules whose factors are simple RG-modules. If $\mathfrak{X}$ is an arbitrary formation and $G \in \mathfrak{X}$, then the factors of this series can be located in some different ways. L.A. Shemetkov indicated the formations (in the finite groups case) where it will be the most convenient location.

Theorem 2 (L.A. Shemetkov [16])**.** *Let G be a finite group and $\mathfrak{X}$ be a local formation. If p be a prime such that a Sylow p-subgroup P of $G^{\mathfrak{X}}$ is abelian, then every chief p-factor of $G^{\mathfrak{X}}$ is $\mathfrak{X}$-eccentric. In particular, if Sylow q-subgroups of $G^{\mathfrak{X}}$ are abelian for every prime q, then every chief factor of $G^{\mathfrak{X}}$ is $\mathfrak{X}$-eccentric.*

Suppose now that G includes a normal abelian subgroup A such that $G/A \in \mathfrak{X}$ (that is $G^{\mathfrak{X}} \leq A$). From some other results of this Shemetkov's paper (we will consider it a little bit later), one can derive the existence of a subgroup C such that $G = G^{\mathfrak{X}}C$ and $C \cap G^{\mathfrak{X}} = \langle 1 \rangle$. Put $E = C \cap A$, then clearly E is normal in G and $A = G^{\mathfrak{X}} \times E$. By Theorem 2, every chief factor of $G^{\mathfrak{X}}$ is $\mathfrak{X}$-eccentric. The same result was obtained in the paper of R. Baer [1]. Observe, that if $\mathfrak{X}$ is a local formation, then every finite $\mathfrak{X}C$-nilpotent group belongs to $\mathfrak{X}$ (see K. Doerk and T.O. Hawkes [[2], Theorem IV.3.2]). Thus we come to the following concept.

We say that the RG-module A has the *Baer-Shemetkov's decomposition* for the formation $\mathfrak{X}$, or A has the *Baer-Shemetkov's $\mathfrak{X} - RG$-decomposition* if

$$A = \mathfrak{X}C_{RG}^{\infty}(A) \oplus \mathfrak{X}E_{RG}^{\infty}(A),$$

where $\mathfrak{X}E_{RG}^{\infty}(A)$ is the maximal $\mathfrak{X} - RG$-hypereccentric RG-submodule of A. It is possible to prove that in this case, $\mathfrak{X}E_{RG}^{\infty}(A)$ includes every $\mathfrak{X} - RG$-hypereccentric RG-submodule and, in particular, $\mathfrak{X}E_{RG}^{\infty}(A)$ is defined uniquely.

If $\mathfrak{X} = \mathfrak{I}$, we will say about the *Z-decomposition*, whereas if $\mathfrak{X} = \mathfrak{F}$, we will say about the *$\mathfrak{F}$-decomposition.* The first natural question for infinite modules was the question of the conditions under which there is the Z-decomposition, and the first work where this issue has been addressed in the paper of B. Hartley and M.J. Tomkinson [7].

Theorem 3 (B. Hartley and M.J. Tomkinson [7])**.** *Let G be a locally nilpotent group and let A be a $\mathbb{Z}G$-module. If A is $\mathbb{Z}$-periodic and the p-component of A are artinian as $\mathbb{Z}$-modules (that is are Chernikov groups) for every prime p, then A has the Z-decomposition.*

In this case, the factor-group $G/C_G(A)$ is actually hypercentral. The following case, namely the case of artinian $\mathbb{Z}G$-module over hypercentral group G, has been considered by D.I. Zaitsev.

Theorem 4 (D.I. Zaitsev [18]). *Let G be a hypercentral group and let A be a $\mathbb{Z}G$-module. If A is an artinian $\mathbb{Z}G$-module, then A has the Z-decomposition.*

Note that this result can be easily extended to an artinian DG-module, where D is a Dedekind domain and G is a hypercentral group.

The following natural step was the consideration of existence of the $\mathfrak{F}$-decomposition for artinian modules. We notice first that in this deliberation the considered group has to be FC-hypercentral instead of hypercentral. The first results in this direction have been obtained by Zaitsev.

