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# Associative words in the symmetric group of degree three

**Ernest Płonka**

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ABSTRACT. Let G be a group. An element $w(x,y)$ of the absolutely free group on free generators $x, y$ is called an associative word in $G$ if the equality $w(w(g_1,g_2),g_3) = w(g_1,w(g_2,g_3))$ holds for all $g_1, g_2 \in G$. In this paper we determine all associative words in the symmetric group on three letters.

## 1. Introduction

Let $G$ be a group and let $F = F(x,y)$ be the absolutely free group on free generators $x, y$. Let $V = V(G)$ be the subgroup of $F$ consisting of all words $w$ such that $w(g_1,g_2) = 1$ for all $g_1, g_2 \in G$. An element $w \in F$ is said to be *associative* in $G$ if the equality

$$w(w(g_1,g_2),g_3) = w(g_1,w(g_2,g_3)) \tag{1.1}$$

holds for all elements $g_1, g_2, g_3 \in G$. The words $1, x, y, xy$ and $yx$ are, of course, associative (trivial words) for any group. It is known that in the absolutely free group ([6,7]) and in the class of all abelian groups ([4]) there are no other associative words. In other groups $(G;\cdot)$ such nontrivial word $w$ can exist, however. Moreover, in some free nilpotent groups there are nontrivial associative words $w(x,y) = x \circ y$ such that $(G;\circ)$ is a group and the group operation $x \cdot y$ can be expressed as a

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word in the group $(G; \circ)$: see [1,2,5,9]. In this paper we are looking for the associative words in the symmetric group of degree three which is metabelian but not nilpotent. We show that each associative word in $S_3$ is equivalent *modulo* $V(S_3)$ to one of the five words mentioned above or to one of $x^3, y^3, x^4, y^4, x^4y^4, [x,y]^{x+y}$, $[y,x]^{x+y}$.

## 2. Preliminaries

We use standard notations:

$$x^{-1}yx = y^x, \qquad [y,x] = y^{-1}x^{-1}yx, \qquad [y,x]^{-1} = [x,y] \qquad x^{\beta y} = (x^y)^\beta,$$

$$x^{\alpha+\beta y+z+0} = x^\alpha (x^\beta)^y x^z$$

for arbitrary group elements $x, y, z$ and all integers $\alpha, \beta$.

Let us recall the following simple facts about the identities in $S_3$.

(i) *The relations*

$$[xy, z] = [x,z]^y[y,z], [x, yz] = [x,z][x,y]^z$$

*are identities in any group.*

(ii) *The commutator subgroup $S_3'$ of $S_3$ consists of all even permutations and the square of each element from $S_3$ is in $S_3'$.*

This yields

(iii) *For all products $C$ of commutators the equality*

$$C^{(1-x)(1+x)} = 1$$

*is an identity in $S_3$.*

(iv) *The equalities*

$$x^6 = [y,x]^3 = 1, [[y,x],[u,v]] = 1, [x^2,[y,z]] = 1,$$
$$[y^2, x] = [y,x]^{y+1}, [y^3, x] = [y,x]^{y-1}, [y^4, x] = [y,x]^{-y-1}$$

*are identities in the group $S_3$.*

From (ii) and (iv) one can derived

(v) *The equality*

$$[y,x]^{xy} = [y,x]^{-1-x-y}$$

*is an identity in $S_3$.*

The following consequence of Corollary 2 in [8] plays very important role in our considerations.

