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RESEARCH ARTICLE

On elementary domains of partial projective representations of groups

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Abstract. We characterize the finite groups containing only elementary domains of factor sets of partial projective representations. A condition for a finite subset A of a group G, which contains the unity of the group, to induce an elementary partial representation of G whose (idempotent) factor set is total is given. Finally, we characterize the elementary partial representation of abelian groups of degrees ≤ 4 with total factor set.

Introduction

Elementary partial representations were introduced and studied by M. Dokuchaev and N. Zhukavets in [6]. It turned out that together with the irreducible (indecomposable) representations they are elementary blocks from which the irreducible (indecomposable) partial representations can be constructed. In particular, given a finite group G and a field K, whose characteristic does not divide the order of G, every partial K-representation of G can be obtained from the elementary partial representations of G and the (usual) irreducible representations of subgroups of G.

Any elementary partial representation Γ is considered as a partial projective representation with idempotent factor set σ, and the domain of

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such a σ is called *elementary domain*. It is known (see [3]), that for any finite group G the domain of any partial factor set of G can be constructed by taking unions of some elementary domains and, conversely, the union of any collection of elementary domains is a domain for the factor set of some partial projective representation of G. The main purpose of this article is to characterize the finite groups which have only elementary domains.

This paper is structured as follows. After the introduction, we recall in Section 1 some background and give a few preliminary results. In Section 2, elementary partial representations are treated and an algorithm for finding them is given. Those considerations allow us in Section 3, to calculate all the elementary domains for cyclic groups of order ≤ 5 and with the help of Theorem 3.2 we conclude that any cyclic group of prime order $p > 5$ has non-elementary domains. With the results of Section 3, we guarantee that the finite groups containing only elementary domains have order $2^m 3^n$, for some $m, n \in \mathbb{N}$. In Sections 4 and 5, it is proved that the only possibilities for m, n are $(m, n) = (0, 0), (0, 1)$ or $(1, 0)$.

In Section 6 a condition for a set $A \in \mathcal{P}_1(G)$ to induce an elementary partial representation with total factor set is given. Finally, in the last section we use the description given in [6], of the elementary partial representations of abelian groups of degrees less than or equal to 4, to find the sets $A \in \mathcal{P}_1(G)$ that induce elementary partial representations whose factor sets are total.

1. Basic definitions and some results

For any group G with identity 1 and any field K, the group algebra KG governs the theory of K-representations of G; in an analogous way the *partial group algebra* $K_{\mathrm{par}}G$, which is the semigroup algebra $KE(G)$, controls the partial representations of G. We recall that $E(G)$ is the monoid generated by the symbols $\{[x] \mid x \in G\}$ with defining relations:

$$[x^{-1}][x][y] = [x^{-1}][xy], \quad [x][y][y^{-1}] = [xy][y^{-1}] \ \text{ and } \ [x][1] = [x],$$

(it follows that $[1][x] = [x]$). This monoid was defined by R.Exel in [7] (see the semigroup $S(G)$). The construction of Exel was completed in [8] as follows. Denote by $\mathcal{P}_1(G)$ the set of all finite subsets of G containing 1. Put

$$\tilde{G}^{\mathcal{R}} = \{(A, g) \in \mathcal{P}_1(G) \times G \mid g \in A\},$$

with multiplication $(A,a)(B,b) = (A \cup aB, ab)$; $\tilde{G}^{\mathcal{R}}$ is the Birget-Rhodes expansion of G (see [9]), and as it was shown in [8] the map

$$E(G) \ni [x] \to (\{1,x\},x) \in \tilde{G}^{\mathcal{R}},$$

determines an isomorphism from $E(G)$ to $\tilde{G}^{\mathcal{R}}$.

We shall identify the elements of $E(G)$ with their images in $\tilde{G}^{\mathcal{R}}$. For an element $x = (A,a) \in E(G)$, the set A will be called the *support* of x.

We shall need to work with a quotient semigroup E_3 of $E(G)$. Denote by N_3 the ideal $\{(A,a) \in E(G) \mid a \in A \subseteq G, |A| \geq 4\}$ and by E_3 the factor semigroup $E(G)/N_3$. Evidently,

$$E_3 = \{(A,a) \in E(G) \mid |A| \leq 3\} \cup 0.$$

Given a semigroup S, the *Green relation* $\mathcal{J}$ is defined for S, as follows:

$$(x,y) \in \mathcal{J} \Longleftrightarrow S^1xS^1 = S^1yS^1.$$

In particular, for any (two sided) ideal I of S, we get $I = \bigcup\limits_{x \in I} J_x$.