Theorem 5 (D.I. Zaitsev [19]). *Let G be a hyperfinite locally soluble group and A a $\mathbb{Z}G$-module. If A is an artinian $\mathbb{Z}G$-module, then A has the $\mathfrak{F}$-decomposition.*

Theorem 6 (D.I. Zaitsev [20]). *Let G be an FC-hypercentral group and let A be a $\mathbb{Z}G$-module. If A has finite composition $\mathbb{Z}G$-series, then A has the $\mathfrak{F}$-decomposition.*

The next result belongs to Z.Y. Duan, who considered the following special case of FC-hypercentral groups.

Theorem 7 (Z.Y. Duan [3]). *Let G be a locally soluble group having an ascending series of normal subgroups, every factor of which is finite or cyclic and let A be a $\mathbb{Z}G$-module. If A is an artinian $\mathbb{Z}G$-module, then A has the $\mathfrak{F}$-decomposition.*

The most general solution for the formation $\mathfrak{F}$ was obtained by L.A. Kurdachenko, B.V. Petrenko and I.Ya. Subbotin.

Theorem 8 (L.A. Kurdachenko, B.V. Petrenko, I.Ya. Subbotin [8]). *Let G be a locally soluble FC-hypercentral group, D a Dedekind domain and A a DG-module. If A is an artinian DG-module, then A has the $\mathfrak{F}$-decomposition.*

These partial cases have played an important role in the study of the general case of existence of Baer-Shemetkov's $\mathfrak{X}$-decomposition. Actually, its solution has allowed us to obtain solutions for many important formations $\mathfrak{X}$. To deal with this, it is convenient to split the entire study into two cases: $\mathfrak{X}$ includes $\mathfrak{F}$ and $\mathfrak{X}$ is a proper formation of finite groups.

A formation $\mathfrak{X}$ is said to be *overfinite* if it satisfies the following conditions:

(i) if $G \in \mathfrak{X}$ and H is a normal subgroup of G having finite index, then $H \in \mathfrak{X}$;

(ii) if G is a group, H is a normal subgroup of G such that G/H is finite and $H \in \mathfrak{X}$, then $G \in \mathfrak{X}$;

(iii) $\mathfrak{I} \leq \mathfrak{X}$.

Clearly, every overfinite formation always includes $\mathfrak{F}$. The most important examples of these formations are polycyclic groups, Chernikov groups, soluble minimax groups, soluble groups of finite special rank, and soluble groups of finite section rank. The most general result here is the following

Theorem 9. *Let G be a locally soluble FC-hypercentral group, D a Dedekind domain and A be an artinian DG-module. If $\mathfrak{X}$ is an overfinite formation of groups, then A has the Baer-Shemetkov's $\mathfrak{X}$-decomposition.*

For the case $D = \mathbb{Z}$, this result was obtained by L.A. Kurdachenko, B.V. Petrenko and I.Ya. Subbotin in the paper [9]. In a general form it was placed in the book [[11], Chapter 10]. It is worth mentioning that since every overfinite formation $\mathfrak{X}$ includes the formation of all finite groups, every FC-hypercentral group is likewise $\mathfrak{X}C$-hypercentral.

Corollary 1. *Let G be a locally soluble FC-hypercentral group, D a Dedekind domain and A be an artinian DG-module. Then A has the Baer-Shemetkov's $\mathfrak{X}$-decomposition for the following formations $\mathfrak{X}$:*

(i) *$\mathfrak{X}$ is the formation of all polycyclic groups;*

(ii) *$\mathfrak{X}$ is the formation of all Chernikov groups;*

(iii) *$\mathfrak{X}$ is the formation of all soluble minimax groups;*

(iv) *$\mathfrak{X}$ is the formation of all soluble groups of finite special rank;*

(v) *$\mathfrak{X}$ is the formation of all soluble groups of finite section rank.*

In connection with Theorem 9, the following question naturally arises: for what formations of finite groups does the Baer-Shemetkov's decomposition exist? Before dealing with it, we first notice that infinite groups behave badly with respect to some properties automatically satisfied by finite groups. For example, suppose that $\mathfrak{X}$ is a formation of finite groups and G is a residually finite group. If G is finite, then $G \in \mathfrak{X}$. However, if G is infinite, the situation can be totally different. To avoid this complication, we consider the following formation of finite groups.

A formation $\mathfrak{X}$ of finite groups is said to be *infinitely hereditary concerning a class of groups* $\mathfrak{H}$ if it satisfies the following condition:

(IH) whenever an $\mathfrak{H}$-group G is residually $\mathfrak{X}$-group, then every finite factor-group of G belongs to $\mathfrak{X}$.

