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Automorphic equivalence of the representations of Lie algebras

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Abstract. In this paper we research the algebraic geometry of the representations of Lie algebras over fixed field k. We assume that this field is infinite and $char\,(k) = 0$. We consider the representations of Lie algebras as 2-sorted universal algebras. The representations of groups were considered by similar approach: as 2-sorted universal algebras - in [3] and [2]. The basic notions of the algebraic geometry of representations of Lie algebras we define similar to the basic notions of the algebraic geometry of representations of groups (see [2]). We prove that if a field k has not nontrivial automorphisms then automorphic equivalence of representations of Lie algebras coincide with geometric equivalence. This result is similar to the result of [4], which was achieved for representations of groups. But we achieve our result by another method: by consideration of 1-sorted objects. We suppose that our method can be more perspective in the further researches.

1. Introduction: representations of Lie algebras as 2-sorted universal algebras

In this paper we research the algebraic geometry of the representations of Lie algebras.

We consider the Lie algebras over the field k. And we say that we have the representation of Lie algebra (L, V) if the elements of the Lie algebra L act on the vector space V over the field k as linear transformations and the mapping $\mathfrak{f} : L \to \mathrm{End}_k\,(V)$ which we define by $\mathfrak{f}\,(l)\,(v) = l \circ v$, where $l \in L$, $v \in V$, $\circ$ is acting of the elements of the algebra L over

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elements of V, is a homomorphism from the Lie algebra L to the Lie algebra $\mathrm{End}_k^{(-)}(V) = \mathfrak{end}_k(V)$. Some time we will omit the symbol $\circ$. In this paper we assume that k is infinite and *char* $(k) = 0$.

We consider a representation of Lie algebra as 2-sorted universal algebra. Particularly the homomorphisms of representations we define by this definition:

Definition 1.1. We say that we have a *homomorphism* (φ, ψ) from the representation (L_1, V_1) to the representation (L_2, V_2) if we have a homomorphism of Lie algebras $\varphi : L_1 \to L_2$ and a linear map $\psi : V_1 \to V_2$ such that

$$\varphi(l) \circ \psi(v) = \psi(l \circ v) \tag{1.1}$$

holds for every $l \in L_1$ and every $v \in V_1$.

We denote $(\varphi, \psi) : (L_1, V_1) \to (L_2, V_2)$.

It means that the field k is fixed in our considerations. But algebras Lie and theirs modules me can change and we can compare the algebraic geometry of representations (L_1, V_1) and (L_2, V_2) such that $L_1 \neq L_2$ and $V_1 \neq V_2$. Therefore the multiplication by scalars of the elements of the algebra Lie L and the elements of its module V we can consider as unary operations: for every scalar $\lambda \in k$ we have two unary operations. But the acting of the elements of the algebra Lie L over the elements of its module V we must consider as one binary 2-sorted operation.

If $(\varphi, \psi) : (L, V) \to (P, Q)$ is a homomorphism of the representations, than $\ker \varphi$ is an ideal of the Lie algebra L, $\ker \psi$ is a L-submodule of the L-module V, $(\ker \varphi, \ker \psi)$ is a representation and a congruence in (L, V).

If $H = (L, V)$ is a representation of Lie algebra and $T_1 \subseteq L$, $T_2 \subseteq V$ we will denote $(T_1, T_2) \subseteq H$. If also $P_1 \subseteq L$, $P_2 \subseteq V$ we will denote $(T_1, T_2) \cap (P_1, P_2) = (T_1 \cap P_1, T_2 \cap P_2)$.

2. Basic notions of the algebraic geometry of representations of Lie algebras

We denote by Ξ the variety of the all representations of Lie algebras over the fixed field k.

Definition 2.1. We say that the representation (L, V) is a *free representation* with the pair of sets of the free generators (X, Y) if $X \subset L$, $Y \subset V$ and for every representation (P, U) and every pair of mappings $\widetilde{\varphi} : X \to P$, $\widetilde{\psi} : Y \to U$ there exists a homomorphism $(\varphi, \psi) : (L, V) \to (P, U)$ such that $\widetilde{\varphi}_{|X} = \varphi_{|X}$, $\widetilde{\psi}_{|X} = \psi_{|X}$.

From this one we will denote the mappings $\widetilde{\varphi}$ and φ, $\widetilde{\psi}$ and ψ by same letters.

We will denote this representation by $W = W(X, Y)$. It is well known that $W(X, Y) = (L(X), A(X)Y)$, where $L(X) = L$ is the free Lie algebra with the set X of free generators, $A(X)$ is the free associative algebra with unit which has the set X of free generators, $A(X)Y = \bigoplus_{y \in Y} A(X)y = V$ is the free $A(X)$ module with the basis Y. In this notation the symbol $\circ$ of the action is omitted. In particular, if $X = \varnothing$ then $L(X) = \{0\}$, $A(X) = k$, if $Y = \varnothing$ then $A(X)Y = \{0\}$, if $X = \{x\}$ then $L(X) = kx$, $A(X) = k[x]$.

X^0, Y^0 will be infinite countable sets of symbols. We consider the category Ξ^0. $\mathrm{Ob}\Xi^0 = \{W(X, Y) \mid |X| < \infty, |Y| < \infty, X \subset X^0, Y \subset Y^0\}$. Morphisms of this category are homomorphisms of its objects. The category Ξ^0 is a small category: $\mathrm{Ob}\Xi^0$ and $\mathrm{Mor}\Xi^0$ are sets. So we can tell about elements and subsets of $\mathrm{Ob}\Xi^0$ and $\mathrm{Mor}\Xi^0$.

We will take our equations from the representations $W = W(X, Y) = (L(X), A(X)Y) \in \mathrm{Ob}\Xi^0$. We have two sorts of equations: the equations in the Lie algebra - $t_1 \in L(X)$ and the action type equations - $t_2 \in A(X)Y$. We can resolve our equations in arbitrary $H = (L, V) \in \Xi$. The homomorphism $(\varphi, \psi) : W(X, Y) \to H$ will be the solution of the equation $t_1 \in L(X)$ if $\varphi(t_1) = 0$ and will be the solution of the equation $t_2 \in A(X)Y$ if $\psi(t_2) = 0$.

We can consider the system of equations $T = (T_1, T_2)$, where $T_1 \subseteq L(X)$, $T_2 \subseteq A(X)Y$. We can consider this system as a set $T = T_1 \cup T_2$ but it is not natural because the subsets T_1 and T_2 have different origins: $T_1 \subseteq L(X)$, $T_2 \subseteq A(X)Y$. So it is natural to consider the system of equations $T = (T_1, T_2)$ as a pair of sets. However for the sake of brevity we will some time write "the set (T_1, T_2)". The set of solutions of the system (T_1, T_2) in the representation $H = (L, V)$ is

$$(T_1, T_2)'_H = \{(\varphi, \psi) \in \mathrm{Hom}(W(X, Y), H) \mid T_1 \subseteq \ker \varphi, T_2 \subseteq \ker \psi\}.$$

Vice versa, for every set $A \subset \mathrm{Hom}(W(X, Y), H)$ we can consider the set

$$A'_H = \left(\bigcap_{(\varphi,\psi) \in A} \ker \varphi, \bigcap_{(\varphi,\psi) \in A} \ker \psi \right).$$

This set will be the maximal system of equations, such that A is a subset of the set of its solutions. Also we can consider the algebraic closer of the system (T_1, T_2):

$$(T_1,T_2)''_H = \left(\bigcap_{(\varphi,\psi)\in(T_1,T_2)'_H} \ker\varphi, \bigcap_{(\varphi,\psi)\in(T_1,T_2)'_H} \ker\psi \right) =$$

$$= \left(\bigcap_{\substack{(\varphi,\psi)\in\mathrm{Hom}(W,H),\\ T_1\subseteq\ker\varphi,\, T_2\subseteq\ker\psi}} \ker\varphi, \bigcap_{\substack{(\varphi,\psi)\in\mathrm{Hom}(W,H),\\ T_1\subseteq\ker\varphi,\, T_2\subseteq\ker\psi}} \ker\psi \right).$$

It will be the maximal system of equations which have the same solutions as (T_1,T_2).

It is clear that $(T_1,T_2)\subseteq(T_1,T_2)''_H$ holds for every $W(X,Y)\in\mathrm{Ob}\Xi^0$, every $(T_1,T_2)\subseteq W(X,Y)$ and every $H\in\Xi$.

Definition 2.2. The set $(T_1,T_2)\subseteq W(X,Y)$ is H-*closed* if $(T_1,T_2)''_H = (T_1,T_2)$.

It is clear that the closed sets are congruences. The family of the all H-closed sets in the free representation $W=W(X,Y)\in\mathrm{Ob}\Xi^0$ we denote by $Cl_H(W)$.

Definition 2.3. $H_1,H_2\in\Xi$. H_1,H_2 are called *geometrically equivalent* if $(T_1,T_2)''_{H_1}=(T_1,T_2)''_{H_2}$ holds for every $W(X,Y)\in\mathrm{Ob}\Xi^0$ and every $(T_1,T_2)\subseteq W(X,Y)$.

We consider $W_1=W(X_1,Y_1), W_2=W(X_2,Y_2)\in\mathrm{Ob}\Xi^0$ and (T_1,T_2) some congruence in W_2. We denote by $\beta=\beta_{W_1,W_2}(T_1,T_2)$ the following relation in $\mathrm{Hom}(W_1,W_2)$: $((\varphi_1,\psi_1),(\varphi_2,\psi_2))\in\beta$ if and only if $\varphi_1(l)\equiv\varphi_2(l)\,(\mathrm{mod}\,T_1)$ holds for every $l\in L(X_1)$ and $\psi_1(v)\equiv\psi_2(v)\,(\mathrm{mod}\,T_2)$ holds for every $v\in A(X_1)Y_1$. This relation is a 2-sorted analog of the relation β from [1, Subsection 3.3]. Now we define as in [1, Subsection 3.4]

Definition 2.4. $H_1,H_2\in\Xi$. H_1,H_2 are called *automorphically equivalent* if these 3 conditions hold:

1) There exists an automorphism $\Phi:\Xi^0\to\Xi^0$.
2) There exists a function $\alpha=\alpha(\Phi)$ such that $\alpha(\Phi)_W : Cl_{H_1}(W)\to Cl_{H_2}(\Phi(W))$ is a bijection for every $W\in\mathrm{Ob}\Xi^0$.
3) $\Phi(\beta_{W_1,W_2}(T_1,T_2))=\beta_{\Phi(W_1),\Phi(W_2)}\left(\alpha(\Phi)_{W_2}(T_1,T_2)\right)$ holds for every $W_1,W_2\in\mathrm{Ob}\Xi^0$, and every $(T_1,T_2)\in Cl_{H_1}(W_2)$.

Here $\Phi((\varphi_1,\psi_1),(\varphi_2,\psi_2))=(\Phi(\varphi_1,\psi_1),\Phi(\varphi_2,\psi_2))$.

It can be proved as in [1, Proposition 8] that if H_1 and H_2 are automorphically equivalent then function α is uniquely determined by automorphism Φ.

3. Some facts about the closed congruences in the free representations of Lie algebras

In this Section we assume that $X_1 \subseteq X_2 \subset X^0$, $Y_1 \subseteq Y_2 \subset Y^0$, $(L, V) = H \in \Xi$. We denote $(L(X_i), A(X_i)Y_i) = W(X_i, Y_i) = W_i$, where $i = 1, 2$.

If $(T_1, T_2) \subseteq W_1$, then, because $W_1 \subseteq W_2$, we can consider the sets

$$(T_1, T_2)'_{W_i,H} = \{(\varphi, \psi) : W_i \to H \mid T_1 \subseteq \ker\varphi, T_2 \subseteq \ker\psi\}$$

and we will denote $\left((T_1, T_2)'_{W_i,H}\right)'_H = (T_1, T_2)''_{W_i,H}$, where $i = 1, 2$. We say that (T_1, T_2) is H-closed in W_i if $(T_1, T_2)''_{W_i,H} = (T_1, T_2)$. In all other sections of this paper it is clear what kind of algebraic closer of the system of equations we consider. But in this Section we must fine distinguish between the different features.

Proposition 3.1. *We assume that* $(T_1, T_2) \subseteq W_2$, $(\mu, \nu) \in (T_1 \cap L(X_1), T_2 \cap A(X_1)Y_1)'_{W_1,H}$. *We denote*

$$[(\mu, \nu)] = \left\{(\varphi, \psi) \in (T_1, T_2)'_H \mid \varphi_{|X_1} = \mu_{|X_1}, \psi_{|Y_1} = \nu_{|Y_1}\right\}.$$

Then

$$\left(\left(\bigcap_{(\varphi,\psi)\in[(\mu,\nu)]} \ker\varphi\right) \cap L(X_1), \left(\bigcap_{(\varphi,\psi)\in[(\mu,\nu)]} \ker\psi\right) \cap A(X_1)Y_1\right) = \\ = (\ker\mu, \ker\nu)$$

holds.