(vi) *If for some* $A, B, C \in Z_3$ *the equality*

$$[y, x]^{A+Bx+Cy} = 1,$$

*holds for all* $x, y \in S_3$*, then* $A = B = C = 0$*.*

**Proposition 2.1.** *Any* $2-$*word in* $S_3$ *is equivalent* $\pmod V$ *to some word of the form*

$$w(x, y) = x^{\alpha} y^{\beta} [y, x]^{A+Bx+Cy} \tag{2.1}$$

*where* $\alpha, \beta \in Z_6$ *and* $A, B, C \in Z_3$*.*

*Proof.* It is enough to apply Hall's classical collection process from [3]. □

**Proposition 2.2.** *The word*

$$w(x, y) = x^{\alpha} y^{\beta} [y, x]^{A+Bx+Cy}$$

*is asociative in* $S_3$*, then* $\alpha, \beta \in \{0, 1, 3, 4\}$*.*

*Proof.* By putting $y = z = 1$ and $x = y = 1$ into (2.2) we get

$$x^{\alpha^2} = x^{\alpha}, \;\; and \;\; x^{\beta^2} = x^{\beta} \tag{2.2}$$

and therefore $\alpha(\alpha - 1) \equiv \beta(\beta - 1) \equiv 0 \pmod 6$. □

**Proposition 2.3.** *If* $w(x, y)$ *is associative in a group* $G$*, then the word* $u(x, y) = w(y, x)$ *is also associative in* $G$*.*

*Proof.* We have

$$u(u(x, y), z) = w(z, w(y, x)) = w(w(z, y), x) = u(x, u(y, z)).$$ □

## 3. Associative words

First of all we show that for some pairs $(\alpha, \beta)$ no word of the form (2.1) is associative in $S_3$. It what follows we shall always assume that $A, B, C \in Z_3$ and sometimes we write $\gamma(s, t)$ instead of $A + Bs + Ct$.

**Theorem 3.1.** *There are no associative words in the group* $S_3$ *which are of the form*

$$x^{\alpha} y^{\beta} [y, x]^{A+Bx+Cy}, \tag{3.1}$$

*where* $(\alpha, \beta)$ *is one of the following pairs*

$$(1, 3), (3, 1), (1, 4), (4, 1), (3, 4), (4, 3), (3, 3)$$

*Proof.* Case $\alpha = 1, \beta = 3$.
Let us begin with an auxiliary result

$$\begin{aligned} w(x,y)^3 &= xy^3[y,x]^\gamma xy^3[y,x]^\gamma xy^3[y,x]^\gamma \\ &= xy^3x^2[y,x]^\gamma y^3[y^3,x][y,x]^{x\gamma}y^3[y,x]^\gamma \\ &= xy^3x^2[y,x]^\gamma[y^3,x]^y[y,x]^{xy\gamma}[y,x]^\gamma \\ &= xy^3x^3[y,x]^{x+1}[y,x]^\gamma[y^3,x]^{(y-1)y}[y,x]^{xy\gamma}[y,x]^\gamma \\ &= xy^3x^3[y,x]^{x+1}[y,x]^\gamma[y^3,x]^{(y-1)y}[y,x]^{xy\gamma}[y,x]^\gamma. \end{aligned}$$

We have thus established

$$(xy^3[y,x]^\gamma)^3 = x^3y^3[y,x]^{-1+x-y+(1-x-y)\gamma}. \tag{3.2}$$

Further we have

$$\begin{aligned} L = w(w(1,y),z) = w(y^3,z) = y^3z^3[z,y^3]^{\gamma(y,z)} = y^3z^3[z,y]^{(y-1)\gamma(y,z)}, \\ R = w(1,w(y,z)) = w(y,z)^3 = (yz^3[z,y]^{\gamma(y,z)})^3 \\ = yz^3(z^3y[y,z^3])yz^3[z,y]^{(yz-1)\gamma(y,z)} = y^3z^3[z,y]^{-1-z+y+(yz-1)\gamma(y,z)}. \end{aligned}$$

Thus $L = R$ is equivalent to the equality

$$[z,y]^{(-1-A-B+C)+(1+A-C)y+(-1-A+B+C))z} = 1.$$

By (vi) we have $-1-A-B+C = 0, 1+A-C = 0$ and $-1-A+B+C = 0$, which has the solution $B = 0, C = A+1$. Therefore the associative word (3.1) has to be of the form $w(x,y) = xy^3[y,x]^{C-1+Cy}$. Let us put $z = x$ into the associative low (1.1). We get

$$\begin{aligned} L = w(w(x,y),x) = xy^3[y,x]^{\gamma(x,y)}w(x,y)x^3[x,xy^3[y,x]^{\gamma(x,y)}]^{\gamma(xy,x)} \\ x^4y^3[y,x]^{x+y}[y,x]^{x\gamma(x,y)}[y,x]^{(1-x+y)\gamma(xy,x)}[y,x]^{(1-x)\gamma(x,y)\gamma(xy,x)}, \end{aligned}$$

which in the case $B = 0, A = C-1$ gives

$$\begin{aligned} L = x^4y^3[y,x]^{x+y+x(C-1+Cy)+(1-x+y)(C-1+Cx)+(1-x)((C+Cx)-1)(C-1+Cy)} \\ x^4y^3[y,x]^{x+y+x(C-1+Cy)+(1-x+y)(C-1+Cx)+(x-1)(C-1+Cy)}. \end{aligned}$$