We recall some results about the $\mathcal{J}$-classes of $E(G)$. For sets $A, B \in \mathcal{P}_1(G)$, define $A \preceq B \Longleftrightarrow \exists x \in G \;\; xA \subseteq B$.

Lemma 1.1. [3] *If* $(A,a),(B,b) \in E(G)$, *then* $A \preceq B \Longleftrightarrow (B,b) \in E(G)(A,a)E(G)$.

The next result gives a characterization of the $\mathcal{J}$-classes of $E(G)$.

Proposition 1.2. [3] *Let* $(A,a) \in E(G)$. *Then*

$$J_{(A,a)} = \{\, (B,b) \in E(G) \,|\, \exists\, x \in G \; xA = B\}. \tag{1}$$

It follows that $J_{(A,\,a)}$ does not depend on $a \in A$, so we shall write J_A instead of $J_{(A,\,a)}$. Further, if $A = \{1,a\}$ we shall denote J_A by J_a and if $A = \{1,a,b\}$, J_A will be denoted by $J_{a,\,b}$.

Definition 1.3. *A set* $A \in \mathcal{P}_1(G)$ *is called an* n*-set if* $|A| = n$.

Now we determine the non-trivial $\mathcal{J}$-classes of the semigroup E_3.

The $\mathcal{J}$- class of a 2-set

Let $A = \{1,a\}$ be a 2-set. Condition (1) means that $(B,b) \in J_a$ if and only if $xA = B$, for some $x \in G$. Then if $B = \{1,b\}$ $(b \neq 1)$ we have $\{x, xa\} = \{1,b\}$, which implies $x = 1$ or $xa = 1$. From this we conclude $b = a$ or $b = a^{-1}$. Consequently, $B = A$ or $B = \{1,a^{-1}\}$, and we obtain

$$J_a = J_b \Longleftrightarrow b \in \{a, a^{-1}\}. \tag{2}$$

The $\mathcal{J}$- class of a 3-set

Consider now a 3-set $A = \{1, a_1, a_2\}$. We know that $(B, b) \in J_{a_1, a_2}$ exactly when $\{x, xa_1, xa_2\} = B$ for some $x \in G$. Since $1 \in B$, we have the following cases

(i) If $x = 1$, then $A = B$,
(ii) if $xa_1 = 1$, then $\{x, xa_1, xa_2\} = \{1, a_1^{-1}, a_1^{-1}a_2\} = B$,
(iii) if $xa_2 = 1$, then $\{x, xa_1, xa_2\} = \{1, a_2^{-1}a_1, a_2^{-1}\} = B$.

Hence

$$J_{a_1, a_2} = J_{a_1^{-1}, a_1^{-1}a_2} = J_{a_2^{-1}a_1, a_2^{-1}}. \tag{3}$$

For $n \in \mathbb{N}$, we denote by C_n the cyclic group of order n.

Now we compute the ideal generated by an idempotent element whose support is given by a 2-set in the cyclic group.

Proposition 1.4. *For $0 < s < t$ the ideal $I_s = \langle(\{1, a^s\}, 1)\rangle$ of $E_3(C_t)$ is of the form $I_s = J_{a^s} \cup \bigcup_{\substack{m=1 \\ m \neq s}}^{t-1} J_{a^m, a^s} \cup 0$.*

Proof. For $1 \leq s \leq t-1$ and $1 \leq m \leq n \leq t-1$ we want to find the triples $\{1, a^m, a^n\}$ (including $\{1, a^m\} = \{1, a^m, a^m\}$ when $m = n$) such that $(\{1, a^m, a^n\}, 1) \in I_s$. By Lemma 1.1, this occurs exactly when there exists $x \in C_t$ such that $\{x, xa^s\} \subseteq \{1, a^m, a^n\}$. We consider the possible cases for x.

Case 1 $x = 1$. Then $\{x, xa^s\} = \{1, a^s\} \subseteq \{1, a^m, a^n\}$. Hence $m = s$ or $n = s$, and we get the triples $\{1, a^m, a^s\}$, where $1 \leq m \leq t-1$.

Case 2 $x = a^m$. Then we have $\{x, xa^s\} = \{a^m, a^{m+s}\} \subseteq \{1, a^m, a^n\}$. Therefore $a^{m+s} \in \{1, a^n\}$, then $m = t - s$ or $n = m + s$, and the triples are $\{1, a^{t-s}, a^n\}$ and $\{1, a^m, a^{m+s}\}$. By (3) $J_{a^{t-s}, a^n} = J_{a^s, a^{n+s}}$ and $J_{a^m, a^{m+s}} = J_{a^{t-m}, a^s}$.