It is worth mentioning that many formations of finite groups are infinitely hereditary concerning the class of FC-hypercentral groups. For example we can mention finite abelian groups, finite nilpotent groups of class at most c, finite soluble groups of derived length at most d, finite soluble groups, and finite groups of exponent dividing n. Moreover, all groups from these five examples, finite nilpotent groups, and finite supersoluble groups are infinitely hereditary concerning both the classes of the FC-groups and hyperfinite groups.

Theorem 10 (L.A. Kurdachenko, J. Otal, I.Ya. Subbotin [10])**.** *Let G be an infinite locally soluble FC-hypercentral group, D a Dedekind domain and A an artinian DG-module. If $\mathfrak{X}$ is a formation, which is infinitely hereditary concerning the class of FC-hypercentral groups, then A has the Baer-Shemetkov's $\mathfrak{X} - DG$-decomposition.*

The problem of existence of Baer-Shemetkov's decomposition is tightly connected to another group theoretical problem, namely the problem of the existence of complements and supplements to the important normal subgroups of the group. L.A. Shemetkov also made a significant contribution to the solution of this problem.

Let G be a group and H be a proper subgroup of G. A proper subgroup K is said to be *a supplement to H in G* if $G = HK$. A supplement K to H is said to be *a complement to H in G* if further $H \cap K = \langle 1 \rangle$.

The notion of a complement has been created in the paper [5] of P. Hall, while the concept of the supplement has been initiated later on in another Hall's paper [6]. Note, that for it P. Hall initially used another term a *partial complement*.

If H is a normal subgroup of G and H has a complement K, then it is said that G *splits over* H and denote this by $G = H \leftthreetimes K$. If all complements to H are conjugate, then it is said that G *conjugately splits over* H.

In finite group theory, the problem on the existence of complements and supplements to normal subgroups has been studied quite deep. Note here an important theorem of Gaschütz [4]:

If an abelian normal subgroup A of a finite group G has a complement in a subgroup H and $(|G:H|,|A|)=1$, then A has a complement in G. Moreover, if H is conjugately splits over A, then G is conjugately splits over A.

The Schur-Zassenhaus's theorem is deduced from this result. Especially interesting is the following corollary of Gaschütz's theorem:

An abelian normal subgroup A of a finite group G has a complement in G if and only if a Sylow p-subgroup of A has a complement in a Sylow p-subgroup of G for each prime p.

L.A. Shemetkov has obtained the following generalization of Schur-Zassenhaus's and Gaschütz's theorems.

Theorem 11 (L.A. Shemetkov [15])**.** *Let G be a finite group and π a set of primes. Let K be a subgroup of G, generated by all π'-elements of G. If p^2 does not divide $|K|$ for every $p\in\pi$, then K has a complement in G.*

Gaschütz's theorem has two following reductions. Let $\mathbf{cl}_G(H)=\{H^g|g\in G\}$, $L=\mathbf{Core}_G(H)=\bigcap_{g\in G}H^g$. Clearly $A\leq L$, L is a normal subgroup of G and $(|G/L|,|A|)=1$.

Let π be a set of all prime divisors of $|A|$ and $\sigma=\Pi(G)\backslash\pi$. Denote by $\mathfrak{X}_\sigma$ the class of all finite σ-groups. Clearly $\mathfrak{X}_\sigma$ is a formation. In notations of Gaschütz's theorem we have $G/L\in\mathfrak{X}_\sigma$. The second reduction is following. We have that $A\leq G^{\mathfrak{X}}_\sigma$, $G^{\mathfrak{X}}_\sigma$ is a normal subgroup of G and $(|G/G^{\mathfrak{X}}_\sigma|,|A|)=1$.

This raises the natural question: For which formations does an analog of Gaschütz's theorem exist? The answer has been obtained by L.A. Shemetkov in the paper [16]. This result is one of the most general results on the existence of complements to $\mathfrak{X}$-residual in finite groups.

Theorem 12 (L.A. Shemetkov [15])**.** *Let G be a finite group and $\mathfrak{X}$ be a local formation.*

Let p be a prime such that a Sylow p-subgroup P of $G^{\mathfrak{X}}$ is abelian. If S is a Sylow p-subgroup of G, then P has a complement in S.