After some calculations we get

$$L = x^4y^3[y,x]^{C+C(x+1)y}.$$

Similarly we have

$$\begin{aligned} R = w(x, w(y, x)) &= xw(y, x)^3 [yx^3 [x, y]^{\gamma(y,x)}, x]^{\gamma(x,xy)} \\ &= xw(y, x)^3 [yx^3, x]^{\gamma(x,xy)} [[x, y]^{\gamma(y,x)}, x]^{\gamma(x,xy)} \\ &= xw(y, x)^3 [y, x]^{x\gamma(x,xy)} [y, x]^{(1-x)\gamma(y,x)\gamma(x,xy)}, \end{aligned}$$

which in the case $B = 0$ and $A = C - 1$ implies

$$\begin{aligned} R &= xw(y, x)^3 [y, x]^{x(C-1+Cxy} [y, x]^{(1-x)(C+Cx-1)(C+1+Cxy)} \\ &= xw(y, x)^3 [y, x]^{-C+Cx+(1+C)y}. \end{aligned}$$

Now from the equality $L = R$ we obtain

$$w(x, y)^3 = x^3 y^3 [y, x]^{-x}.$$

We get a contradiction, because formula (3.2) for $\gamma = C - 1 + Cy$ gives

$$w(x, y)^3 = x^3 y^3 [y, x]^{C+1-x+(C+1)y}.$$

Case $\alpha = 1, \beta = 4$.
We have

$$L = w(w(1, y), z) = w(y^4, z) = y^4 z^4 [z, y^4]^{\gamma(1,z)} = y^4 z^4 [z, y]^{-(y+1)\gamma(1,z)}.$$

Similarly

$$\begin{aligned} R = w(1, w(y, z)) &= (yz^4 [z, y]^{y\gamma(y,z)} yz^4 [z, y]^{\gamma(y,z)})^2 \\ = (y^2 z^4 [z^4, y][z, y]^{y\gamma(y,z)} yz^4 [z, y]^{\gamma(y,z)})^2 &= y^4 z^4 [z, y]^{(z+1)-(y+1)\gamma(y,z)}. \end{aligned}$$

Hence $L = R$ yields $[z, y]^{1+z} = 1$ which, by (vi), is not an identity in $S_3$ .
Case $\alpha = 3, \beta = 4$.
We have

$$L = w(w(1, y), z) = w(y^4, z) = z^4 [z, y^4]^{\gamma(1,z)} = z^4 [z, y]^{-(y+1)\gamma(1,z)}.$$

Since $y^2$ commutes both $z^4$ and $[z, y]^{\gamma(y,z)}$, we can use of the previous case. We obtain

$$\begin{aligned} R = w(1, w(y, z)) = w(y, z)^4 = (y^2 yz^4 [z, y]^{\gamma(y,z)} y^2 yz^4 [z, y]^{\gamma(y,z)})^2 = \\ = y^2 (yz^4 [z, y]^{\gamma(y,z)} yz^4 [z, y]^{\gamma(y,z)})^2 = z^4 [z, y]^{-(z+1)-(y+1)\gamma(y,z)}. \end{aligned}$$

Thus the condition $L = R$ yields the equality

$$[z, y]^{z+1} = 1,$$

which is not an identity in $S_3$.