Proof. If $t_1 \in \left(\bigcap_{(\varphi,\psi)\in[(\mu,\nu)]} \ker\varphi\right) \cap L(X_1)$, then $\mu(t_1) = \varphi(t_1) = 0$ for every φ such that $(\varphi, \psi) \in [(\mu, \nu)]$. If $t_2 \in \left(\bigcap_{(\varphi,\psi)\in[(\mu,\nu)]} \ker\psi\right) \cap A(X_1)Y_1$, then $\nu(t_2) = \psi(t_2) = 0$ for every ψ such that $(\varphi, \psi) \in [(\mu, \nu)]$.

If $t_1 \in \ker\mu$, then $t_1 \in L(X_1)$, so $\varphi(t_1) = \mu(t_1) = 0$ holds for every φ such that $(\varphi, \psi) \in [(\mu, \nu)]$. If $t_2 \in \ker\nu$, then $t_2 \in A(X_1)Y_1$, so $\psi(t_2) = \nu(t_2) = 0$ holds for every ψ such that $(\varphi, \psi) \in [(\mu, \nu)]$. □

Proposition 3.2. *If* $(T_1, T_2) \subseteq W_2$ *is* H*-closed, then*

$$(T_1, T_2) \cap W_1 = (T_1 \cap L(X_1), T_2 \cap A(X_1)Y_1)$$

is H-closed in W_1.

Proof.

$$\left(T_1,T_2\right)''_H=\left(\bigcap_{(\varphi,\psi)\in(T_1,T_2)'_H}\ker\varphi,\bigcap_{(\varphi,\psi)\in(T_1,T_2)'_H}\ker\psi\right)=\left(T_1,T_2\right).$$

$$\left(T_1\cap L\left(X_1\right),T_2\cap A\left(X_1\right)Y_1\right)''_{W_1,H}=$$

$$\left(\bigcap_{(\mu,\nu)\in(T_1\cap L(X_1),T_2\cap A(X_1)Y_1)'_{W_1,H}}\ker\mu,\bigcap_{(\mu,\nu)\in(T_1\cap L(X_1),T_2\cap A(X_1)Y_1)'_{W_1,H}}\ker\nu\right).$$

We will consider $(\varphi,\psi)\in(T_1,T_2)'_H$. There exists only one $(\mu,\nu)\in\mathrm{Hom}\left(W_1,H\right)$ such that $\varphi_{|X_1}=\mu_{|X_1}$, $\psi_{|Y_1}=\nu_{|Y_1}$. If $t_1\in T_1\cap L\left(X_1\right)$, then $\mu\left(t_1\right)=\varphi\left(t_1\right)=0$, if $t_2\in T_2\cap A\left(X_1\right)Y_1$, then $\nu\left(t_2\right)=\psi\left(t_2\right)=0$. Hence $(\mu,\nu)\in\left(T_1\cap L\left(X_1\right),T_2\cap A\left(X_1\right)Y_1\right)'_{W_1,H}$. So by Proposition 3.1

$$\left(\left(\bigcap_{(\varphi,\psi)\in[(\mu,\nu)]}\ker\varphi\right)\cap L\left(X_1\right),\left(\bigcap_{(\varphi,\psi)\in[(\mu,\nu)]}\ker\psi\right)\cap A\left(X_1\right)Y_1\right)=$$
$$=\left(\ker\mu,\ker\nu\right).$$

The set $(T_1,T_2)'_H$ can by presented as union of the disjoint sets $[(\mu,\nu)]$, where $(\mu,\nu)\in\mathrm{Hom}\left(W_1,H\right)$ such that exists $(\varphi,\psi)\in(T_1,T_2)'_H$, for which $(\varphi,\psi)\in[(\mu,\nu)]$ holds.

$$\left(T_1\cap L\left(X_1\right),T_2\cap A\left(X_1\right)Y_1\right)''_{W_1,H}\subseteq$$

$$\subseteq\left(\bigcap_{\substack{(\mu,\nu)\in(T_1\cap L(X_1),T_2\cap A(X_1)Y_1)'_H,\\ \exists(\varphi,\psi)\in(T_1,T_2)'_H|(\varphi,\psi)\in[(\mu,\nu)]}}\ker\mu,\bigcap_{\substack{(\mu,\nu)\in(T_1\cap L(X_1),T_2\cap A(X_1)Y_1)'_H,\\ \exists(\varphi,\psi)\in(T_1,T_2)'_H|(\varphi,\psi)\in[(\mu,\nu)]}}\ker\nu\right).$$

$$\bigcap_{\substack{(\mu,\nu)\in(T_1\cap L(X_1),T_2\cap A(X_1)Y_1)'_H,\\ \exists(\varphi,\psi)\in(T_1,T_2)'_H|(\varphi,\psi)\in[(\mu,\nu)]}}\ker\mu=$$

$$=\bigcap_{\substack{(\mu,\nu)\in(T_1\cap L(X_1),T_2\cap A(X_1)Y_1)'_H,\\ \exists(\varphi,\psi)\in(T_1,T_2)'_H|(\varphi,\psi)\in[(\mu,\nu)]}}\left(\left(\bigcap_{(\varphi,\psi)\in[(\mu,\nu)]}\ker\varphi\right)\cap L\left(X_1\right)\right)=$$

$$= \left(\bigcap_{(\varphi,\psi)\in(T_1,T_2)'_H} \ker\varphi \right) \cap L(X_1) = T_1 \cap L(X_1).$$

$$\bigcap_{\substack{(\mu,\nu)\in(T_1\cap L(X_1),T_2\cap A(X_1)Y_1)'_H,\\ \exists(\varphi,\psi)\in(T_1,T_2)'_H|(\varphi,\psi)\in[(\mu,\nu)]}} \ker\nu =$$

$$= \bigcap_{\substack{(\mu,\nu)\in(T_1\cap L(X_1),T_2\cap A(X_1)Y_1)'_H,\\ \exists(\varphi,\psi)\in(T_1,T_2)'_H|(\varphi,\psi)\in[(\mu,\nu)]}} \left(\left(\bigcap_{(\varphi,\psi)\in[(\mu,\nu)]} \ker\psi \right) \cap A(X_1)Y_1 \right) =$$

$$= \left(\bigcap_{(\varphi,\psi)\in(T_1,T_2)'_H} \ker\psi \right) \cap A(X_1)Y_1 = T_2 \cap A(X_1)Y_1. \quad \square$$

Proposition 3.3. *If* $(T_1, T_2) \subseteq W_1$ *is* H*-closed in* W_1*, then*

$$(T_1, T_2) = (T_1, T_2)''_{W_2,H} \cap W_1.$$

Proof. In $(T_1, T_2)'_{W_2,H} = \{(\varphi,\psi) \in \mathrm{Hom}(W_2, H) \mid \ker\varphi \supseteq T_1, \ker\psi \supseteq T_2\}$ we can define equivalence: $(\varphi_1,\psi_1) \sim (\varphi_2,\psi_2)$ if and only if $\varphi_{1|X_1} = \varphi_{2|X_1}$, $\psi_{1|Y_1} = \psi_{2|Y_1}$. As in the proof of Proposition 3.2, for every class of this equivalence there exist only one $(\mu,\nu) \in (T_1,T_2)'_{W_1,H}$ such that this class coincide with $[(\mu,\nu)]$. Vice versa, for every $(\mu,\nu) \in (T_1,T_2)'_{W_1,H}$ there exist only one class of elements of the set $(T_1,T_2)'_{W_2,H}$, which coincide with $[(\mu,\nu)]$.

$$(T_1,T_2)''_{W_2,H} \cap W(X_1,Y_1) =$$

$$\left(\left(\bigcap_{(\varphi,\psi)\in(T_1,T_2)'_{W_2,H}} \ker\varphi \right) \cap L(X_1), \left(\bigcap_{(\varphi,\psi)\in(T_1,T_2)'_{W_2,H}} \ker\psi \right) \cap A(X_1)Y_1 \right).$$

By Proposition 3.1 we have

$$\left(\bigcap_{(\varphi,\psi)\in(T_1,T_2)'_{W_2,H}} \ker\varphi \right) \cap L(X_1) =$$

$$= \bigcap_{(\mu,\nu)\in(T_1,T_2)'_{W_1,H}} \left(\left(\bigcap_{(\varphi,\psi)\in[(\mu,\nu)]} \ker\varphi \right) \cap L(X_1) \right) =$$

$$= \bigcap_{(\mu,\nu)\in(T_1,T_2)'_{W_1,H}} \ker\mu = T_1.$$

$$\left(\bigcap_{(\varphi,\psi)\in(T_1,T_2)'_{W_2,H}} \ker\psi\right) \cap A(X_1)Y_1 =$$

$$= \bigcap_{(\mu,\nu)\in(T_1,T_2)'_{W_1,H}} \left(\left(\bigcap_{(\varphi,\psi)\in[(\mu,\nu)]} \ker\psi\right) \cap A(X_1)Y_1\right) =$$

$$= \bigcap_{(\mu,\nu)\in(T_1,T_2)'_{W_1,H}} \ker\nu = T_2. \quad \square$$

Theorem 3.1. *If* $(L_1, V_1) = H_1, (L_2, V_2) = H_2 \in \Xi$ *and* $Cl_{H_1}(W_2) = Cl_{H_2}(W_2)$, *then* $Cl_{H_1}(W_1) = Cl_{H_2}(W_1)$.

Proof. We consider $(T_1, T_2) \in Cl_{H_1}(W_1)$. By Proposition 3.3 $(T_1, T_2) = (T_1, T_2)''_{W_2,H_1} \cap W_1$. $(T_1, T_2)''_{W_2,H_1} \in Cl_{H_1}(W_2) = Cl_{H_2}(W_2)$. Therefore, by Proposition 3.2, $(T_1, T_2)''_{W_2,H_1} \cap W_1 = (T_1, T_2) \in Cl_{H_2}(W_1)$. $\square$

4. Representations of Lie algebras and Lie algebras with projection-derivation

It is well known that if we have a representation of the Lie algebra (L, V) then in the k-linear space $M = L \oplus V$ we can define the structure of Lie algebra if we define the new Lie brackets $[,]_M$ by this formula

$$[l_1 + v_1, l_2 + v_2]_M = [l_1, l_2] + l_1 \circ v_2 - l_2 \circ v_1, \tag{4.1}$$

where $l_1, l_2 \in L$, $v_1, v_2 \in V$.

We will denote by p the projection of M on the linear subspace V. $p(l + v) = v$ for every $l \in L$, $v \in V$. We have

$$p[l_1 + v_1, l_2 + v_2]_M = p([l_1, l_2] + l_1 \circ v_2 - l_2 \circ v_1) = l_1 \circ v_2 - l_2 \circ v_1,$$

$$[p(l_1 + v_1), l_2 + v_2]_M + [l_1 + v_1, p(l_2 + v_2)]_M =$$
$$= [v_1, l_2 + v_2]_M + [l_1 + v_1, v_2]_M = -l_2 \circ v_1 + l_1 \circ v_2$$

for every $l_1, l_2 \in L$, $v_1, v_2 \in V$. Therefore in the new Lie algebra p will be a derivation. The projection p we consider as an additional unary operation defined on the Lie algebra M. We call these algebras: Lie algebras with projection-derivation and denote (M, p).

Vice versa, if we assume that we have a Lie algebra with projection-derivation (M, p) then we have the decomposition of the k-linear space $M = \ker p \oplus \mathrm{im} p$. If we denote $\ker p = L$, $\mathrm{im} p = V$, then we can prove this proposition:

Proposition 4.1. *If we consider L with the Lie brackets inducted from M then L is a Lie algebra. If we define*

$$l \circ v = [l, v] \tag{4.2}$$

for every $l \in L$ and every $v \in V$ then (L, V) is a representations of the Lie algebra L over the linear space V, for every $v_1, v_2 \in V$ the $[v_1, v_2] = 0$ holds.

Proof. If $l_1, l_2 \in \ker p$ then $p[l_1, l_2] = [p(l_1), l_2] + [l_1, p(l_2)] = 0$, so $L = \ker p$ is a Lie algebra.

If $l \in \ker p$, $v \in \mathrm{im} p$ then $p[l, v] = [p(l), v] + [l, p(v)] = [l, v]$, so $[l, v] = l \circ v \in \mathrm{im} p$.