Case 3 $x = a^n$. In this case we obtain $\{x, xa^s\} = \{a^n, a^{n+s}\} \subseteq \{1, a^m, a^n\}$, and consequently $a^{n+s} \in \{1, a^m\}$, this implies $n = t - s$ or $m = n+s-t$. In this case the triples are $\{1, a^m, a^{t-s}\}$ and $\{1, a^{n+s-t}, a^n\}$. On the other hand by (3) $J_{a^m, a^{t-s}} = J_{a^{s+m}, a^s}$ and $J_{a^{n+s-t}, a^n} = J_{a^s, a^{t-n}}$.

Then the triples appearing in the last two cases are such that their $\mathcal{J}$-classes are equal to $\mathcal{J}$-classes of triples in Case 1. From this we conclude that $I_s = J_{a^s} \cup \bigcup_{\substack{m=1 \\ m \neq s}}^{t-1} J_{a^m, a^s} \cup 0$. □

Corollary 1.5. *With the notations of the Proposition 1.4, the only $\mathcal{J}$-class of a 2-set contained in I_s is J_s.*

Given a subgroup H of G, we denote by $N_G(H)$ the normalizer of H in G, and by $M_n(KH)$ the ring of $(n\times n)$-matrices over the group algebra of KH. We have the next.

Theorem 1.6. [5] *Let K be a field, G a finite group and let $\mathcal{C}$ denote a full set of representatives of the conjugacy classes of subgroups of G. Then the partial group algebra of G over K is of the form:*

$$K_{\mathrm{par}}G \cong \bigoplus_{\substack{H\in\mathcal{C}\\ 1\le m\le (G:H)}} c_m(H)M_m(KH), \tag{4}$$

where $c_m(H) =$

$$\frac{1}{m}(G\colon N_G(H))\left(\binom{(G:H)-1}{m-1} - \sum_{\substack{H<B\le G\\ (B:H)\mid m}} \frac{m/(B:H)c_{m/(B:H)}(B)}{(G:N_G(B))}\right)$$

and $c_m(H)M_m(KH)$ means the direct sum of $c_m(H)$ copies of $M_m(KH)$.

We shall need some examples from [1] which are simple consequences of Theorem 1.6.

Corollary 1.7. *(i) For any prime number p,*

$$K_{\mathrm{par}}(C_p) \cong \bigoplus_{m=1}^{p-1} \frac{1}{m}\binom{p-1}{m-1} M_m(K)\oplus KC_p,$$

(ii) $K_{\mathrm{par}}(C_9)\cong K\oplus 4M_2(K)\oplus 14M_4(K)\oplus 14M_5(K)\oplus 4M_7(K)\oplus M_8(K)\oplus KC_3\oplus M_2(KC_3)\oplus 9M_3(K)\oplus 9M_6(K)\oplus KC_9,$

(iii) $K_{\mathrm{par}}(C_3\times C_3)\cong K\oplus 4M_2(K)\oplus 14M_4(K)\oplus 14M_5(K)\oplus 4M_7(K)\oplus M_8(K)\oplus 4KC_3\oplus 4M_2(KC_3)\oplus 8M_3(K)\oplus 8M_6(K)\oplus K(C_3\times C_3),$

(iv) $K_{\mathrm{par}}(C_2\times C_2)\cong K\oplus M_3(K)\oplus 3KC_2\oplus K(C_2\times C_2),$

(v) $K_{\mathrm{par}}(S_3)\cong K\oplus M_2(K)\oplus 3M_3(K)\oplus M_4(K)\oplus M_5(K)\oplus 3KC_2\oplus 3M_2(KC_2)\oplus KS_3.$

Denote by $\psi : K_{par}G \to \oplus M_l(KH)$ the isomorphism established in the proof of Theorem 1.6 (see Section 2). Let $Pr = Pr_l$ be the projection of $\oplus M_l(KH)$ onto the matrix algebra $M_l(KH)$. Consider also the map $[\ ]\colon G\ni g\mapsto [g]\in K_{\mathrm{par}}G$. A function of the form

$$\Gamma = Pr\circ\psi\circ[\ ] : G\to M_l(KH) \tag{5}$$

is called an *elementary partial representation* of G and we shall say that the set $D = \{(x,y) \in G \times G \,|\, \Gamma(x)\,\Gamma(y) \neq 0\}$ is an *elementary domain.*

To define partial projective representations we need the concept of K-cancellative monoid. A *K-semigroup* is a semigroup S with 0 together with a map $K \times S \ni (\alpha, x) \to \alpha x \in S$, satisfying $\alpha(\beta x) = (\alpha\beta)x$, $\alpha(xy) = (\alpha x)y = x(\alpha y)$, $1_K x = x$ and $0_K x = 0$ for any $\alpha, \beta \in K, x, y \in S$.
By a *K-cancellative semigroup* we mean a K-semigroup S such that for any $\alpha, \beta \in K$ and $0 \neq x \in S$ one has $\alpha x = \beta x \implies \alpha = \beta$.