Case $\alpha = 3, \beta = 3$. We have

$$L = w(w(x, 1), z) = x^3 z^3 [z, x^3]^{\gamma(x,z)} = x^3 z^3 [z, x]^{(x-1)(A+Bx+Cz)},$$
$$R = w(x, w(1, z)) = x^3 z^3 [z^3, x]^{\gamma(x,z)} = x^3 z^3 [z, x]^{(z-1)(A+Bx+Cz)}.$$

Thus the equality $L = R$ implies, in view of (vi),

$$C - B \equiv A - C + B \equiv B - A - C \equiv 0 \pmod 3$$

which yields $A = 0$ and $B - C = 0$. So every word of the form $w(x, y) = x^3 y^3 [x, y]^{B(x+y)}$ satisfies the equation $w(w(x, 1), z) = w(x, w(1, z))$ but none of them is associative. Indeed,for such words we have

$$\begin{aligned} L = w(w(1, y), z) = w(y^3, z) &= y^3 z^3 [z^3, y^3]^{B(y+z)} \\ &= y^3 z^3 [z, y]^{B(y-1)(z-1)(y+z)} = y^3 z^3, \\ R = w(1, w(y, z)) &= w(y, z)^3 \\ = y^3 z^3 [z, y]^{B(y+z)} y^3 z^3 &[z, y]^{B(y+z)} y^3 z^3 [z, y]^{B(y+z)} \\ = y^3 z^3 [z, y]^{B(y+z)} &y^3 z^3 y^3 z^3 [z, y]^{B(yz+1)(y+z)} \\ = y^3 z^3 [z, y]^{B(y+z)} &[z, y]^{(z-1)(y-1)} [z, y]^{-B(y+z)} \\ &= y^3 z^3 [z, y]^{(B+1)(y+z)}. \end{aligned}$$

Thus $L = R$ implies the equation $[z, y]^{y+z} = 1$, which is not an identity in $S_3$.

Now by Proposition 2.3 we know that if the word $w(x, y)$ of the form (2.1) is associative in $S_3$, then $w(y, x)$ is also associative in $S_3$. Since

$$w(y, x) = y^i x^j [x, y]^{A+Bx+Cz} = x^j y^i [y, x]^{A'+B'x+C'y}$$

for some $A', B', C' \in Z_3$ the proof of Theorem 3.1 is complete. □

In the following lemmas we consider the cases of pairs $(\alpha, \beta)$ for which there exist associative words in $S_3$.

**Lemma 3.2.** *The word*

$$w(x, y) = x[y, x]^{A+Bx+Cy} = x[y, x]^{\gamma(x,y)} \tag{3.3}$$

*is associative in $S_3$ if and only if $A = B = C = 0$.*

*Proof.* Using the identities (ii), (iv) and (v) we have

$$\begin{aligned} w(x, w(y,y)) &= x[y,x]^{\gamma(x,y)}, \\ w(w(x,y),y) &= x[y,x]^{\gamma(x,y)}[y, x[y,x]^{\gamma(x,y)}]^{\gamma(x,y)} \\ &= x[y,x]^{\gamma(x,y)}[y,x]^{\gamma(x,y)}[y,x]^{(1-y)\gamma(x,y)\gamma(x,y)}. \end{aligned}$$

Taking into account (iii) we see that if $w$ is associative, then

$$[y,x]^{A+Bx+Cy+(1-y)(A-C+Bx)^2} = 1,$$

which, by (vi) ensures the following system of congruences

$$\begin{cases} A + (A-C)^2 + B^2 + 2(A-C)B \equiv 0 \pmod 3, \\ B + 2(A-C)B + 2(A-C)B \equiv 0 \pmod 3, \\ C - (A-C)^2 - B^2 + 2(A-C)B \equiv 0 \pmod 3. \end{cases}$$

The solution of the system are four triples $(A, B, C)$ of the form $(0,0,0)$, $(2,2,0)$, $(2,0,1)$ and $(0,1,1)$. In order to exclude the last three cases we put $y = x$ into (3.3). Then we get

$$\begin{aligned} L &= w(w(x,x),z) = x[z,x]^{\gamma(x,z)} \\ R &= w(x, w(x,z)) = x[x[z,x]^{\gamma(x,z)}, x]^{\gamma(x,x)} \\ &= x[z,x]^{(x-1)\gamma(x,x)\gamma(x,z)}. \end{aligned}$$