If $l_1, l_2 \in \ker p$, $v \in \mathrm{im} p$ then

$$[l_1, l_2] \circ v = [[l_1, l_2], v] = -[[l_2, v], l_1] - [[v, l_1], l_2] = \\ = [l_1, [l_2, v]] - [l_2, [l_1, v]] = l_1 \circ (l_2 \circ v) - l_2 \circ (l_1 \circ v).$$

Also we have for $v_1, v_2 \in \mathrm{im} p$ that $p[v_1, v_2] = [p(v_1), v_2] + [v_1, p(v_2)] = [v_1, v_2] + [v_1, v_2]$. $char(k) \neq 2$, so $[v_1, v_2] \in \mathrm{im} p$, $p[v_1, v_2] = [v_1, v_2]$ and $[v_1, v_2] = 0$. □

Proposition 4.2. *We assume that $(\varphi, \psi) : (L_1, V_1) \to (L_2, V_2)$ is a homomorphism of representations. Then $f = \varphi \oplus \psi : (L_1 \oplus V_1, p_{V_1}) \to (L_2 \oplus V_2, p_{V_2})$, which define by formula $f(l + v) = \varphi(l) + \psi(v)$ for every $l \in L_1$, $v \in V_1$ is a homomorphism of the Lie algebras with projection-derivation and $\ker f = \ker \varphi \oplus \ker \psi$. Vice versa, if $f : (M_1, p_1) \to (M_2, p_2)$ is a homomorphism of the Lie algebras with projection-derivation then $(r_2 f \kappa_1, p_2 f \iota_1) : (\ker p_1, \mathrm{im} p_1) \to (\ker p_2, \mathrm{im} p_2)$, where $r_2 = id_{M_2} - p_2$ and $\kappa_1 : \ker p_1 \hookrightarrow M_1$, $\iota_1 : \mathrm{im} p_1 \hookrightarrow M_1$ are embeddings, is a homomorphism of the representations of the Lie algebras and $\ker r_2 f \kappa_1 = \ker f \cap \ker p_1$, $\ker p_2 f \iota_1 = \ker f \cap \mathrm{im} p_1$.*

Proof. For the sake of brevity hear and in other proves we denote the various Lie brackets, projections and embeddings by similar symbols. It should not cause confusion because we not cause confusion when, for example, in the various groups denote multiplication, taking the inverse element and unit by similar symbols.

If $(\varphi,\psi):(L_1,V_1)\to(L_2,V_2)$ is a homomorphism of representations then $f=\varphi\oplus\psi$ is a linear mapping. If $l_1,l_2\in L_1$, $v_1,v_2\in V_1$ then

$$\begin{aligned} f\left[l_1+v_1,l_2+v_2\right] &= f\left(\left[l_1,l_2\right]+l_1\circ v_2-l_2\circ v_1\right)= \\ &= \varphi\left[l_1,l_2\right]+\psi\left(l_1\circ v_2\right)-\psi\left(l_2\circ v_1\right)= \\ &= \left[\varphi\left(l_1\right),\varphi\left(l_2\right)\right]+\varphi\left(l_1\right)\circ\psi\left(v_2\right)-\varphi\left(l_2\right)\circ\psi\left(v_1\right). \end{aligned}$$

$$\begin{aligned} \left[f\left(l_1+v_1\right),f\left(l_2+v_2\right)\right] &= \left[\varphi\left(l_1\right)+\psi\left(v_1\right),\varphi\left(l_2\right)+\psi\left(v_2\right)\right]= \\ &= \left[\varphi\left(l_1\right),\varphi\left(l_2\right)\right]+\varphi\left(l_1\right)\circ\psi\left(v_2\right)-\varphi\left(l_2\right)\circ\psi\left(v_1\right). \end{aligned}$$

If $l\in L_1$, $v\in V_1$ then

$$fp\left(l+v\right)=f\left(v\right)=\psi\left(v\right),$$

$$pf\left(l+v\right)=p\left(\varphi\left(l\right)+\psi\left(v\right)\right)=\psi\left(v\right).$$

So f is a homomorphism of the Lie algebras with projection-derivation.

It is clear that $\ker f\supseteq\ker\varphi\oplus\ker\psi$. If $l\in L_1$, $v\in V_1$ and $f\left(l+v\right)=\varphi\left(l\right)+\psi\left(v\right)=0$, then, because $\varphi\left(l\right)\in L_2$, $\psi\left(v\right)\in V_2$, $\varphi\left(l\right)=0$, $\psi\left(v\right)=0$. So $\ker f=\ker\varphi\oplus\ker\psi$.

Now we assume that $f:(M_1,p_1)\to(M_2,p_2)$ is a homomorphism of the Lie algebras with projection-derivation. $pr=p\left(id-p\right)=p-p^2=0$, so $rf\kappa:\ker p\to\ker p$. Also is clear that $pf\iota:\mathrm{im}p\to\mathrm{im}p$.

It is clear that $rf\kappa$ and $pf\iota$ are linear mappings. For every $l\in\ker p$ we have $pf\left(l\right)=fp\left(l\right)=0$. So we have for every $l_1,l_2\in\ker p$

$$\begin{aligned} rf\kappa\left[l_1,l_2\right] &= r\left[f\left(l_1\right),f\left(l_2\right)\right]=\left(id-p\right)\left[f\left(l_1\right),f\left(l_2\right)\right]= \\ &= \left[f\left(l_1\right),f\left(l_2\right)\right]-\left[pf\left(l_1\right),f\left(l_2\right)\right]-\left[f\left(l_1\right),pf\left(l_2\right)\right]=\left[f\left(l_1\right),f\left(l_2\right)\right]. \end{aligned}$$

$$\begin{aligned} &\left[rf\kappa\left(l_1\right),rf\kappa\left(l_2\right)\right]=\left[\left(id-p\right)f\left(l_1\right),\left(id-p\right)f\left(l_2\right)\right]= \\ &=\left[f\left(l_1\right),f\left(l_2\right)\right]-\left[pf\left(l_1\right),f\left(l_2\right)\right]-\left[f\left(l_1\right),pf\left(l_2\right)\right]+\left[pf\left(l_1\right),pf\left(l_2\right)\right]= \\ &=\left[f\left(l_1\right),f\left(l_2\right)\right]. \end{aligned}$$

So $rf\kappa$ is a homomorphism of the Lie algebras. If $l\in\ker p$, $v\in\mathrm{im}p$, then

$$\begin{aligned} rf\kappa\left(l\right)\circ pf\iota\left(v\right) &= \left[rf\left(l\right),pf\left(v\right)\right]= \\ &= \left[f\left(l\right),pf\left(v\right)\right]-\left[pf\left(l\right),pf\left(v\right)\right]=\left[f\left(l\right),pf\left(v\right)\right]. \end{aligned}$$

$$pf\iota\left(l\circ v\right)=pf\left[l,v\right]=p\left[f\left(l\right),f\left(v\right)\right]=$$
$$=\left[pf\left(l\right),f\left(v\right)\right]+\left[f\left(l\right),pf\left(v\right)\right]=\left[f\left(l\right),pf\left(v\right)\right].$$

So $(rf\kappa, pf\iota)$ is a homomorphism of the representations of the Lie algebras.

It is clear that $\ker f\cap\ker p\subseteq\ker rf\kappa$. If $l\in\ker p$ and $rf\kappa\left(l\right)=0$ then $l=r\left(l\right)$ and $f\left(l\right)=fr\left(l\right)=rf\kappa\left(l\right)=0$. So $\ker rf\kappa=\ker f\cap\ker p$. Analogously $\ker pf\iota=\ker f\cap \mathrm{im}p$. □

We denote by Θ the variety of all Lie algebras with projection-derivation. The elements of this variety are Lie algebras with all operations and axioms of Lie algebras and with one additional unary operation: projection p, which fulfills two axioms of linear map and two additional axioms:

1) $p\left(p\left(m\right)\right)=p\left(m\right)$ holds for every $m\in M$,
2) $p\left[m_1,m_2\right]=\left[p\left(m_1\right),m_2\right]+\left[m_1,p\left(m_2\right)\right]$ holds for every $m_1,m_2\in M$,

where $M\in\Theta$.

We can consider the varieties Ξ and Θ as categories. The objects of these categories are universal algebras from these varieties and morphisms are homomorphisms. We have a functor $\mathcal{F}:\Xi\to\Theta$, such that

$$\mathcal{F}\left(L,V\right)=\left(L\oplus V,p_V\right)$$

for $(L,V)\in\mathrm{Ob}\Xi$,

$$\mathcal{F}\left(\left(\varphi,\psi\right):\left(L_1,V_1\right)\to\left(L_2,V_2\right)\right)=\varphi\oplus\psi:\left(L_1\oplus V_1,p_{V_1}\right)\to\left(L_2\oplus V_2,p_{V_2}\right)$$

for $(\varphi,\psi)\in\mathrm{Mor}\Xi$.

Also we have a functor $\mathcal{F}^{-1}:\Theta\to\Xi$, such that

$$\mathcal{F}^{-1}\left(M,p\right)=\left(\ker p,\mathrm{im}p\right)$$

for $(M,p)\in\mathrm{Ob}\Theta$,

$$\mathcal{F}^{-1}\left(f:\left(M_1,p_1\right)\to\left(M_2,p_2\right)\right)=\left(rf\kappa,pf\iota\right):\left(\ker p_1,\mathrm{im}p_1\right)\to\left(\ker p_2,\mathrm{im}p_2\right)$$

for $f\in\mathrm{Mor}\Theta$.

By Propositions 4.1 and 4.2 $\mathcal{F}\mathcal{F}^{-1}=id_\Theta$, $\mathcal{F}^{-1}\mathcal{F}=id_\Xi$ so these functors are isomorphisms of categories.

Theorem 4.1. *If $(F,p)=F\left(m_1,\ldots,m_n\right)$ is a free Lie algebras with projection-derivation with free generators $\{m_1,\ldots,m_n\}$ then $\mathcal{F}^{-1}\left(F,p\right)=\left(L,V\right)$ is a free representation with the pair of sets of the free generators (X,Y), where $X=\{r\left(m_1\right),\ldots,r\left(m_n\right)\}$ and $Y=\{p\left(m_1\right),\ldots,p\left(m_n\right)\}$.*

Proof. It is clear that $X \subset \ker p$, $Y \subset \operatorname{im} p$. We will consider an arbitrary $(Q,U) \in \Xi$. We assume that we have 2 mappings:

$$\varphi : \{r(m_1), \ldots, r(m_n)\} \ni r(m_i) \to q_i \in Q$$

and

$$\psi : \{p(m_1), \ldots, p(m_n)\} \ni p(m_i) \to u_i \in U.$$

So we have a mapping

$$f : \{m_1, \ldots, m_n\} \ni m_i \to q_i + u_i = \varphi r(m_i) + \psi p(m_i) \in Q \oplus U.$$

Hence, by our assumption about (F,p), this mapping can be extended to the homomorphism

$$f : (F,p) \to \mathcal{F}(Q,U) = (Q \oplus U, p_U).$$

So there is a homomorphism

$$\mathcal{F}^{-1}(f) = (rf\kappa, pf\iota) : \mathcal{F}^{-1}(F,p) = (\ker p, \operatorname{im} p) \to (Q,U).$$

$$rf\kappa(r(m_i)) = (r)^2 f(m_i) = r(\varphi r(m_i) + \psi p(m_i)) = \varphi r(m_i),$$

$$pf\iota(p(m_i)) = p(\varphi r(m_i) + \psi p(m_i)) = \psi p(m_i)$$

holds for $1 \leq i \leq n$, because $\varphi r(m_i) \in Q$, $\psi p(m_i) \in U$. □

We will denote

$$\Xi' = \left\{ W(X,Y) \in \mathrm{Ob}\Xi^0 \mid |X| = |Y| \right\}.$$

Theorem 4.2. *If* $W = (L,V) = \left(L(x_1, \ldots, x_n), \bigoplus_{i=1}^{n} A(x_1, \ldots, x_n) y_i \right) \in \Xi'$ *then* $\mathcal{F}(W) = (F,p)$ *is a free Lie algebra with projection-derivation which has* n *free generators* $m_i = x_i + y_i$, $1 \leq i \leq n$.

Proof. For $1 \leq i \leq n$ we have that $x_i \in L$, $y_i \in V$, so $m_i = x_i + y_i \in F = L \oplus V$. We will consider an arbitrary $(N,p) \in \Theta$. We assume that we have a mapping

$$f : \{m_1, \ldots, m_n\} \ni m_i \to n_i \in N.$$

We will construct two other mappings

$$\varphi : \{x_1, \ldots, x_n\} \ni x_i \to r(n_i) \in \ker p \subset N$$

and

$$\psi : \{y_1, \ldots, y_n\} \ni y_i \to p(n_i) \in \operatorname{im} p \subset N.$$

By our assumption about (L, V), these mappings can be extended to the homomorphism

$$(\varphi, \psi) : (L, V) \to \mathcal{F}^{-1}(N, p) = (\ker p, \operatorname{im} p).$$

So there is a homomorphism

$$\mathcal{F}(\varphi, \psi) = (\varphi \oplus \psi) : \mathcal{F}(L, V) = (F, p) \to (N, p).$$

$$(\varphi \oplus \psi)(m_i) = (\varphi \oplus \psi)(x_i + y_i) = \varphi(x_i) + \psi(y_i) = \\ = r(n_i) + p(n_i) = n_i = f(m_i)$$

holds for $1 \leq i \leq n$, because $x_i \in L$, $y_i \in V$. □

5. Automorphisms of the category Ξ^0 and of the category Θ^0

If we have a category $\mathfrak{K}$, which objects are universal algebras and morphisms are homomorphism, then automorphism Φ of this category transform the homomorphism $id_A \in \operatorname{Mor}\mathfrak{K}$, where $A \in \operatorname{Ob}\mathfrak{K}$, to homomorphism $id_{\Phi(A)}$, because homomorphism id_A uniquely defined by its "algebraic" property: id_A is a unit of the semigroup $\operatorname{End}A$. Therefore we have a

Proposition 5.1. *If* $A, B \in \operatorname{Ob}\mathfrak{K}$, $A \cong B$, $\Phi \in \operatorname{Aut}\mathfrak{K}$ *then* $\Phi(A) \cong \Phi(B)$.