Suppose that the Sylow q-subgroups of $G^{\mathfrak{X}}$ are abelian for every prime divisor q of $|G/G^{\mathfrak{X}}|$. Then $G^{\mathfrak{X}}$ has a complement in G. In particular, if an $\mathfrak{X}$-residual $G^{\mathfrak{X}}$ of a group G is abelian, then it has a complement in G.

The existence of this general result for finite groups was a good incentive for finding of its analog for infinite groups. However, in the infinite case, the situation is much more complicated, it is difficult to speak of an analog of the overall result. However, there is quite a lot of interesting results for various special cases. A major role in the development of this subject belongs to D.I. Zaitsev. L.A. Shemetkov was one of those who drew Zaitsev's attention on this subject matter. One of the first natural question here is the existence of supplements to locally nilpotent residual whenever this residual is abelian. For some natural finiteness conditions D.I. Zaitsev has obtained the following results.

Theorem 13 (D.I. Zaitsev [22]). *Let G be a group and A be an abelian normal subgroup of G. Suppose that the following conditions hold:*

(i) *A as $\mathbb{Z}G$-module is artinian;*

(ii) *$C_A(G) = \langle 1 \rangle$;*

(iii) *G/A is locally nilpotent and $G/C_G(A)$ is hypercentral.*

Then G splits conjugately over A.

If R is a ring and G is a group, we denote by $\omega(RG)$ the augmentation ideal of the group ring RG.

Theorem 14 (D.I. Zaitsev [21]). *Let G be a group and A be an abelian normal subgroup of G. Suppose that the following conditions hold:*

(i) *A as $\mathbb{Z}G$-module is noetherian;*
(ii) *$A = A(\omega(\mathbb{Z}(G/A)))$;*
(iii) *G/A is locally nilpotent and $G/C_G(A)$ is hypercentral.*

Then G splits conjugately over A.

Under some restrictions, some conditions of existence of complements to the locally nilpotent residual have been obtained by D.J.S. Robinson.

Theorem 15 (D.J.S. Robinson [13]). *Let G be a group and A be an abelian normal subgroup of G. Suppose that the following conditions hold:*

(i) *A as $\mathbb{Z}G$-module is noetherian;*
(ii) *G includes a normal subgroup $H \geq A$ such that H/A is hypercentral;*
(iii) *$A = A(\omega(\mathbb{Z}(H/A)))$;*
(iv) *upper FC-hypercenter of $G/C_G(A)$ includes $HC_G(A)/C_G(A)$.*

Then G splits conjugately over A.

Theorem 16 (D.J.S. Robinson [14]). *Let G be a group and A be an abelian normal subgroup of G. Suppose that the following conditions hold:*

(i) *A as $\mathbb{Z}G$-module is artinian;*

(ii) *G includes a normal subgroup $H \geq A$ such that H/A is hypercentral;*

(iii) *$C_A(H) = \langle 1 \rangle$;*

(iv) *upper FC-hypercenter of $G/C_G(A)$ includes $HC_G(A)/C_G(A)$.*

Then G splits conjugately over A.

Note that Theorem 15 has been proved using only group theory methods, while the proof of Theorem 16 was based on homology theory. A purely group-theoretical proof of this Theorem 16 has been presented in the book [[11], Chapter 17]. The latest general results regarding complements to the locally nilpotent residual have been obtained in the paper [12].

We say that a normal subgroup L of a group G is said to be G-*hyperfinite* if L has an ascending series

$$\langle 1 \rangle = L_0 \leq L_1 \leq ... \leq L_\alpha \leq L_{\alpha+1} \leq ... \leq L_\gamma = L$$

of G-invariant subgroups whose factors are finite.

Theorem 17 (L.A. Kurdachenko, J. Otal, I.Ya. Subbotin [12]). *Let G be a group and A be an abelian normal subgroup of G. Suppose that G has an ascendant subgroup $H \geq A$ such that H/A is finitely generated and nilpotent. If A is H-hyperfinite and H-hypereccentric, then G conjugately splits over A.*

Finally we would like to mention that D.I. Zaitsev initiated the search for the conditions of existance of complements to the locally finite residual. He obtained the following two results.

Theorem 18 (D.I. Zaitsev [19]). *Let G be a group and A be an abelian normal subgroup of G. Suppose that the following conditions hold:*

(i) *A as $\mathbb{Z}G$-module is artinian;*

(ii) *every non-identity G-invariant subgroup of A is infinite;*

(iii) *G/A is locally soluble and hyperfinite.*

Then G splits conjugately over A.