Thus the condition $L = R$ together with (iii) gives the equality

$$[z,x]^{(x-1)(A-B-C)(A+Bx+Cz)} = [z,x]^{A+Bx+Cz}.$$

The equality is, by (vi), an identity in $S_3$ if and only if the triples $(A, B, C)$ satisfies the following system of congruences

$$\begin{cases} (A-B-C)(B-A-C) \equiv A \pmod 3, \\ (A-B-C)(A-B-C) \equiv B \pmod 3, \\ (A-B-C)(B-A-C) \equiv C \pmod 3. \end{cases}$$

The proof of the lemma is complete, because none of the triples $(2,2,0)$, $(2,0,1)$ and $(0,1,1)$ do satisfy the system. □

By Proposition 2.3 we have also

**Corollary 3.3.** *The word*

$$y[y,x]^{A+Bx+Cy}$$

*satisfies the associativity low if and only if* $A = B = C = 0$.

**Lemma 3.4.** *The word*

$$w(x,y) = xy[y,x]^{A+Bx+Cy}$$

*is associative in* $S_3$ *if and only if* $B = C = A = 0$ *or* $A - 1 = B = C = 0$.

*Proof.* We have

$$\begin{aligned} w(w(x,y),z) &= xy[y,x]^{\gamma(x,y)}z[z,xy[y,x]^{\gamma(x,y)}]^{\gamma(xy,z)} \\ &= xyz[y,x]^{z\gamma(x,y)+(z-1)\gamma(x,y)\gamma(xy,z)}[z,x]^{y\gamma(xy,z)}[z,y]^{\gamma(xy,z)} \end{aligned}$$

and

$$\begin{aligned} w(x,w(y,z)) &= xyz[z,y]^{\gamma(y,z)}[yz[z,y]^{\gamma(y,z)},x]^{\gamma(x,yz)} \\ &= xyz[y,x]^{z\gamma(x,yz)}[z,x]^{\gamma(x,yz)}[z,y]^{\gamma(y,z)+(x-1)\gamma(y,z)\gamma(xy,z)}. \end{aligned}$$

Hence we get

$$\begin{aligned} &(w(x,w(y,z)))^{-1}w(w(x,y),z) \\ &= [y,x]^{(1-z)\{-Cy+\gamma(x,y)\gamma(xy,z)\}}[z,x]^{(1-y)(-A)}[z,y]^{(1-x)\{-By+\gamma(y,z)\gamma(xy,z)\}} \end{aligned} \tag{3.4}$$

By putting $z = y$ into (3.4) we obtain

$$\begin{aligned} &(w(x,w(y,y)))^{-1}w(w(x,y),y) = \\ &[y,x]^{(1-y)\{-A-Cy+(A+Bx+Cy)(A+Bxy+Cy)\}}, \end{aligned}$$

which in view of (iii) and (v) can be rewritten as

$$[y,x]^{(1-y)\{(A-C)^2-(A-C)-B^2)\}}.$$

Now we put $y = x$ into (3.4). This gives

$$\begin{aligned} &w(x,w(x,z))^{-1}w(w(x,x),z) = \\ &[z,x]^{(1-x)\{(A-B)^2-(A-B)-C^2\}\}} \end{aligned}$$

In view of (vi) if the word $w(x,y)$ is associative in $S_3$, then the following system of congruences

$$\begin{cases}(A-C)^2-(A-C)-B^2\equiv 0 \pmod 3,\\(A-B)^2-(A-B)-C^2\equiv 0 \pmod 3).\end{cases}$$

has to satisfy. The solution of the system is $B=C=0$ and $A=0$ or $A=1$. Since the words $xy$ and $yx$ are associative, Lemma 3.4 follows. □

**Lemma 3.5.** *The* $2-$*word*

$$w(x,y)=x^3[y,x]^{A+Bx+Cy} \tag{3.5}$$

*is associative in* $S_3$ *if and only if* $A=B=C=0$.

*Proof.* Clearly, the word $x^3$ is associative in the group $S_3$. We have

$$R=w(x,w(1,z))=x^3,$$
$$L=w(w(x,1)z)=w(x^3,z)=x^3[z,x]^{\gamma(x,z)}=x^3[z,x]^{(x-1)(A+Bx+Cz)}$$
$$[z,x]^{(-A+B-C)+(A-B-C)x+Cz}.$$

So the equality $R=L$ is equivalent to the conditions $C=0$ and $A=B$.