Theorem 5.1. *The category* Ξ^0 *has* 2*-sorted IBN propriety: if* $W(X_1, Y_1)$, $W(X_2, Y_2) \in \operatorname{Ob}\Xi^0$ *and* $W(X_1, Y_1) \cong W(X_2, Y_2)$, *then* $|X_1| = |X_2|$ *and* $|Y_1| = |Y_2|$.

Proof. We consider $W(X, Y) = (L(X), A(X)Y) = (L, V) \in \operatorname{Ob}\Xi^0$. L/L^2 is a k-linear space and $\dim L/L^2 = |X|$.

In the associative algebra $A(X)$ we will consider $\langle L \rangle$ - two-sided ideal generated by the set $L = L(X) \subset A(X)$. This ideal coincide with $\langle X \rangle$ - two-sided ideal generated by the set X, because every element of L can be generated by elements of X. In the $A(X)$-module $V = A(X)Y$ we will consider submodule $\langle L \rangle V = \operatorname{Span}_k(av \mid a \in \langle L \rangle, v \in V) = \langle X \rangle V$. $\dim V/\langle X \rangle V = \dim V/\langle L \rangle V = |Y|$.

We assume that we have two objects of Ξ^0: $W(X_1, Y_1) = (L(X_1), A(X_1)Y_1) = (L_1, V_1)$ and $W(X_2, Y_2) = (L(X_2), A(X_2)Y_2) = (L_2, V_2)$ - and there is an isomorphism $(\varphi, \psi) : (L_1, V_1) \to (L_2, V_2)$. It means that $\varphi : L_1 \to L_2$ is an isomorphism and $L_1/L_1^2 \cong L_2/L_2^2$, so $|X_1| = |X_2|$.

By (1.1) we have that $\psi : V_1 \to V_2$ is an isomorphism of the L_1-modules, when acting of L_1 over V_2 defined by $l \circ v = \varphi(l) v$, where $l \in L_1$, $v \in V_2$. $A(X_1)$ and $A(X_2)$ are universal enveloped algebras of L_1 and L_2 respectively. Therefore the isomorphism $\varphi : L_1 \to L_2$ can be extended to the isomorphism of algebras with unit $\varphi : A(X_1) \to A(X_2)$. $A(X_1)$ is generated as algebra with unit by elements of L_1, so ψ is also an isomorphism of the $A(X_1)$-modules. Therefore there is an isomorphism of $A(X_1)$-modules $V_1 / \langle L_1 \rangle V_1 \cong V_2 / \langle L_2 \rangle V_2$, because $\psi(\langle L_1 \rangle V_1) = \langle L_2 \rangle V_2$. So $\dim V_1 / \langle L_1 \rangle V_1 = \dim V_2 / \langle L_2 \rangle V_2$ and $|Y_1| = |Y_2|$. □

This is a well-known

Definition 5.1. We consider a category $\mathfrak{K}$ and the family of objects $\{A_i\}_{i \in I} \subseteq \mathrm{Ob}\mathfrak{K}$. The pair $\left(Q \in \mathrm{Ob}\mathfrak{K}, \{\eta_i : A_i \to Q\}_{i \in I} \subseteq \mathrm{Mor}\mathfrak{K}\right)$ called *coproduct* of $\{A_i\}_{i \in I}$ if for every $B \in \mathrm{Ob}\mathfrak{K}$ and every $\{\alpha_i : A_i \to B\}_{i \in I} \subseteq \mathrm{Mor}\mathfrak{K}$ there exists only one $\alpha : Q \to B \in \mathrm{Mor}\mathfrak{K}$ such that $\alpha_i = \alpha \eta_i$.

The coproduct is defined up to isomorphism. It is clear that if $\Phi \in \mathrm{Aut}\mathfrak{K}$ then

$$\Phi\left(\coprod_{i \in I} A_i\right) \cong \coprod_{i \in I} \Phi(A_i). \tag{5.1}$$

It is easy to check that if $W(X_1, Y_1), W(X_2, Y_2) \in \mathrm{Ob}\Xi^0$ then $W(X_1, Y_1) \sqcup W(X_2, Y_2) \cong W(X_3, Y_3)$, where

$$|X_3| = |X_1| + |X_2|, |Y_3| = |Y_1| + |Y_2|. \tag{5.2}$$

Similar to [3] we define

Definition 5.2. We say that the free representation $W(X, Y) \in \mathrm{Ob}\Xi^0$ is a *cyclic* if $|X| = 1$ and $|Y| = 1$.

Proposition 5.2. *For every $\Phi \in \mathrm{Aut}\Xi^0$ the $\Phi(W(\varnothing, \varnothing)) = W(\varnothing, \varnothing)$ holds and if $W(x, y) \in \mathrm{Ob}\Xi^0$ is a cyclic representation then $\Phi(W(x, y))$ is also a free cyclic representation.*

Proof. We consider the factor set $\mathrm{Ob}\Xi^0 / \cong$ (skeleton of the set $\mathrm{Ob}\Xi^0$). The elements of this set we denote by

$$[W(X, Y)] = \left\{W(X_1, Y_1) \in \mathrm{Ob}\Xi^0 \mid W(X, Y) \cong W(X_1, Y_1)\right\}.$$

By Theorem 5.1 we can define the mapping

$$g : \mathrm{Ob}\Xi^0 / \cong \ni [W(X, Y)] \to (|X|, |Y|) \in \mathbb{N} \oplus \mathbb{N}$$

and this mapping is bijection. $\mathbb{N}\oplus\mathbb{N}$ we consider as a set with the operation of the addition according the components. $\mathbb{N}\oplus\mathbb{N}$ will be a commutative semigroup with this operation. $\mathrm{Ob}\Xi^0/\cong$ we consider as a set with the operation of the coproduct. $\mathrm{Ob}\Xi^0/\cong$ with this operation also will be a commutative semigroup. By (5.2) g is an isomorphism.

We consider an arbitrary $\Phi \in \mathrm{Aut}\Xi^0$. By Proposition 5.1 in the set $\mathrm{Ob}\Xi^0/\cong$ we can define the factor mapping

$$\widetilde{\Phi} : \mathrm{Ob}\Xi^0/\cong\ni [W(X,Y)] \to [\Phi(W(X,Y))] \in \mathrm{Ob}\Xi^0/\cong .$$

$\widetilde{\Phi}$ is a bijection, because $\Phi \in \mathrm{Aut}\Xi^0$. By (5.1) $\widetilde{\Phi}$ is an isomorphism.

Therefore $g\widetilde{\Phi}g^{-1}$ is an automorphism of $\mathbb{N}\oplus\mathbb{N}$. $(0,0)$ is a unit of the semigroup $\mathbb{N}\oplus\mathbb{N}$, so $g\widetilde{\Phi}g^{-1}(0,0) = (0,0)$ or $\widetilde{\Phi}g^{-1}(0,0) = g^{-1}(0,0)$. $g^{-1}(0,0) = \{W(\varnothing,\varnothing)\}$, therefore $\Phi(W(\varnothing,\varnothing)) = W(\varnothing,\varnothing)$.

The semigroup $\mathbb{N}\oplus\mathbb{N}$ has only one minimal set of generators: $\{(1,0),(0,1)\}$. So the automorphism $g\widetilde{\Phi}g^{-1}$ must preserve this set. Therefore we have two cases: or $\widetilde{\Phi}g^{-1}(1,0) = g^{-1}(0,1)$ and $\widetilde{\Phi}g^{-1}(0,1) = g^{-1}(1,0)$, or $\widetilde{\Phi}g^{-1}(1,0) = g^{-1}(1,0)$ and $\widetilde{\Phi}g^{-1}(0,1) = g^{-1}(0,1)$. $g^{-1}(1,0) = \{W(x,\varnothing) \mid x \in X^0\}$, $g^{-1}(0,1) = \{W(\varnothing,y) \mid y \in Y^0\}$. In the first case we have that $\Phi(W(x,\varnothing)) = W(\varnothing,y_1)$, $\Phi(W(\varnothing,y)) = W(x_1,\varnothing)$, where $x, x_1 \in X^0$, $y, y_1 \in Y^0$. Therefore

$$\Phi(W(x,y)) \cong \Phi(W(x,\varnothing) \sqcup W(\varnothing,y)) \cong \\ \cong W(\varnothing,y_1) \sqcup W(x_1,\varnothing) \cong W(x_1,y_1).$$

So $\Phi(W(x,y)) = W(x_2,y_2)$, where $x_2 \in X^0$, $y_2 \in Y^0$. In the second case we achieve the similar result. □

Corollary 1. *If* $\Phi \in \mathrm{Aut}\Xi^0$, *then* $\Phi(\Xi') = \Xi'$.

Proof. If $X \subset X^0, Y \subset Y^0$ and $|X| = |Y| = n > 1$ then $X = \{x_1,\ldots,x_n\}$, $Y = \{y_1,\ldots,y_n\}$ and we have that

$$\Phi(W(X,Y)) \cong \Phi\left(\coprod_{i=1}^{n} W(x_i,y_i)\right) \cong \\ \cong \coprod_{i=1}^{n} \Phi(W(x_i,y_i)) \cong \coprod_{i=1}^{n} W(x'_i,y'_i) \cong W(X',Y'),$$

where $X' = \{x'_1,\ldots,x'_n\} \subset X^0$, $Y' = \{y'_1,\ldots,y'_n\} \subset Y^0$ and $|X'| = |Y'| = n$. □

In the category Θ we can consider subcategory Θ^0. We take a infinite countable sets of symbols M^0. The objects of Θ^0 will be the free algebras in Θ with the set of free generators M such that $M \subset M^0$, $|M| < \infty$. We will denote these algebras by $F(M)$. The morphisms of Θ^0 will be the homomorphisms of these algebras.

By using of the Theorems 4.1 and 4.2 $\mathcal{F}(\Xi') = \mathrm{Ob}\Theta^0$ and $\mathcal{F}^{-1}(\mathrm{Ob}\Theta^0) = \Xi'$, so, by Corollary 1 from the Proposition 5.2 we prove the

Theorem 5.2. *If* $\Phi \in \mathrm{Aut}\Xi^0$ *then* $\mathcal{F}\Phi_{|\Xi'}\mathcal{F}^{-1} \in \mathrm{Aut}\Theta^0$.

6. Automorphic equivalence in the variety Ξ and in the variety Θ

Proposition 6.1. *If* $(T_1, T_2) \subset W = W(X, Y) \in \Xi'$, $H = (L, V) \in \Xi$, (T_1, T_2) *is an* H*-closed congruence, then* $T_1 \oplus T_2 \subset \mathcal{F}(W(X, Y))$ *is an* $\mathcal{F}(H)$*-closed congruence. If* $T \subset F(M) \in \mathrm{Ob}\Theta^0$, $N = (N, p) \in \Theta$, T *is an* N*-closed congruence, then* $(T \cap \ker p, T \cap \mathrm{im} p) \subset \mathcal{F}^{-1}(F(M))$ *is an* $\mathcal{F}^{-1}(N)$*-closed congruence. The mappings*

$$\mathcal{F}_{W,H} : Cl_H(W) \ni (T_1, T_2) \to T_1 \oplus T_2 \in Cl_{\mathcal{F}(H)}(\mathcal{F}(W))$$

and

$$\mathcal{F}^{-1}_{F(M),N} : Cl_N(F(M)) \ni T \to (T \cap \ker p, T \cap \mathrm{im} p) \in \\ \in Cl_{\mathcal{F}^{-1}(N)}\left(\mathcal{F}^{-1}(F(M))\right)$$

are bijections.