Example 1. *For any group algebra KG and $n \in \mathbb{N}$, $M_n(KG)$ is a K-cancellative monoid.*

Definition 1.8. [2, Theorem 3] *Let M be a K-cancellative monoid. A map $\Gamma\colon G \to M$ is a partial projective representation of G if and only if:*

- *For all $x, y \in G$,*
$$\Gamma(x^{-1})\Gamma(xy) = 0 \Leftrightarrow \Gamma(x)\Gamma(y) = 0 \Leftrightarrow \Gamma(xy)\Gamma(y^{-1}) = 0;$$
- *There exists a unique partially defined map $\sigma\colon G \times G \to K^*$, with domain:* $\operatorname{dom}\sigma = \{(x,y) \,|\, \Gamma(x)\Gamma(y) \neq 0\}$, *such that for all $(x,y) \in$* $\operatorname{dom}\sigma$
$$\Gamma(x^{-1})\Gamma(x)\Gamma(y) = \sigma(x,y)\Gamma(x^{-1})\Gamma(xy),$$
$$\Gamma(x)\Gamma(y)\Gamma(y^{-1}) = \sigma(x,y)\Gamma(xy)\Gamma(y^{-1}).$$

The map σ is called a *factor set of* Γ or a *partial factor set* of G. It shall be convenient to set $\sigma(x,y) = 0$ for each $(x,y) \in G \times G$ with $\Gamma(x)\Gamma(y) = 0$, and maintain the notation $\operatorname{dom}\sigma$ for the set of pairs $(x,y) \in G \times G$ with $\Gamma(x)\Gamma(y) \neq 0$.

The partial factor sets σ of G form a commutative inverse semigroup which we denote by $pm(G)$ (see [2]), and its quotient semigroup by a natural congruence $\sim$ is the *partial Schur multiplier* $pM(G)$ of G (see [2, p. 259]). The semigroups $pm(G)$ and $pM(G)$ are disjoint unions of abelian groups called *components;* described as follows.
Consider the next transformations in $G \times G$

$$g\colon (x,y) \mapsto (xy, y^{-1}), \;\; h\colon (x,y) \mapsto (y^{-1}, x^{-1}) \text{ and } t\colon (x,y) \mapsto (x, 1). \tag{6}$$

The maps in (6) satisfy the relations

$$g^2 = h^2 = 1, \; (gh)^3 = 1, \; t^2 = t, \; gt = t, \; tght = thgh, \; tht = 0, \tag{7}$$

where 0 is the map $(x,y) \mapsto (1,1)$. Take the abstract semigroup $\mathcal{T}$ generated by the symbols g, h, t with the relations (7). The maps in (6) determine an action of $\mathcal{T}$ on $G \times G$. Denote by $C(G)$ the semilattice of all non-empty $\mathcal{T}$-subsets of $G \times G$ with respect to the set theoretic inclusion and intersection.

Theorem 1.9. [2] *The semigroups $pm(G)$ and $pM(G)$ are semilattices of abelian groups*

$$pm(G) = \bigcup_{X \in C(G)} pm_X(G), \qquad pM(G) = \bigcup_{X \in C(G)} pM_X(G),$$

where $pm_X(G) = \{\sigma \in pm(G) \mid \operatorname{dom} \sigma = X\}$ and $pM_X(G) = \dfrac{pm_X(G)}{\sim}$.

Denote by $Y^*(E_3)$ the semilattice of two sided ideals of E_3 different from E_3 and $\emptyset$, with respect to the set theoretic union and inclusion. The following result shall be useful.

Proposition 1.10. [4] *The semigroups $(C(G), \cap)$ and $(Y^*(E_3), \cup)$ are isomorphic.*

The next assertion states that every domain of a partial factor set of G can be constructed from the elementary ones.

Theorem 1.11. [3] *Let G be a finite group. The domain of any partial factor set of G is a union of elementary domains and, conversely, any union of elementary domains is a domain for the factor set of a partial projective representation of G.*

Theorem 1.11 shows the importance of the elementary domains and suggests the problem of a characterization of those finite groups which have only elementary domains.

2. On the behavior of elementary partial representations

In order to study the behavior of the elementary partial representations of G, we will consider a groupoid $\beta(G)$ associated to G.

The groupoid $\beta(G)$ is the small category with objects $(A, 1)$ and morphisms (A, g), where $g \in G$ and A is a finite subset of G containing 1 and g^{-1}. The composition $(A, g) \cdot (B, h)$ in $\beta(G)$ is defined for the pairs (A, g) and (B, h), such that $A = hB$, in which we define $(hB, g) \cdot (B, h) = (B, gh)$.