Further we have

$$w(w(x,x),z)=x^3[z,x^3]^{A+Bx+Cz}=x^3[z,x]^{(x-1)(A+Bx+Cz)},$$
$$w(x,w(x,z))=x^3[x^3[z,x]^{A+Bx+Cz},x]^{A+Bx+Cx}$$
$$=x^3[z,x]^{(A-B-C)(x-1)(A-B+Cz)}$$

Hence the equality $w(w(x,x),z)=w(x,w(x,z))$ after using (v) and (vi), yields the system of equalities

$$\begin{cases}(A-B-C)(B-A-C)\equiv 2A+B-C \pmod 3,\\(A-B-C)^2\equiv A-B-C \pmod 3,\\C(A-B-C)\equiv C \pmod 3).\end{cases}$$

The system has four solutions for $(A,B,C)$:(0,0,0),(1,0,0),(1,1,0) and (2,2,0). We check that the last three triple do not produce associative words of the form $w(x,y)=x^3[y,x]^\gamma(x,y)$. To do this let us calculate

$$w(x,y)^3=x^3[y,x]^{\gamma(x,y)}(x^3[y,x]^{\gamma(x,y)}x^3)[y,x]^{\gamma(x,y)}$$
$$=x^3[y,x]^{\gamma(x,y)}[y,x]^{x\gamma(x,y)}[y,x]^{\gamma(x,y)}=x^3[y,x]^{(x-1)\gamma(x,y)}$$

Taking this into account we get

$$L(A,B,C) = w(w(x,y),y) = w(x,y)^3[y, w(x,y)^{\gamma(x,y)} = \\ x^3[y,x]^{(x-1)\gamma(x,y)+(1-y)\gamma(x,y)\gamma(x,y)},$$
$$R(A,B,C) = w((x, w(y,y)) = x^3[y^3,x]^{\gamma(x,y)} = x^3[y,x]^{(y-1)\gamma(x,y)}$$

Now it easy to check the following equalities

$$L(1,0,0) = x^3[y,x]^{x-y}, R(1,0,0) = x^3[y,x]^{y-1}$$
$$L(1,1,0) = x^3[y,x]^{x+y}, R(1,1,0) = x^3[y,x]^{x+1}$$
$$L(2,2,0) = x^3[y,x]^{-x-y}, R(2,2,0) = x^3[y,x]^{-x-1}.$$

The proof is thus complete. □

**Lemma 3.6.** *The* $2-$*word*

$$w(x,y) = x^4[y,x]^{A+Bx+Cy}$$

*is associative in* $S_3$ *if and only if* $A = B = C = 0$.

*Proof.* We put $z = 1$ into the associativity law and we make use of the formulas (i), (ii), (iii) and (iv). We have

$$L = w(w(x,y),1) = w(x,y)^4 = (x^4[y,x]^{A+Bx+Cy}$$
$$R = w(x, w(y,1)) = w(x,y^4) = x^4[y^4,x]^{\gamma(x,y)} = x^4[y,x]^{-(y+1)\gamma(x,y)}$$
$$= x^4[y,x]^{(B-A-C)+y(B-A-C)}.$$

Therefore the equality $L = R$ ensures $B = 0$ and $A = C$. Taking this into account we get

$$w(w(x,x),z) = w(x^4,z) = x^4[z,x^4]^{\gamma(1,z)} = x^4[z,x]^{-(x+1)(A+Az)} = x^4$$
$$w(x, w(x,z)) = x^4[x^4[z,x]^{\gamma(x,z)}, x]^{\gamma(1,z)} = x^4[z,x]^{A(1-x)},$$

which shows that $A = B = C = 0$ and Lemma 3.6 follows. □

**Lemma 3.7.** *The word*

$$w(x,y) = x^4y^4[y,x]^{A+Bx+Cy}$$

*is associative in* $S_3$ *if and only if* $A = B = C = 0$.