Proof. If $(\varphi, \psi) \in (T_1, T_2)'_H$, then by Proposition 4.2

$$\ker \mathcal{F}(\varphi, \psi) = \ker(\varphi \oplus \psi) = \ker\varphi \oplus \ker\psi \supseteq T_1 \oplus T_2,$$

so

$$\mathcal{F}\left((T_1, T_2)'_H\right) = \left\{f = \varphi \oplus \psi \mid (\varphi, \psi) \in (T_1, T_2)'_H\right\} \subseteq (T_1 \oplus T_2)'_{\mathcal{F}(H)}.$$

We will consider $l + v \in (T_1 \oplus T_2)''_{\mathcal{F}(H)} = \bigcap_{f \in (T_1 \oplus T_2)'_{\mathcal{F}(H)}} \ker f$, where $l \in \ker p$, $v \in \mathrm{im} p$. $l + v \in \bigcap_{(\varphi,\psi) \in (T_1, T_2)'_H} \ker(\varphi \oplus \psi)$ holds, so for every $(\varphi, \psi) \in (T_1, T_2)'_H$ we have $(\varphi \oplus \psi)(l + v) = \varphi(l) + \psi(v) = 0$. $\varphi(l) \in \ker p$, $\psi(v) \in \mathrm{im} p$, so $\varphi(l) = 0$, $\psi(v) = 0$. Hence $l \in \bigcap_{(\varphi,\psi) \in (T_1, T_2)'_H} \ker\varphi =$

T_1, $v \in \bigcap_{(\varphi,\psi)\in(T_1,T_2)'_H} \ker\psi = T_2$. Therefore $l+v \in T_1 \oplus T_2$ and $(T_1 \oplus T_2)''_{\mathcal{F}(H)} \subseteq T_1 \oplus T_2$. It means that $T_1 \oplus T_2$ is an $\mathcal{F}(H)$-closed congruence.

If $f \in T'_N$ then $\mathcal{F}^{-1}(f) = (rf\kappa, pf\iota)$ and by Proposition 4.2 $\ker rf\kappa = \ker f \cap \ker p \supseteq T \cap \ker p$, $\ker pf\iota = \ker f \cap \operatorname{im} p \supseteq T \cap \operatorname{im} p$ holds. So $(rf\kappa, pf\iota) \in (T\cap\ker p, T\cap \operatorname{im}p)'_{\mathcal{F}^{-1}(N)}$ and $\mathcal{F}^{-1}(T'_N) \subseteq (T\cap\ker p, T\cap\operatorname{im}p)'_{\mathcal{F}^{-1}(N)}$. Therefore

$$(T\cap\ker p, T\cap\operatorname{im}p)''_{\mathcal{F}^{-1}(N)} \subseteq \left(\bigcap_{f\in T'_N}\ker rf\kappa, \bigcap_{f\in T'_N}\ker pf\iota\right) =$$

$$= \left(\bigcap_{f\in T'_N}(\ker f\cap\ker p), \bigcap_{f\in T'_N}(\ker f\cap\operatorname{im}p)\right) =$$

$$= \left(\left(\bigcap_{f\in T'_N}\ker f\right)\cap\ker p, \left(\bigcap_{f\in T'_N}\ker f\right)\cap\operatorname{im}p\right) =$$

$$= (T\cap\ker p, T\cap\operatorname{im}p),$$

so $(T\cap\ker p, T\cap\operatorname{im}p)$ is an $\mathcal{F}^{-1}(N)$-closed congruence.

If $(T_1,T_2)\in Cl_H(W)$ and $T = T_1\oplus T_2 \subset \mathcal{F}(W)$, then $T_1 = T\cap\ker p$, $T_2 = T\cap\operatorname{im}p$, so $\mathcal{F}^{-1}_{\mathcal{F}(W),\mathcal{F}(H)}\mathcal{F}_{W,H} = id_{Cl_H(W)}$.

If $N_1 = (N_1,p_1), N_2 = (N_2,p_2) \in \Theta$ and $f : N_1 \to N_2$ is a homomorphism then $\ker f$ is p-invariant. So, $T \in Cl_N(F(M))$ is also p-invariant and $T = (T\cap\ker p)\oplus(T\cap\operatorname{im}p)$. Therefore $\mathcal{F}_{\mathcal{F}^{-1}(F(M)),\mathcal{F}^{-1}(N)}\mathcal{F}^{-1}_{F(M),N} = id_{Cl_N(F(M))}$. □

If $F_1 = F(M_1), F_2 = F(M_2) \in \mathrm{Ob}\Theta^0$ and T is a congruence in F_2 then $\beta_{F_1,F_2}(T)$ will be a relation in $\mathrm{Hom}(F_1,F_2)$ which we define as in [1, Subsection 3.3]: $(f_1,f_2)\in\beta_{F_1,F_2}(T)$ if and only if $f_1(n)\equiv f_2(n) \pmod T$ holds for every $n\in F_1$.

Proposition 6.2. *If* $W_1 = W(X_1,Y_1), W_2 = W(X_2,Y_2) \in \Xi'$, $H \in \Xi$, $(T_1,T_2) \in Cl_H(W_2)$ *then* $\mathcal{F}(\beta_{W_1,W_2}(T_1,T_2)) = \beta_{\mathcal{F}(W_1),\mathcal{F}(W_2)}(\mathcal{F}_{W_2,H}(T_1,T_2))$. *If* $F_1 = F(M_1), F_2 = F(M_2) \in \mathrm{Ob}\Theta^0$, $N = (N,p) \in \Theta$, $T \in Cl_N(F_2)$ *then* $\mathcal{F}^{-1}(\beta_{F_1,F_2}(T)) = \beta_{\mathcal{F}^{-1}(F_1),\mathcal{F}^{-1}(F_2)}\left(\mathcal{F}^{-1}_{F_2,N}(T)\right)$.

Proof. $\mathcal{F}(W_i) = \left(L(X_i)\oplus A(X_i)Y_i, p_{|A(X_i)Y_i}\right)$ where $i = 1,2$. $\mathcal{F}_{W_2,H}(T_1,T_2) = T_1\oplus T_2 \subseteq L(X_2)\oplus A(X_2)Y_2 = \mathcal{F}(W_2)$. If

$((\varphi_1,\psi_1),(\varphi_2,\psi_2)) \in \beta_{W_1,W_2}(T_1,T_2)$ then $\varphi_1(l) \equiv \varphi_2(l) (\operatorname{mod} T_1)$ holds for every $l \in L(X_1)$ and $\psi_1(v) \equiv \psi_2(v) (\operatorname{mod} T_2)$ holds for every $v \in A(X_1) Y_1$. $\mathcal{F}(\varphi_i,\psi_i) = \varphi_i \oplus \psi_i \in \operatorname{Hom}(\mathcal{F}(W_1),\mathcal{F}(W_2))$ where $i = 1,2$. For every $n \in \mathcal{F}(W_1)$ we have $n = l + v$, where $l \in L(X_1)$, $v \in A(X_1) Y_1$. So $(\varphi_1 \oplus \psi_1)(n) = \varphi_1(l) + \psi_1(v) \equiv \varphi_2(l) + \psi_2(v) (\operatorname{mod} T_1 \oplus T_2)$, $\varphi_2(l) + \psi_2(v) = (\varphi_2 \oplus \psi_2)(n)$ and

$$(\mathcal{F}(\varphi_1,\psi_1),\mathcal{F}(\varphi_2,\psi_2)) \in \beta_{\mathcal{F}(W_1),\mathcal{F}(W_2)}(T_1 \oplus T_2).$$

We assume that $(f_1,f_2) \in \beta_{\mathcal{F}(W_1),\mathcal{F}(W_2)}(T_1 \oplus T_2)$. $\mathcal{F}^{-1}(f_i) = (rf_i\kappa, pf_i\iota) \in \operatorname{Hom}(W_1,W_2)$ where $i = 1,2$. If $l \in L(X_1)$ then $f_1(l) - f_2(l) \in T_1 \oplus T_2$ and $rf_1\kappa(l) - rf_2\kappa(l) \in T_1$. Analogously we have $pf_1\iota(v) - pf_2\iota(v) \in T_2$ for every $v \in A(X_2) Y_2$, so

$$\left(\mathcal{F}^{-1}(f_1),\mathcal{F}^{-1}(f_2)\right) = ((rf_1\kappa, pf_1\iota),(rf_2\kappa, pf_2\iota)) \in \beta_{W_1,W_2}(T_1,T_2).$$

Therefore

$$\mathcal{F}(\beta_{W_1,W_2}(T_1,T_2)) = \beta_{\mathcal{F}(W_1),\mathcal{F}(W_2)}(\mathcal{F}_{W_2,H}(T_1,T_2)).$$

From this fact and from proving of Proposition 6.1 we can conclude that

$$\mathcal{F}^{-1}(\beta_{F_1,F_2}(T)) = \beta_{\mathcal{F}^{-1}(F_1),\mathcal{F}^{-1}(F_2)}\left(\mathcal{F}^{-1}_{F_2,N}(T)\right). \qquad \square$$

Theorem 6.1. *If* $H_1 = (L_1,V_1), H_2 = (L_2,V_2) \in \Xi$ *are automorphically equivalent then* $N_1 = \mathcal{F}(H_1), N_2 = \mathcal{F}(H_2)$ *are automorphically equivalent.*

Proof. We have an automorphism $\Phi \in \operatorname{Aut}\Xi^0$ and the system of bijections $\alpha(\Phi)_W : Cl_{H_1}(W) \to Cl_{H_2}(\Phi(W))$ for every $W \in \operatorname{Ob}\Xi^0$. Also the equation

$$\Phi(\beta_{W_1,W_2}(T_1,T_2)) = \beta_{\Phi(W_1),\Phi(W_2)}\left(\alpha(\Phi)_{W_2}(T_1,T_2)\right)$$

holds for every $W_1,W_2 \in \operatorname{Ob}\Xi^0$, and every $(T_1,T_2) \in Cl_{H_1}(W_2)$.

By Proposition 5.2 there is an automorphism $\Psi = \mathcal{F}\Phi_{|\Xi'}\mathcal{F}^{-1} \in \operatorname{Aut}\Theta^0$. By Proposition 6.1 the mapping:

$$\alpha(\Psi)_F = \mathcal{F}_{\Phi\mathcal{F}^{-1}(F),H_2}\alpha(\Phi)_{\mathcal{F}^{-1}(F)}\mathcal{F}^{-1}_{F,N_1} : Cl_{N_1}(F) \to Cl_{N_2}(\Psi(F))$$

is a bijection for every $F \in \operatorname{Ob}\Theta^0$. By Proposition 6.2 we have for every $F_1,F_2 \in \operatorname{Ob}\Theta^0$, and every $T \in Cl_{N_1}(F_2)$ that

$$\Psi\left(\beta_{F_1,F_2}(T)\right) = \mathcal{F}\Phi\mathcal{F}^{-1}\left(\beta_{F_1,F_2}(T)\right) =$$
$$= \mathcal{F}\Phi\left(\beta_{\mathcal{F}^{-1}(F_1),\mathcal{F}^{-1}(F_2)}\left(\mathcal{F}^{-1}_{F_2,N_1}(T)\right)\right) =$$
$$= \mathcal{F}\left(\beta_{\Phi\mathcal{F}^{-1}(F_1),\Phi\mathcal{F}^{-1}(F_2)}\left(\alpha(\Phi)_{\mathcal{F}^{-1}(F_2)}\left(\mathcal{F}^{-1}_{F_2,N_1}(T)\right)\right)\right) =$$
$$= \beta_{\mathcal{F}\Phi\mathcal{F}^{-1}(F_1),\mathcal{F}\Phi\mathcal{F}^{-1}(F_2)}\left(\mathcal{F}_{\Phi\mathcal{F}^{-1}(F_2),H_2}\alpha(\Phi)_{\mathcal{F}^{-1}(F_2)}\left(\mathcal{F}^{-1}_{F_2,N_1}(T)\right)\right) =$$
$$= \beta_{\Psi(F_1),\Psi(F_2)}\left(\alpha(\Psi)_{F_2}(T)\right). \quad \square$$

7. Automorphisms of the category of the finitely generated free algebras of the some variety of 1-sorted algebras

In this Section we explain the method of verbal operations which we will use for the studying of the relation between the automorphic equivalence and geometric equivalence in the our variety Θ. We use results of the [4] and [5].

In this Section the word "algebra" means "universal algebra". Also so on in this Section Θ will be an arbitrary variety of 1-sorted algebras. As in the Section 5 we define the category Θ^0 of the finitely generated free algebras of our variety Θ. The infinite countable sets of symbols which will be the generators of our free algebras we will denote in this Section by X^0.

Definition 7.1. An automorphism Υ of a category $\mathfrak{K}$ is *inner*, if it is isomorphic as a functor to the identity automorphism of the category $\mathfrak{K}$.

It means that for every $A \in \mathrm{Ob}\mathfrak{K}$ there exists an isomorphism $s_A^{\Upsilon} : A \to \Upsilon(A)$ such that for every $\alpha \in \mathrm{Mor}_{\mathfrak{K}}(A, B)$ the diagram

$$\begin{array}{ccc} A & \xrightarrow{s_A^{\Upsilon}} & \Upsilon(A) \\ \alpha \downarrow & & \downarrow \Upsilon(\alpha) \\ B & \xrightarrow[s_B^{\Upsilon}]{} & \Upsilon(B) \end{array}$$

commutes. The group of the all automorphisms of the category Θ^0 we denote by $\mathfrak{A}$. The subgroup of the all inner automorphisms of Θ^0 we denote by $\mathfrak{Y}$. This is a normal subgroup of $\mathfrak{A}$: $\mathfrak{Y} \lhd \mathfrak{A}$.