Clearly the identity morphisms in $\beta(G)$ are $(A,1)$ with $A \subseteq G$, and the inverse of (A,g) is (gA, g^{-1}).

It is useful to represent the groupoid $\beta(G)$ by an oriented graph $E_{\beta(G)}$, whose vertexes are the identity morphisms $(A,1)$, and each morphism in $\beta(G)$ gives an oriented arrow $(A,g)\colon (A,1) \longrightarrow (gA,1)$, from the vertex $s(A,g) = (A,1)$ to the vertex $r(A,g) = (gA,1)$. Note that each connected component of $E_{\beta(G)}$ is a subgroupoid (with the same composition).

We conclude that the vertexes belonging to the connected component of $(A,1)$ are of the form $(g_i^{-1}A,1)$, where $A = \bigcup_{i=1}^{m} (\operatorname{St} A)g_i$ is a disjoint union of cosets and $\operatorname{St} A = \{g \in G \mid gA = A\}$. In particular, the number of vertexes in the connected component of $(A,1)$ is $m = \frac{|A|}{|\operatorname{St} A|}$.

For an arbitrary groupoid β, denote by $\beta^{(2)} \subseteq \beta \times \beta$ the set of all composable pairs. The *groupoid algebra $K\beta$* is a vector space over K with base β and multiplication given by

$$\gamma_1\gamma_2 = \begin{cases} \gamma_1 \cdot \gamma_2 & \text{if } (\gamma_1,\gamma_2) \in \beta^{(2)}, \\ 0 & \text{if } (\gamma_1,\gamma_2) \notin \beta^{(2)}, \end{cases}$$

extended by linearity to $K\beta$.

Proposition 2.1. [1] *Let β be a groupoid such that E_β is connected and has m vertexes. Let $H = \{\gamma \in \beta \mid s(\gamma) = r(\gamma) = x_1\}$ be the isotropy group of a vertex $x_1 \in E_\beta$. Then $K\beta \cong M_m(KH)$ as K-algebras.*

The isomorphism given in [1] is as follows:
Set $\xi_{1,i} = \{\gamma \in \beta \mid s(\gamma) = x_1,\, r(\gamma) = x_i\}$, where $1 \le i \le m$. Fix $\gamma_i \in \xi_{1,i}$, then any $\gamma \in \beta$ with $s(\gamma) = x_i$ and $r(\gamma) = x_j$ can be written uniquely in the form $\gamma = \gamma_j h \gamma_i^{-1}$, with $h \in H$. The map

$$K\beta \ni \gamma \mapsto e_{j,i}(h) \in M_m(KH) \tag{8}$$

extended by linearity, gives the desired isomorphism.

Remark 1. *If a groupoid β is a finite union of subgroupoids β_i, the groupoid algebra $K\beta$ is a direct sum $\oplus_i K\beta_i$. Thus by Proposition 2.1, there is an isomorphism $\oplus\, \phi_i\colon K\beta \to \oplus M_{m_i}(KH_i)$.*

In the groupoid $\beta(G)$, when identifying the set A with the vertex $(A,1)$ of the graph $E_{\beta(G)}$, and the arrow (A,g) of $E_{\beta(G)}$ with the element $g \in G$, we obtain that $\operatorname{St} A$ coincides with the isotropy group of $(A,1)$.

From now on we always suppose that the group G is finite.

In [1] it was shown that the mapping $\lambda_p\colon G \to K\beta(G)$ defined by:

$$\lambda_p(g) = \sum_{1, g^{-1} \in A} (A, g) \tag{9}$$

is a partial representation of G. We also recall the next.

Theorem 2.2. [1] *For a finite group G and any field K, there is a K-algebra isomorphism $\alpha\colon K_{\text{par}}G \to K\beta(G)$ such that $\alpha([g]) = \lambda_p(g)$.*

From Remark 1 and Theorem 2.2, we obtain an isomorphism $\oplus\,\phi_i \circ \alpha\colon K_{\text{par}}(G) \to \oplus M_{m_i}(KH_i)$. Thus the elementary partial representations of G have the form

$$\Gamma = (Pr_j \circ \oplus\,\phi_i) \circ (\alpha \circ [\ ]) = \phi_j \circ \lambda_p\colon G \to M_{m_j}(KH_j). \tag{10}$$

Let $A \in \mathcal{P}_1(G)$ such that $H = H_j = \operatorname{St} A$ and let $m = m_j$ be the index of H in A, write $A = \bigcup_{i=1}^{m} Hg_i$ as union of disjoint cosets ($g_1 = 1$) and $\phi = \phi_j$. Then by (8)

$$\Gamma\,(g) = \sum_{g^{-1} \in g_i^{-1}A} \phi((g_i^{-1}A, g)) \ = \sum_{g = g_t^{-1}h_{t,\,i}g_i} e_{t,\,i}(h_{t,\,i}). \tag{11}$$