Theorem 19 (D.I. Zaitsev [23]). *Let G be a group and A be an abelian normal subgroup of G. Suppose that the following conditions hold:*

(i) *A as $\mathbb{Z}G$-module is noetherian;*

(ii) *A does not include proper G-invariant subgroups of finite index;*

(iii) *G/A is locally soluble and hyperfinite.*

Then G splits conjugately over A.

References

[1] Baer R., *Durch Formationen bestimmte Zerlegungen von Normalteilern endlicher Gruppen*, J. Algebra, N.**20**, 1972, pp. 38-56.

[2] Doerk K., Hawkes T., *Finite soluble groups*, Walter de Gruyter, 1992.

[3] Duan Z.Y., *Extensions of abelian-by-hyper (cyclic or finite) groups*, Communications in Algebra, N.**20**, 1992, pp. 2305-2321.

[4] Gaschütz W., *Zur Erweitrungstheorie der endlichen Gruppen*, J. Reine und Angew. Math., N.**190**, 1952, pp. 93-107.

[5] Hall P., *Complemented groups*, Journal London Math. Soc., N.**12**, 1937, pp. 201-204.

[6] Hall P., *The construction of soluble groups*, J. Reine Angew. Math., N.**182**, 1940, pp. 206-214.

[7] Hartley B., Tomkinson M.J., *Splitting over nilpotent and hyperfinite residuals*, Math. Proc. Cambridge Phil. Soc., N.**78**, 1975, pp. 215-226.

[8] Kurdachenko L.A., Petrenko B.V., Subbotin I.Ya., *On generalized hypercenters in artinian modules*, Comm. Algebra, N.**25**, 1997, pp. 1023-1046.

[9] Kurdachenko L.A., Petrenko B.V., Subbotin I.Ya., *Direct decompositions in artinian modules over FC-hypercentral groups*, Matematica contemporanea, N.**14**, 1998, pp. 89-99.

[10] Kurdachenko L.A., Otal J., Subbotin I.Ya., *Baer like decomposition of modules*, Illinois Journal Math., N.**47**, 2003, pp. 329-343.

[11] Kurdachenko L.A., Otal J., Subbotin I.Ya., *Artinian modules over group rings*, Frontiers in Mathematics. BIRKHÄUSER: Basel, 2007, 248 p.

[12] Kurdachenko L.A., Otal J., Subbotin I.Ya., *Some criteria for existence of supplements to normal subgroups and their applications*, International J. Algebra and Computation, N.**20**, 2010, pp. 689-719.

[13] Robinson D.J.S., *Splitting theorems for infinite groups*, Sympos. Mat. Ist. Naz. Alta Mat., N.**17**, 1975, pp. 441-470.

[14] Robinson D.J.S., *Cohomology of locally nilpotent groups*, J. Pure Applied Algebra, N.**48**, 1987, pp. 281-300.

[15] Shemetkov L.A., *Factorizations of finite groups*, Doklady AN USSR, N.**178(3)**, 1968, pp. 559-562.

[16] Shemetkov L.A., *On formation properties of finite groups*, Doklady AN USSR, N.**204(6)**, 1972, pp. 1324-1327.

[17] Shemetkov L.A., *Formations of finite groups*, NAUKA, Moskow, 1978.

[18] Zaitsev D.I., *On the existence of direct complements in groups with operators*, in *Investigations in Group Theory*, Math. Inst. Kyiv, 1976, pp. 26-44.

[19] Zaitsev D.I., *The splitting extensions of abelian groups*, in *The structure of groups and the properties of their subgroups*, Math. Inst. Kyiv, 1986, pp. 22-31.

[20] Zaitsev D.I., *On direct decompositions of abelian groups with operators*, Ukrain. Math. J., N.**40**, 1988, pp. 303-309.

[21] Zaitsev D.I., *On extensions of abelian groups*, in *The constructive description of groups with prescribed properties of subgroups*, Math. Inst. Kyiv, 1980, pp. 16-40.

[22] Zaitsev D.I., *On splitting of extensions of the abelian groups*, in *The investigations of groups with the prescribed properties of systems of subgroups*, Math. Inst. Kyiv, 1981, pp. 14-25.

[23] Zaitsev D.I., *The hyperfinite extensions of abelian groups*, in *The investigations of groups with the restrictions on subgroups*, Math. Inst. Kyiv, 1988, pp. 17-26.

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