We know from [1] that if automorphic equivalence of algebras $H_1, H_2 \in \Theta$ provided by inner automorphism then H_1 and H_2 are geometrically equivalent. Hear variety Θ can by even a variety of many-sorted algebras.

So for studying of the difference between the automorphic equivalence and geometric equivalence of the algebras from Θ, we must calculate the quotient group $\mathfrak{A}/\mathfrak{Y}$.

In the 1-sorted case there is a reason to define

Definition 7.2. An automorphism Φ of the category Θ^0 is called *strongly stable* if it satisfies the conditions:

A1) Φ preserves all objects of Θ^0,

A2) there exists a system of bijections $\left\{ s_F^\Phi : F \to F \mid F \in \mathrm{Ob}\Theta^0 \right\}$ such that Φ acts on the morphisms $\alpha : D \to F$ of Θ^0 by this way:

$$\Phi(\alpha) = s_F^\Phi \alpha \left(s_D^\Phi \right)^{-1}, \tag{7.1}$$

A3) $s_F^\Phi \mid_X = id_X$, for every free algebra $F = F(X) \in \mathrm{Ob}\Theta^0$.

The subgroup of the all strongly stable automorphisms of Θ^0 we denote by $\mathfrak{S}$.

We say that the variety Θ has IBN propriety if for every $F(X), F(Y) \in \mathrm{Ob}\Theta^0$ we have $F(X) \cong F(Y)$ only if $|X| = |Y|$. In this case we have the decomposition

$$\mathfrak{A} = \mathfrak{Y}\mathfrak{S} \tag{7.2}$$

so $\mathfrak{A}/\mathfrak{Y} = \mathfrak{S}/\mathfrak{S} \cap \mathfrak{Y}$.

The system of bijections $\left\{ s_F^\Phi = s_F : F \to F \mid F \in \mathrm{Ob}\Theta^0 \right\}$ mentioned in definition of the strongly stable automorphism fulfills these two conditions:

B1) for every homomorphism $\alpha : A \to B \in \mathrm{Mor}\Theta^0$ the mappings $s_B \alpha s_A^{-1}$ and $s_B^{-1} \alpha s_A$ are homomorphisms;

B2) $s_F \mid_X = id_X$ for every free algebra $F \in \mathrm{Ob}\Theta^0$.

These bijections uniquely defined by the strongly stable automorphism Φ, because for every $F \in \mathrm{Ob}\Theta^0$ and every $f \in F$ we have

$$s_F^\Phi(f) = s_F^\Phi \alpha(x) = \left(s_F^\Phi \alpha \left(s_D^\Phi \right)^{-1} \right)(x) = (\Phi(\alpha))(x), \tag{7.3}$$

where $D = D(x) \in \mathrm{Ob}\Theta^0$ is a 1-generated free algebra and $\alpha : D \to F$ homomorphism such that $\alpha(x) = f$.

On the other side by system of bijections $\left\{ s_F : F \to F \mid F \in \mathrm{Ob}\Theta^0 \right\}$ which fulfills conditions B1) and B2) we can define the strongly stable automorphism Φ, which preserves all objects of Θ^0 and acts on the morphisms $\alpha : D \to F$ of Θ^0 by formula (7.1) with $s_F^{\Phi} = s_F$. By this way we construct an one-to-one and onto correspondence between the set of the all strongly stable automorphisms of the category Θ^0 and the set of the all systems of bijections which fulfill conditions B1) and B2).

We denote the signature of the algebras from the variety Θ by Ω. The arity of the operation $\omega \in \Omega$ we denote by n_ω and by F_ω we denote $F(x_1, \ldots, x_{n_\omega}) \in \mathrm{Ob}\Theta^0$. $\omega(x_1, \ldots, x_{n_\omega}) \in F_\omega$. If we have system of bijections $\left\{s_F : F \to F \mid F \in \mathrm{Ob}\Theta^0\right\}$ which fulfills conditions B1) and B2) then

$$w_\omega(x_1, \ldots, x_{n_\omega}) = s_{F_\omega}(\omega(x_1, \ldots, x_{n_\omega})) \in F_\omega. \tag{7.4}$$

We will consider the system of words $W = \{w_\omega \mid \omega \in \Omega\}$. In every $H \in \Theta$ we can define new operations $\{\omega^* \mid \omega \in \Omega\}$ by using of the system of words W:

$$\omega^*(h_1, \ldots, h_{n_\omega}) = w_\omega(h_1, \ldots, h_{n_\omega}) \tag{7.5}$$

for every $h_1, \ldots, h_{n_\omega} \in H$. We denote by H_W^* the new algebra which coincide as set with H but has other operations: $\{\omega^* \mid \omega \in \Omega\}$ instead $\{\omega \mid \omega \in \Omega\}$. The system of words $W = \{w_\omega \mid \omega \in \Omega\}$ fulfills these two conditions:

Op1) $w_\omega(x_1, \ldots, x_{n_\omega}) \in F_\omega$ for every $\omega \in \Omega$,

Op2) for every $F = F(X) \in \mathrm{Ob}\Theta^0$ there exists an isomorphism $\sigma_F : F \to F_W^*$ such that $\sigma_F \mid_X = id_X$ because the bijections $\left\{s_F \mid F \in \mathrm{Ob}\Theta^0\right\}$ will be isomorphisms $\sigma_F : F \to F_W^*$.

On the other side if we have a system of words $W = \{w_\omega \mid \omega \in \Omega\}$ which fulfills conditions Op1) and Op2), then we have that $F_W^* \in \Theta$, so the isomorphisms $\sigma_F : F \to F_W^*$ are uniquely determined by the system of words W. This system of isomorphisms $\left\{\sigma_F : F \to F_W^* \mid F \in \mathrm{Ob}\Theta^0\right\}$ is a system of bijections which fulfills conditions B1) and B2) with $s_F = \sigma_F$. By this way we construct an one-to-one and onto correspondence between the set of the all system of bijections which fulfills conditions B1) and B2) and the set of the all system of words which fulfills conditions Op1) and Op2).

Therefore we can calculate the group $\mathfrak{S}$ if we can find the all system of words which fulfill conditions Op1) and Op2). For calculation of the group $\mathfrak{S} \cap \mathfrak{Y}$ we also have a

Criterion 7.1. *The strongly stable automorphism Φ of the category Θ^0 which corresponds to the system of words W is inner if and only if for* *every* $F \in \mathrm{Ob}\Theta^0$ *there exists an isomorphism* $c_F : F \to F_W^*$ *such that* $c_F\alpha = \alpha c_D$ *fulfills for every* $(\alpha : D \to F) \in \mathrm{Mor}\Theta^0$.

8. Strongly stable automorphisms of the category Θ^0

The variety Θ, which was defined in the Section 4, is a variety of 1-sorted universal algebras. If $F(M) \in \mathrm{Ob}\Theta^0$ then by Theorem 4.1 $|M| = \dim\left(\ker p/(\ker p)^2\right)$, so variety Θ possesses the IBN property: for free algebras $F(M_1), F(M_2) \in \Theta$ we have $F(M_1) \cong F(M_2)$ if and only if $|M_1| = |M_2|$. So we have for our variety Θ^0 the decomposition (7.2) and for calculation of the group $\mathfrak{A}/\mathfrak{Y} = \mathfrak{S}/\mathfrak{S} \cap \mathfrak{Y}$ we can use the method described in the Section 7.

The signature of our variety Θ is $\Omega = \{0, \lambda(\lambda \in k), +, [,], p\}$, where 0 is 0-ary operation of the taking 0, λ for every $\lambda \in k$ is the 1-nary operation of the multiplication by this scalar, p is the 1-nary operation of projection, $+$ is the addition and $[,]$ are the Lie brackets. We must find for the calculation of the group $\mathfrak{S}$ all the system of words

$$W = \left\{w_0, w_\lambda(\lambda \in k), w_+, w_{[,]}, w_p\right\} \tag{8.1}$$

which fulfill conditions Op1) and Op2) and after use the Criterion 7.1 for the calculation of the group $\mathfrak{S} \cap \mathfrak{Y}$. By this way we will prove the

Theorem 8.1. *If* $\mathrm{Aut}k = \{id_k\}$ *then the group* $\mathfrak{A}/\mathfrak{Y}$ *is a trivial.*

Proof. If $(F, p) = F(m_1, \ldots, m_n) \in \mathrm{Ob}\Theta^0$ then by Theorems 4.1 and 4.2 $(F, p) = \mathcal{F}\mathcal{F}^{-1}(F, p) = (L \oplus V, p_V)$, where $p = p_V$,

$$L = \ker p = L(r(m_1), \ldots, r(m_n))$$

is a free Lie algebra with the free generators $r(m_1), \ldots, r(m_n)$,

$$V = \mathrm{im}p = \bigoplus_{i=1}^{n} A(r(m_1), \ldots, r(m_n))\, p(m_i)$$

is a free module with the basis $\{p(m_1), \ldots, p(m_n)\}$ over algebra $A(r(m_1), \ldots, r(m_n))$, which is an associative algebra with unit generated by the free generators $r(m_1), \ldots, r(m_n)$. Hear we must understand that by formula (4.2)

$$r(m_{i_1}) \ldots r(m_{i_s})\, v = [r(m_{i_1}), [\ldots, [r(m_{i_s}), v]]],$$

where $v \in V$, $1 \leq i_1, \ldots, i_s \leq n$, $s \in \mathbb{N}$, if $s = 0$ then $1v = v$. So by linearity we can understand what means $f(r(m_1), \ldots, r(m_n))v$, for every associative polynomial from n variables $f \in A(x_1, \ldots, x_n)$.

We assume that $\Psi \in \mathfrak{S}$ corresponds to the system of bijections $\left\{s_F^{\Psi} = s_F : F \to F \mid F \in \mathrm{Ob}\Theta^0\right\}$ and to the system of words (8.1) and the words of this system correspond to the operations from Ω by formula (7.4) with $s_{F_\omega} = s_{F_\omega}^{\Psi}$. W fulfills conditions Op1) and Op2). In particular by condition Op2) all axioms of the variety Θ must fulfill for operations defined by system of words W. In this proof we have more convenient to denote by an other symbols than the symbols of Ω the operations defined by the words from W according the (7.5).

$w_0 = 0$ because $w_0 \in F(\varnothing)$ and $F(\varnothing) = \{0\}$.

We denote by $\lambda *$ the operation defined by the word $w_\lambda \in F(m)$ ($\lambda \in k$), where $F(m)$ is a 1-generated object of the category Θ^0. $(F(m), p) = (L \oplus V, p_V)$, where $L = L(r(m)) = \mathrm{sp}_k\{r(m)\}$, $V = A(r(m))p(m) = k[r(m)]p(m)$, so

$$\lambda * m = w_\lambda(m) = \varphi(\lambda)r(m) + q_\lambda(r(m))p(m),$$

where $\varphi(\lambda) \in k$, $q_\lambda \in k[x]$. If $\lambda \neq 0$ then $\lambda^{-1} * (\lambda * m) = m$ must fulfill.

$$\begin{aligned}\lambda^{-1} * (\lambda * m) &= \lambda^{-1} * (\varphi(\lambda)r(m) + q_\lambda(r(m))p(m)) = \\ &= \varphi\left(\lambda^{-1}\right) r(\varphi(\lambda)r(m) + q_\lambda(r(m))p(m)) + \\ &+ q_{\lambda^{-1}}(r(\varphi(\lambda)r(m) + q_\lambda(r(m))p(m)))\,p(\varphi(\lambda)r(m) + q_\lambda(r(m))p(m)) = \\ &= \varphi\left(\lambda^{-1}\right)\varphi(\lambda)r(m) + q_{\lambda^{-1}}(\varphi(\lambda)r(m))(q_\lambda(r(m))p(m)).\end{aligned}$$

On the other side $m = r(m) + p(m)$. So $\varphi\left(\lambda^{-1}\right)\varphi(\lambda) = 1$ and $\varphi(\lambda) \neq 0$.

$$q_{\lambda^{-1}}(\varphi(\lambda)r(m))(q_\lambda(r(m))p(m)) = s(r(m))p(m),$$

where $s \in k[x]$. If $\deg q_{\lambda^{-1}} = n$, $\deg q_\lambda = t$ then $\deg s = n + t$, but $\deg s = 0$ must hold, so $n = 0$, $t = 0$. Therefore $q_\lambda = \psi(\lambda) \in k$,

$$\lambda * m = \varphi(\lambda)r(m) + \psi(\lambda)p(m). \tag{8.2}$$

For $\lambda = 0$ it also fulfills with $\varphi(0) = \psi(0) = 0$, because $0 * m = 0$.