*Proof.* We have

$$L = w(w(1,y),z) = w(y^4,z) = y^4z^4[z,y^4]^{\gamma(1,z)}$$
$$= y^4z^4[z,y]^{-(y+1)(A+B+Cz)} = y^4z^4[z,y]^{(C-A-B)+(C-A-B)y}$$

and

$$R = w(1,w(y,z)) = w(y,z)^4 = y^4z^4[z,y]^{A+By+Cz}.$$

Hence $C = 0$ and $A = B$. Taking this into account we check

$$L = w(w(x,y),x) = x^2y^4[y,x]^{\gamma(x,y)}[x,x^4y^4[y,x]^{\gamma(x,y)}]^{\gamma(1,x)}$$
$$= x^2y^4[y,x]^{\gamma(x,y)}[y,x]^{(y+1)A(x+1))}[y,x]^{(1-x)A(x+1)\gamma(x,y)}$$
$$= x^2y^4[y,x]^{A(1+x)}$$

and also

$$R = w(x,w(y,x)) = x^4y^4x^4[x,y]^{\gamma(y,x)}[y^4x^4[x,y]^{\gamma(y,x)},x]^{\gamma(x,1)}$$
$$x^4y^4x^4[x,y]^{\gamma(y,x)}[y,x]^{-(y-1)\gamma(x,1)}[x,y]^{(x-1)\gamma(x,1)\gamma(x,y)}$$
$$= x^2y^4[y,x]^{A(1+y)}.$$

By (vi) $L = R$ if and only if $A = 0$. Clearly, $x^4y^4$ is associative word in $S_3$. The proof is thus completed. □

**Lemma 3.8.** *If the word*

$$w(x,y) = [y,x]^{A+Bx+Cy} = [y,x]^{\gamma(x,y)}$$

*is associative, then $A = B - C = 0$. Conversely, the word*

$$w(x,y) = [y,x]^{B(x+y)} \tag{3.6}$$

*satisfies the associativity law for all $B \in Z_3$.*

*Proof.* Using the identities (i), (ii),(ii) and (iv) we have

$$L = w(w(x,y),z)) = [z,[y,x]^{\gamma(x,y)}]^{\gamma(1,z)} = [y,x]^{(1-z)(A+Bx+Cy)(A+B+Cz)}$$

and similarly

$$R = w(x,w(y,z)) = [w(y,z),x]^{\gamma(x,1)}$$
$$= [z,y]^{(x-1)(A+By+Cz)(A+Bx+C)}.$$

Thus if $w$ is an associative word in $S_3$, then in the case $y = x$, we get

$$[z, x]^{(x-1)(A-B+C)(A-B+Cz)} = 1, \tag{3.7}$$

because of (iii) and (v). Similarly, in the case $z = y$ we obtain the equation

$$[y, x]^{(1-y)(A+B-C)(A-C+Bx)} = 1. \tag{3.8}$$

Now (3.7), (3.8) and (vi) imply the system of congruences

$$\begin{cases} (A + C - B)^2 \equiv 0 \pmod 3, \\ (A + C - B)(A - B - C) \equiv 0 \pmod 3, \\ (A + C - B)C \equiv 0 \pmod 3, \\ (A + B - C)^2 \equiv 0 \pmod 3, \\ (A + B - C)B \equiv 0 \pmod 3, \\ (A + B - C)(B + C - A) \equiv 0 \pmod 3, \end{cases}$$

which have the solution $A = B - C = 0$.

Conversely, we check that the word $w(x, y) = [y, x]^{Bx+By}$ is associative. Indeed, by (ii) and (iii) we have

$$w(w(x, y), z) = [z, [y, x]^{B(x+y)}]^{B(1+z)} = [y, x]^{B^2(1-z)(1+z)(x+y)} = 1$$

and

$$w(x, w(y, z)) = [[z, y]^{B(z+y)}, x]^{B(1+x)} = [z, y]^{B^2(y+z)(x-1)(x+1)} = 1,$$

as required. □

We have thus established our main result

**Theorem 3.9.** *There are precisely (modulo $V(S_3)$) twelve associative words in the group $S_3$. Namely $1, x, x^3, x^4, y, y^3, y^4, xy, yx, x^4y^4, [y, x]^{x+y}$ and $[x, y]^{x+y}$.*

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## Contact information

**Ernest Płonka**

Institute of Mathematics, Silesian University of Technology, ul. Kaszubska 23, 44-100 Gliwice, Poland

*E-Mail:* `eplonka@polsl.pl`

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