It shall be convenient to state an algorithm that helps us to determine elementary partial representations. For any $g \in G$, consider the set:

$$\mathcal{I}_g = \{i \in \{1, \ldots, m\} \,|\, g_i g \in A\}. \tag{12}$$

The map Γ can be defined as follows:

If $\mathcal{I}_g = \emptyset$ set:

$$\Gamma(g) = 0. \tag{13}$$

If $\mathcal{I}_g \neq \emptyset$, then for any $i \in \mathcal{I}_g$ there exists a unique $j = j_{i,g}$ in $\{1, \ldots, m\}$ and $h = h_{i,g} \in H$ such that $g_i g = hg_j$. By (11) we have

$$\Gamma(g) = \sum_{i \in \mathcal{I}_g} e_{i,j}(h). \tag{14}$$

Remark 2. *Every elementary partial representation, $\Gamma : G \to M_m(KH)$ is monomial over H. That is, for every $g \in G$ each row and each column of the matrix $\Gamma(g)$ contains at most one non-zero entry, which belongs to H.*

Summarizing, in order to obtain the elementary domains of a finite group G, we need to find all its elementary partial representations. For this, we consider the groupoid $\beta(G)$ and take $\mathcal{X} \subset \beta(G)$ a connected component of some vertex A with stabilizer H. If $\mathcal{X}$ has m vertexes, formula (10) tells us that we must compose the map λ_p defined in (9) with the isomorphism $\phi : K\beta(G) \to M_m(KH)$ established in Proposition 2.1. The isomorphism ϕ depends on the choice of the arrows $(A, g_1), \ldots, (A, g_m)$. The next lemma tells us that the elementary domain associated to $\Gamma = \phi \circ \lambda_p$ does not depend on the choice of the labeling elements $g_1, \ldots, g_m$.

Lemma 2.3. [6] *Let $\mathcal{X}$ be a connected component of $\beta(G)$ with m vertexes. Fix a vertex A with stabilizer H and pick two different collections $\{g_1, \ldots, g_m\}$, $\{g'_1, \ldots, g'_m\}$ such that the set $\{g_1A, \ldots, g_mA\}$ coincides with $\{g'_1A, \ldots, g'_mA\}$ and gives all the vertexes of $\mathcal{X}$.*
Let $\phi_1 : K\mathcal{X} \to M_m(KH)$ and $\phi_2 : K\mathcal{X} \to M_m(KH)$ be the isomorphisms determined by $\{g_1, \ldots, g_m\}$ and $\{g'_1, \ldots, g'_m\}$ respectively. Then there exists an invertible matrix $C \in M_m(KH)$ whose non-zero entries are all in H such that $\phi_1(x) = C^{-1}\phi_2(x)C,$ for all $x \in K\beta$.

We denote by $C_D(G)$ the set formed by the elementary domains. Using Lemma 2.3, Theorems 1.6,1.9 and Proposition 1.10 we obtain the next.

Theorem 2.4. *For a finite group G, the following inequalities hold*

$$|C_D(G)| \leq \sum_{\substack{H \in \mathcal{C} \\ 1 \leq m \leq (G:H)}} c_m(H) \leq |C(G)| = |Y^*(E_3)|, \tag{15}$$

where $\mathcal{C}$, H and $c_m(H)$ are as in Theorem 1.6.

3. Elementary domains and cyclic groups

We start this section by calculating all the elementary domains of the cyclic groups of orders ≤ 5. In particular, we shall see that C_4 and C_5 have non-elementary domains.

We denote by $D_i = \{(x, y) \in G \times G \mid \Gamma_i(x)\Gamma_i(y) \neq 0\}$ the elementary domain associated to the elementary partial representation Γ_i and make the "identifications" $A \equiv (A, 1)$ and $g \equiv (A, g)$.