$\mu * (\lambda * m) = (\mu\lambda) * m$ must fulfill for every $\mu, \lambda \in k$ so

$$\begin{aligned}\mu * (\lambda * m) &= \mu * (\varphi(\lambda)r(m) + \psi(\lambda)p(m)) = \\ &= \varphi(\mu)r(\varphi(\lambda)r(m) + \psi(\lambda)p(m)) + \psi(\mu)p(\varphi(\lambda)r(m) + \psi(\lambda)p(m)) = \\ &= \varphi(\mu)\varphi(\lambda)r(m) + \psi(\mu)\psi(\lambda)p(m).\end{aligned}$$

On the other side

$$(\mu\lambda) * m = \varphi(\mu\lambda) r(m) + \psi(\mu\lambda) p(m).$$

Hence

$$\varphi(\mu)\varphi(\lambda) = \varphi(\mu\lambda), \psi(\mu)\psi(\lambda) = \psi(\mu\lambda). \tag{8.3}$$

We denote by $\perp$ the operation defined by the word $w_+ \in F(m_1, m_2)$, where $F(m_1, m_2)$ is a 2-generated object of the category Θ^0.

$$m_1 \perp m_2 = l(r(m_1), r(m_2)) + q_1(r(m_1), r(m_2)) p(m_1) + \\ + q_2(r(m_1), r(m_2)) p(m_2),$$

where $l \in L(x_1, x_2)$, $q_1, q_2 \in A(x_1, x_2)$. We can write

$$l(r(m_1), r(m_2)) = \alpha_1 r(m_1) + \alpha_2 r(m_2) + \widetilde{l}(r(m_1), r(m_2)),$$

where $\widetilde{l} \in L^2(x_1, x_2)$, $\alpha_1, \alpha_2 \in k$. And

$$q_i(r(m_1), r(m_2)) p(m_i) = \widetilde{q_i}(r(m_1), r(m_2)) p(m_i) + \beta_i p(m_i),$$

where $\widetilde{q_i}$ is a polynomial from $A(x_1, x_2)$ such that all its monomials have entries of x_1 or x_2, $\beta_i \in k$, $i = 1, 2$.

$m_1 \perp 0 = m_1$ must fulfill. $m_1 = r(m_1) + p(m_1)$. But

$$m_1 \perp 0 = \alpha_1 r(m_1) + \widetilde{q_1}(r(m_1), 0) p(m_1) + \beta_1 p(m_1).$$

Therefore $\alpha_1 = \beta_1 = 1$. From $0 \perp m_2 = m_2$ we conclude that $\alpha_2 = \beta_2 = 1$.

In $F(m)$ the $(\lambda + \mu) * m = (\lambda * m) \perp (\mu * m)$ must fulfill for every $\mu, \lambda \in k$.

$$(\lambda + \mu) * m = \varphi(\lambda + \mu) r(m) + \psi(\lambda + \mu) p(m).$$

Also we have that

$$(\lambda * m) \perp (\mu * m) = r(\lambda * m) + r(\mu * m) + \widetilde{l}(r(\lambda * m), r(\mu * m)) + \\ + \widetilde{q_1}(r(\lambda * m), r(\mu * m)) p(\lambda * m) + p(\lambda * m) + \\ + \widetilde{q_2}(r(\lambda * m), r(\mu * m)) p(\mu * m) + p(\mu * m).$$

$$r(\lambda * m) = r(\varphi(\lambda) r(m) + \psi(\lambda) p(m)) = \varphi(\lambda) r(m),$$

$$p(\lambda * m) = p(\varphi(\lambda) r(m) + \psi(\lambda) p(m)) = \psi(\lambda) p(m).$$

So

$$\begin{aligned}&(\lambda * m) \perp (\mu * m) = \\&\quad = \varphi(\lambda) r(m) + \varphi(\mu) r(m) + \widetilde{l}(\varphi(\lambda) r(m), \varphi(\mu) r(m)) + \\&\quad + \widetilde{q}_1(\varphi(\lambda) r(m), \varphi(\mu) r(m)) \psi(\lambda) p(m) + \psi(\lambda) p(m) + \\&\qquad + \widetilde{q}_2(\varphi(\lambda) r(m), \varphi(\mu) r(m)) \psi(\mu) p(m) + \psi(\mu) p(m).\end{aligned}$$

Hence

$$\varphi(\lambda + \mu) = \varphi(\lambda) + \varphi(\mu), \psi(\lambda + \mu) = \psi(\lambda) + \psi(\mu). \tag{8.4}$$

Therefore φ, ψ are homomorphisms $k \to k$.

$$\varphi(1) = \psi(1) = 1, \tag{8.5}$$

because $1 * m = m$, so $\ker \varphi = \ker \psi = 0$ and $\operatorname{Im} \varphi \cong \operatorname{Im} \psi \cong k$. Hence $\operatorname{Im} \varphi, \operatorname{Im} \psi$ are infinite sets.

$\lambda * (m_1 \perp m_2) = (\lambda * m_1) \perp (\lambda * m_2)$ must fulfill for every $\lambda \in k$.

$$\begin{aligned}\lambda * (m_1 \perp m_2) = \varphi(\lambda) \Big(r(m_1) + r(m_2) + \widetilde{l}(r(m_1), r(m_2))\Big) + \\+ \psi(\lambda) (\widetilde{q}_1(r(m_1), r(m_2)) p(m_1) + p(m_1)) + \\+ \psi(\lambda) (\widetilde{q}_2(r(m_1), r(m_2)) p(m_2) + p(m_2)).\end{aligned} \tag{8.6}$$

$$\begin{aligned}&(\lambda * m_1) \perp (\lambda * m_2) = \\&\quad = r(\lambda * m_1) + r(\lambda * m_2) + \widetilde{l}(r(\lambda * m_1), r(\lambda * m_2)) + \\&\quad + \widetilde{q}_1(r(\lambda * m_1), r(\lambda * m_2)) p(\lambda * m_1) + p(\lambda * m_1) + \\&\quad + \widetilde{q}_2(r(\lambda * m_1), r(\lambda * m_2)) p(\lambda * m_2) + p(\lambda * m_2) = \\&\quad = \varphi(\lambda) r(m_1) + \varphi(\lambda) r(m_2) + \widetilde{l}(\varphi(\lambda) r(m_1), \varphi(\lambda) r(m_2)) + \\&\quad + \widetilde{q}_1(\varphi(\lambda) r(m_1), \varphi(\lambda) r(m_2)) \psi(\lambda) p(m_1) + \psi(\lambda) p(m_1) + \\&\quad + \widetilde{q}_2(\varphi(\lambda) r(m_1), \varphi(\lambda) r(m_2)) \psi(\lambda) p(m_2) + \psi(\lambda) p(m_2).\end{aligned} \tag{8.7}$$

We decompose $\widetilde{l}$, $\widetilde{q}_1$ and $\widetilde{q}_2$ to the homogeneous components according the degrees (sum of degrees of variables x_1 and x_2) of monomials: $\widetilde{l} = l_2 + \ldots + l_{n_0}$, $\widetilde{q}_i = q_{i,1} + \ldots + q_{i,n_i}$, $n_0 = \deg \widetilde{l}$, $n_i = \deg \widetilde{q}_i$, $i = 1, 2$. We have by comparison of (8.6) and (8.7) that $\varphi(\lambda) l_j = (\varphi(\lambda))^j l_j$ for $2 \leq j \leq n_0$ and $\psi(\lambda) q_{i,j} = \psi(\lambda) (\varphi(\lambda))^j q_{i,j}$ for $1 \leq j \leq n_i$, $i = 1, 2$. We denote $n = \max\{n_0, n_1, n_2\}$. We take $\mu = \varphi(\lambda) \in \operatorname{Im} \varphi \setminus \{0\}$ such that $\varphi(\lambda)^j \neq 1$ for every $j = 1, \ldots, n$. $\psi(\lambda) \neq 0$, so $l_j = 0$, $q_{i,j} = 0$, hence

$\widetilde{l} = 0$, $\widetilde{q}_1 = \widetilde{q}_2 = 0$ and

$$m_1 \perp m_2 = r(m_1) + r(m_2) + p(m_1) + p(m_2) = m_1 + m_2. \tag{8.8}$$

We denote by $W^{\Psi^{-1}}$ the system of words which fulfills conditions Op1) and Op2) and corresponds to the automorphism Ψ^{-1}. By $w_\lambda^{\Psi^{-1}}(m)$ $(\lambda \in k)$ we denote the word from $W^{\Psi^{-1}}$ which corresponds to the operation of the multiplication by the scalar λ. We denote by $\lambda \underset{\Psi^{-1}}{*}$ the operation defined by word $w_\lambda^{\Psi^{-1}}(m)$. By (8.2), (8.3), (8.4) and (8.5) $w_\lambda^{\Psi^{-1}}(m) = \rho(\lambda) r(m) + \sigma(\lambda) p(m)$, where ρ, σ are monomorphisms of the field k. By (8.8) for w_+ we have only one possibility for every system of words which fulfills conditions Op1) and Op2): $w_+(m_1, m_2) = m_1 + m_2$.

By $\left\{s_F^{\Psi^{-1}}\right\}$ we denote the systems of bijections corresponding to automorphism Ψ^{-1}. $\Psi^{-1}\Psi = \Psi\Psi^{-1} = I$, where I is the identical automorphism. By consideration of the formula (7.1) we can conclude that to the automorphism $\Psi^{-1}\Psi$ corresponds the systems of bijections $\left\{s_F^{\Psi^{-1}} s_F^{\Psi} \mid F \in \mathrm{Ob}\Theta^0\right\}$. On the other side to the automorphism I corresponds the systems of bijections $\left\{id_F \mid F \in \mathrm{Ob}\Theta^0\right\}$. So, we have

$$s_{F(m)}^{\Psi^{-1}} s_{F(m)}^{\Psi}(\lambda m) = s_{F(m)}^{I}(\lambda m) = \lambda m = \lambda r(m) + \lambda p(m).$$

On the other side, by using of the formula (7.4),

$$\begin{gathered} s_{F(m)}^{\Psi^{-1}} s_{F(m)}^{\Psi}(\lambda m) = \\ = s_{F(m)}^{\Psi^{-1}}(\varphi(\lambda) r(m) + \psi(\lambda) p(m)) = \varphi(\lambda) \underset{\Psi^{-1}}{*} r(m) + \psi(\lambda) \underset{\Psi^{-1}}{*} p(m) = \\ = (\rho\varphi(\lambda) rr(m) + \sigma\varphi(\lambda) pr(m)) + (\rho\psi(\lambda) rp(m) + \sigma\psi(\lambda) pp(m)) = \\ = \rho\varphi(\lambda) r(m) + \sigma\psi(\lambda) p(m). \end{gathered}$$

Therefore $\rho\varphi = \sigma\psi = id_k$. Analogously $\varphi\rho = \psi\sigma = id_k$. Therefore $\varphi, \psi \in \mathrm{Aut}k$.

Now we consider the case when $\mathrm{Aut}k = \{id_k\}$. We denote by $\times$ the operation defined by the word $w_{[,]} \in F(m_1, m_2)$.

$$\begin{gathered} m_1 \times m_2 = u(r(m_1), r(m_2)) + \\ + t_1(r(m_1), r(m_2)) p(m_1) + t_2(r(m_1), r(m_2)) p(m_2), \end{gathered}$$

where $u \in L(x_1, x_2)$, $t_1, t_2 \in A(x_1, x_2)$. $(\lambda m_1) \times m_2 = \lambda(m_1 \times m_2)$ must fulfill for every $\lambda \in k$.

$$\lambda\left(m_1 \times m_2\right) = \lambda u\left(r\left(m_1\right), r\left(m_2\right)\right) + \lambda t_1\left(r\left(m_1\right), r\left(m_2\right)\right) p\left(m_1\right) + \\ + \lambda t_2\left(r\left(m_1\right), r\left(m_2\right)\right) p\left(m_2\right). \quad (8.9)$$

$$\left(\lambda m_1\right) \times m_2 = u\left(\lambda r\left(m_1\right), r\left(m_2\right)\right) + t_1\left(\lambda r\left(m_1\right), r\left(m_2\right)\right) \lambda p\left(m_1\right) + \\ + t_2\left(\lambda r\left(m_1\right), r\left(m_2\right)\right) p\left(m_2\right). \quad (8.10)$$

We decompose $u = u_0 + u_1 + \ldots + u_{s_0}$, $t_i = t_{i,0} + t_{i,1} + \ldots + t_{i,s_i}$, $i = 1, 2$ by homogeneous components according the degree of x_1. By comparison of (8.9) and (8.10) we have that $\lambda u_j = \lambda^j u_j$ for $0 \leq j \leq s_0$, $\lambda t_{1,j} = \lambda^{j+1} t_{1,j}$, for $0 \leq j \leq s_1$, $\lambda t_{2,j} = \lambda^j t_{2,j}$, for $0 \leq j \leq s_2$. We denote $s = \max\left\{s_0, s_1, s_2\right\}$. We take λ such that $\lambda^j \neq \lambda$ for $j = 0, 2, \ldots, s+1$ and conclude that $u = u_1$, $t_1 = t_{1,0}$, $t_2 = t_{2,1}$.

Also $m_1 \times \left(\lambda m_2\right) = \lambda\left(m_1 \times m_2\right)$ must fulfill for every $\lambda \in k$.

$$m_1 \times \left(\lambda m_2\right) = u_1\left(r\left(m_1\right), \lambda r\left(m_2\right)\right) + t_{1,0}\left(r\left(m_1\right), \lambda r\left(m_2\right)\right) p\left(m_1\right) + \\ + t_{2,1}\left(r\left(m_1\right), \lambda r\left(m_2\right)\right) \lambda p\left(m_2\right). \quad (8.11)$$

Now we decompose $u_1 = u_{1,0} + u_{1,1} + \ldots + u_{1,s_3}$, $t_{1,0} = t_{1,0,0} + t_{1,0,1} + \ldots + t_{1,0,s_4}$, $t_{2,1} = t_{2,1,0} + t_{2,1,1} + \ldots + t_{2,1,s_5}$, by homogeneous components according the degree of x_2. And by comparison of (8.9) and (8.11) as above we conclude that $u = u_1 = u_{1,1}$, $t_1 = t_{1,0} = t_{1,0,1}$, $t_2 = t_{2,1} = t_{2,1,0}$. Therefore by (4.2)

$$m_1 \times m_2 = \alpha\left[r\left(m_1\right), r\left(m_2\right)\right] + \beta\left[r\left(m_2\right), p\left(m_1\right)\right] + \gamma\left[r\left(m_1\right), p\left(m_2\right)\right],$$

where $\alpha, \beta, \gamma \in k$.

$m_1 \times m_2 = -m_2 \times m_1$ must fulfill.

$$m_2 \times m_1 = \alpha\left[r\left(m_2\right), r\left(m_1\right)\right] + \beta\left[r\left(m_1\right), p\left(m_2\right)\right] + \gamma\left[r\left(m_2\right), p\left(m_1\right)\right] =$$

$$-\alpha\left[r\left(m_1\right), r\left(m_2\right)\right] + \gamma\left[r\left(m_2\right), p\left(m_1\right)\right] + \beta\left[r\left(m_1\right), p\left(m_2\right)\right].$$

Therefore $\gamma = -\beta$ and

$$m_1 \times m_2 = \alpha\left[r\left(m_1\right), r\left(m_2\right)\right] + \beta\left[r\left(m_2\right), p\left(m_1\right)\right] - \beta\left[r\left(m_1\right), p\left(m_2\right)\right].$$

In the case 1 we assume that $\beta \neq 0$.

The Jacobi identity

$$J\left(m_1, m_2, m_3\right) = \left(m_1 \times m_2\right) \times m_3 + \left(m_2 \times m_3\right) \times m_1 + \left(m_3 \times m_1\right) \times m_2 = 0 \quad (8.12)$$

must fulfill in $F\left(m_1, m_2, m_3\right)$.

$$
\begin{gathered}
(m_1 \times m_2) \times m_3 = \\
= (\alpha [r(m_1), r(m_2)] + \beta [r(m_2), p(m_1)] - \beta [r(m_1), p(m_2)]) \times m_3 = \\
= \alpha [\alpha [r(m_1), r(m_2)], r(m_3)] + \\
+ \beta [r(m_3), \beta [r(m_2), p(m_1)] - \beta [r(m_1), p(m_2)]] - \\
- \beta [\alpha [r(m_1), r(m_2)], p(m_3)] = \\
= \alpha^2 [[r(m_1), r(m_2)], r(m_3)] + \beta^2 [r(m_3), [r(m_2), p(m_1)]] - \\
- \beta^2 [r(m_3), [r(m_1), p(m_2)]] - \beta\alpha [[r(m_1), r(m_2)], p(m_3)].
\end{gathered}
$$

$$
F(m_1, m_2, m_3) = L(r(m_1), r(m_2), r(m_3)) \oplus \oplus \left(\bigoplus_{i=1}^{3} A(r(m_1), r(m_2), r(m_3)) p(m_i) \right),
$$

so $J(m_1, m_2, m_3) = \sum_{i=0}^{3} J_i$, where

$$
J_0 \in L(r(m_1), r(m_2), r(m_3)),
$$

$$
J_i \in A(r(m_1), r(m_2), r(m_3)) p(m_i),
$$

$i = 1, 2, 3$ and must fulfill $J_i = 0$, $i = 0, \ldots, 3$.

$$
\begin{gathered}
J_1 = \beta^2 [r(m_3), [r(m_2), p(m_1)]] - \beta^2 [r(m_2), [r(m_3), p(m_1)]] - \\
- \beta\alpha [[r(m_2), r(m_3)], p(m_1)] = \\
= \beta^2 [r(m_3), [r(m_2), p(m_1)]] - \beta^2 [r(m_2), [r(m_3), p(m_1)]] - \\
- \beta\alpha [r(m_2), [r(m_3), p(m_1)]] + \beta\alpha [r(m_3), [r(m_2), p(m_1)]]
\end{gathered}
$$

by (4.2) and definition of representation of Lie algebra. So $\beta^2 + \beta\alpha = 0$ must fulfill and we have that $\beta = -\alpha$, $\alpha \neq 0$. It is easy to check that $\beta = -\alpha$ enough for (8.12). Therefore

$$
m_1 \times m_2 = \alpha([r(m_1), r(m_2)] + [r(m_1), p(m_2)] - - [r(m_2), p(m_1)]) = \alpha [m_1, m_2] \tag{8.13}
$$

by (4.1) and (4.2).

In the case 2, if $\beta = 0$ we have that

$$
m_1 \times m_2 = \alpha [r(m_1), r(m_2)]. \tag{8.14}
$$

If $\alpha = 0$, then $F(m_1, m_2) \times F(m_1, m_2) = \{0\}$, but $[F(m_1, m_2), F(m_1, m_2)] \neq \{0\}$. By condition Op2) $F(m_1, m_2) \cong (F(m_1, m_2))_W^*$. From this contradiction we conclude that hear also $\alpha \neq 0$.

We denote by $\mathfrak{p}$ the operation defined by the word $w_p \in F(m)$.

$$\mathfrak{p}(m) = \delta r(m) + q_{\mathfrak{p}}(r(m))\, p(m),$$

where $\delta \in k$, $q_{\mathfrak{p}} \in k[x]$. $\mathfrak{p}(\lambda * m) = \lambda * \mathfrak{p}(m)$ must fulfill for every $\lambda \in k$.

$$\lambda * \mathfrak{p}(m) = \lambda\delta r(m) + \lambda q_{\mathfrak{p}}(r(m))\, p(m). \tag{8.15}$$

$$\mathfrak{p}(\lambda * m) = \delta r(\lambda * m) + q_{\mathfrak{p}}(r(\lambda * m))\, p(\lambda * m) = \\ = \delta\lambda r(m) + q_{\mathfrak{p}}(\lambda r(m))\, \lambda p(m). \tag{8.16}$$

As above we decompose $q_{\mathfrak{p}}$ by homogeneous components according the degree of x and conclude as above by comparison of (8.15) and (8.16) that $\deg q_{\mathfrak{p}} = 0$ and

$$\mathfrak{p}(m) = \delta r(m) + \varepsilon p(m)$$

where $\varepsilon \in k$.

$\mathfrak{p}(\mathfrak{p}(m)) = \mathfrak{p}(m)$ must fulfill in $F(m)$.

$$\mathfrak{p}(\mathfrak{p}(m)) = \delta r(\delta r(m) + \varepsilon p(m)) + \varepsilon p(\delta r(m) + \varepsilon p(m)) = \delta^2 r(m) + \varepsilon^2 p(m).$$

Therefore $\delta^2 = \delta$, $\varepsilon^2 = \varepsilon$.

If $\delta = \varepsilon = 1$ then $\mathfrak{p}(m) = r(m) + p(m) = m$ and $\mathfrak{p}((F(m))_W^*) = (F(m))_W^*$ but $p(F(m)) \neq (F(m))$ contrary to $F(m) \cong (F(m))_W^*$. So it is impossible that $\delta = \varepsilon = 1$.

If $\delta = \varepsilon = 0$, then $\mathfrak{p}(m) = 0$ and $\mathfrak{p}((F(m))_W^*) = 0$ but $p(F(m)) \neq 0$ contrary to $F(m) \cong (F(m))_W^*$. As above we conclude that $\delta = \varepsilon = 0$ is impossible.

If $\delta = 1$, $\varepsilon = 0$. Then $\mathfrak{p}(m) = r(m)$, i.e. $\mathfrak{p} = r$. $\mathfrak{p}$ must be a derivation of $(F(m))_W^*$. In the case 2, by (8.14), we have that

$$\mathfrak{p}(m_1 \times m_2) = r(\alpha[r(m_1), r(m_2)]) = \alpha[r(m_1), r(m_2)],$$

$$\mathfrak{p}(m_1) \times m_2 + m_1 \times \mathfrak{p}(m_2) = r(m_1) \times m_2 + m_1 \times r(m_2) = \\ = \alpha[rr(m_1), r(m_2)] + \alpha[r(m_1), rr(m_2)] = 2\alpha[r(m_1), r(m_2)].$$

$char\,(k) = 0$, so $\mathfrak{p}$ is not a derivation. In the case 1, by (8.13), we have that

$$\mathfrak{p}(m_1 \times m_2) = r(\alpha([r(m_1), r(m_2)] + [r(m_1), p(m_2)] - \\ - [r(m_2), p(m_1)])) = \alpha[r(m_1), r(m_2)],$$

$$\mathfrak{p}(m_1) \times m_2 + m_1 \times \mathfrak{p}(m_2) = r(m_1) \times m_2 + m_1 \times r(m_2) = \\ = \alpha([r(r(m_1)), r(m_2)] + [r(r(m_1)), p(m_2)] - [r(m_2), p(r(m_1))])+ \\ + \alpha([r(m_1), r(r(m_2))] + [r(m_1), p(r(m_2))] - [r(r(m_2)), p(m_1)]) = \\ = \alpha([r(m_1), r(m_2)] + [r(m_1), p(m_2)]) + \alpha([r(m_1), r(m_2)] - [r(m_2), p(m_1)]) = \\ = \alpha(2[r(m_1), r(m_2)] + [r(m_1), p(m_2)] - [r(m_2), p(m_1)]).$$

In this case $\mathfrak{p}$ also is not a derivation.

Therefore we have only one possibility: $\delta = 0$, $\varepsilon = 1$. It means $\mathfrak{p}\,(m) = p\,(m)$, i.e., $\mathfrak{p} = p$.

And in the case 2, by (8.14), we have that

$$p\,(F\,(m_1, m_2) \times F\,(m_1, m_2)) = 0$$

but

$$p\,[r\,(m_1)\,, p\,(m_2)] = [r\,(m_1)\,, p\,(m_2)] \neq 0,$$

so

$$p\,[F\,(m_1, m_2)\,, F\,(m_1, m_2)] \neq 0$$

contrary to $F\,(m_1, m_2) \cong (F\,(m_1, m_2))^*_W$. Therefore the case 2 is impossible.

Hence

$$m_1 \times m_2 = \alpha\,([r\,(m_1)\,, r\,(m_2)] + [r\,(m_1)\,, p\,(m_2)] - \\ - [r\,(m_2)\,, p\,(m_1)]) = \alpha\,[m_1, m_2]\,,$$

where $\alpha \neq 0$.

From this fact, as in [4, end of the subsection 2.5], we conclude that $\Psi \in \mathfrak{Y}$. So $\mathfrak{S} = \mathfrak{S} \cap \mathfrak{Y}$ and $\mathfrak{A}/\mathfrak{Y} = \{1\}$. □

9. The main theorem

Theorem 9.1. *If* $\mathrm{Aut}k = \{id_k\}$ *then automorphic equivalence of representations of Lie algebras coincides with the geometric equivalence.*

Proof. We assume that $H_1 = (L_1, V_1), H_2 = (L_2, V_2) \in \Xi$ are automorphically equivalent. By Theorem 6.1 we have that $N_1 = \mathcal{F}(H_1), N_2 = \mathcal{F}(H_2)$ are automorphically equivalent. By [1, Proposition 9] and Theorem 8.1 we can conclude from this fact that N_1, N_2 are geometrically equivalent. It means that $Cl_{N_1}(F) = Cl_{N_2}(F)$ for every $F \in \mathrm{Ob}\Theta^0$.

We will consider the arbitrary $W_1 = (L(X_1), A(X_1)Y_1) \in \mathrm{Ob}\Xi^0$. There are $X_2 \subset X^0$, $Y_2 \subset Y^0$ such that $X_1 \subseteq X_2$, $Y_1 \subseteq Y_2$ and $W_2 = (L(X_2), A(X_2)Y_2) \in \Xi'$. By Theorem 4.2 there exists $F \in \mathrm{Ob}\Theta^0$ such that $F = \mathcal{F}(W_2)$. By Proposition 6.1 we can conclude from $Cl_{N_1}(F) = Cl_{N_2}(F)$ that $Cl_{H_1}(W_2) = Cl_{H_2}(W_2)$. And by Theorem 3.1 we can conclude that $Cl_{H_1}(W_1) = Cl_{H_2}(W_1)$. So H_1 and H_2 are geometrically equivalent. □

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