The cyclic group $C_2 = \langle a \mid a^2 = 1\rangle$. In this case the groupoid is $\beta(C_2) = \{(\{1\}, 1), (\{1, a\}, 1), (\{1, a\}, a)\}$ whose vertexes are $(\{1\}, 1)$ and $(\{1, a\}, 1)\}$. We have the arrows

$1\colon \{1\} \to \{1\}$, $1\colon \{1,a\} \to \{1,a\}$ and $a\colon \{1,a\} \to \{1,a\}$. Hence E_β has two connected components,

$$E_{\beta_1} = \{(\{1\},1)\} \quad \text{and} \quad E_{\beta_2} = \{(\{1,a\},1),(\{1,a\},a)\}.$$

Observe that $\mathrm{St}\,\{1\} = \{1\}$ and $\mathrm{St}\,\{1,a\} = \{1,a\}$. Therefore using (13) and (14) we obtain the elementary partial representations of C_2 :

$$\Gamma_1 = G \to K\{1\} \quad \text{and} \quad \Gamma_2 = G \to K\{1,a\},$$

where $\Gamma_1(1) = 1$, $\Gamma_1(a) = 0$, $\Gamma_2(1) = 1$, $\Gamma_2(a) = a$. Then $D_1 = \{(1,1)\}$ and $D_2 = C_2 \times C_2$. By Theorem 1.11, C_2 contains only elementary domains.

The cyclic group $C_3 = \langle a \,|\, a^3 = 1\rangle$. The associated groupoid is

$$\beta(C_3) = \{(\{1\},1),\ (\{1,a\},1),\ (\{1,a\},a^2),\ (\{1,a^2\},1),\ (\{1,a^2\},a),\ (\{1,a,a^2\},1),\ (\{1,a,a^2\},a),\ (\{1,a,a^2\},a^2)\}.$$

The non-trivial oriented arrows of E_β are:
$a^2\colon \{1,a\} \to \{1,a^2\}$, $a\colon \{1,a^2\} \to \{1,a\}$, $a, a^2\colon C_3 \to C_3$.
Therefore there are three connected components of E_β, as follows: $E_{\beta_1} = \{(\{1\},1)\}$, $E_{\beta_2} = \{(\{1,a\},1),\ (\{1,a\},a^2),\ (\{1,a^2\},1),\ (\{1,a^2\},a)\}$ and $E_{\beta_3} = \{(C_3,1),\ (C_3,a),\ (C_3,a^2)\}$.
Set $A_1 = \{1\}$, $A_2 = \{1,a\}$ and $A_3 = C_3$, then $\mathrm{St}\,A_i = A_i$, where $i \in \{1,3\}$ and $\mathrm{St}\,A_2 = 1$. By (13) and (14) the elementary partial representations of C_3 are:
$\Gamma_1\colon C_3 \to K1$, $\Gamma_2\colon C_3 \to M_2(K1)$ and $\Gamma_3\colon C_3 \to KC_3$, given by :

$\Gamma_1 : 1 \mapsto 1,\ a \mapsto 0,\ a^2 \mapsto 0$, $\Gamma_3 : 1 \mapsto 1,\ a \mapsto a,\ a^2 \mapsto a^2$, and
$\Gamma_2 : 1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ a \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ a^2 \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$

Finally we get
$D_1 = \{(1,1)\}$, $D_2 = \{(1,1),\ (1,a),\ (1,a^2),\ (a,1),\ (a^2,1),\ (a,a^2),\ (a^2,a)\}$ and $D_3 = C_3 \times C_3$. Since $D_1 \subset D_2 \subset D_3$, Theorem 1.11 implies that C_3 contains only elementary domains.

The cyclic group $C_4 = \langle a \,|\, a^4 = 1\rangle$. The non-trivial oriented arrows are:

$a^3\colon \{1,a\} \to \{1,a^3\}$, $a\colon \{1,a^3\} \to \{1,a\}$,

$a^2\colon \{1,a^2\} \to \{1,a^2\}$,

$a^2\colon \{1,a,a^2\} \to \{1,a^2,a^3\}$, $a^2\colon \{1,a^2,a^3\} \to \{1,a,a^2\}$,

$a\colon \{1,a,a^3\} \to \{1,a,a^2\},\ \ a^3\colon \{1,a,a^2\} \to \{1,a,a^3\},$

$a\colon \{1,a^2,a^3\} \to \{1,a,a^3\},\ \ a^3\colon \{1,a,a^3\} \to \{1,a^2,a^3\}.$

Set $A_1 = \{1\}$, $A_2 = \{1,a\}$, $A_3 = \{1,a^2\}$, $A_4 = \{1,a,a^3\}$ and $A_5 = C_4$. For $H_i = \mathrm{St}\, A_i$, we obtain $H_1 = H_2 = H_4 = 1$, $H_3 = A_3$ and $H_5 = C_4$. Hence there are five elementary partial representations of C_4, as follows:

$\Gamma_1\colon C_4 \to K, \quad 1 \mapsto 1_K,\ a \mapsto 0,\ a^2 \mapsto 0,\ a^3 \mapsto 0,$

$\Gamma_2\colon C_4 \to M_2(K),\ 1 \mapsto Id,\ a \mapsto e_{12},\ a^2 \mapsto 0,\ a^3 \mapsto e_{21},$

$\Gamma_3\colon C_4 \to KC_2, \quad 1 \mapsto 1,\ \ a \mapsto 0,\ a^2 \mapsto 1, a^3 \mapsto 0,$

$\Gamma_4\colon C_4 \to M_3(K),\ 1 \mapsto Id,\ a \mapsto e_{21} + e_{13},\ \ a^2 \mapsto e_{23} + e_{32},$
$a^3 \mapsto e_{12} + e_{31},$

$\Gamma_5\colon C_4 \to KC_4, \quad 1 \mapsto 1,\ a \mapsto a,\ a^2 \mapsto a^2,\ a^3 \mapsto a^3,$

and the elementary domains of C_4 are:

$D_1 = \{(1,1)\}$, $D_2 = \{(1,1), (1,a), (1,a^3), (a,1), (a^3,1), (a,a^3), (a^3,a)\}$, $D_3 = \{(1,1), (1,a^2), (a^2,1), (a^2,a^2)\}$ and $D_4 = D_5 = C_4 \times C_4$. Observe that $D = D_2 \cup D_3$ is a non-elementary domain of C_4.

The cyclic group $C_5 = \langle a \mid a^5 = 1\rangle$. We can show that the elementary domains of C_5 are:

$$
\begin{aligned}
D_1 =&\{(1,1)\},\ D_2 = \{(1,1), (1,a), (1,a^4), (a,1), (a^4,1), (a,a^4), (a^4,a)\},\\
D_3 =&\{(1,1), (1,a^2), (1,a^3), (a^2,1), (a^3,1), (a^3,a^2), (a^2,a^3)\},\\
D_4 =&\{(1,C_5), (C_5,1), (a,a), (a,a^3), (a,a^4), (a^4,a), (a^3,a), (a^3,a^2),\\
&(a^2,a^3), (a^2,a^4), (a^4,a^2), (a^4,a^4)\},\\
D_5 =&\{(1,C_5), (C_5,1), (a,a^2), (a^2,a), (a,a^4), (a^4,a), (a^3,a^2), (a^2,a^2),\\
&(a^2,a^3), (a^3,a^4), (a^4,a^3), (a^3,a^3)\} \ \text{ and } D_6 = D_7 = C_5 \times C_5.
\end{aligned}
$$

Note that $D = D_2 \cup D_3$ is not an elementary domain of C_5.

By our calculations, the groups C_1, C_2 and C_3 contain only elementary domains but for the groups C_4 and C_5 there are domains that are not elementary. We shall prove that for any prime number p greater than 5 the cyclic group C_p contains non-elementary domains. First, we give a technical lemma.

Lemma 3.1. *Let p be a prime number, $p > 5$. Then*

$$2^{\frac{(p-1)(p-2)}{6}} + 2^{\frac{p-1}{2}} - 1 > \sum_{m=1}^{p-1} \frac{1}{m} \binom{p-1}{m-1} + 1.$$

Proof. It is enough to prove that $2^{\frac{(p-1)(p-2)}{6}} + 2^{\frac{p-1}{2}} - 3 > \sum_{m=2}^{p-1} \frac{1}{m}\binom{p-1}{m-1}$.
Since $\sum_{m=2}^{p-1} \frac{1}{m}\binom{p-1}{m-1} < \frac{1}{2}\sum_{m=2}^{p-1}\binom{p-1}{m-1} = 2^{p-2}-1$, it is sufficient to establish the inequality $2^{\frac{(p-1)(p-2)}{6}} + 2^{\frac{p-1}{2}} - 3 > 2^{p-2} - 1$.
Since p is a prime number greater than 5, we have $\frac{(p-1)(p-2)}{6} \geq p-2$.
Consequently $2^{\frac{(p-1)(p-2)}{6}} + 2^{\frac{p-1}{2}} - 3 \geq 2^{p-2} + 2^{\frac{p-1}{2}} - 3 > 2^{p-2} - 1$. □

Theorem 3.2. *If p is a prime number greater than 5, then there are non-elementary domains for C_p.*

Proof. By (i) of Corollary 1.7 and Theorem 2.4, it is enough to prove that the cardinality of $Y^*(E_3)$ is greater than $\sum_{m=1}^{p-1} \frac{1}{m}\binom{p-1}{m-1} + 1$.

For this, we calculate the number of the $\mathcal{J}$-classes of $E_3(C_p)$, where $C_p = \langle a \,|\, a^p = 1\rangle$. Recall that by (2) $J_{a^m} = J_{a^{p-m}}$ for $1 \leq m \leq p-1$. Therefore there are $\frac{p-1}{2}$ different $\mathcal{J}$-classes induced by the 2-sets. Let

$$I_1 = \langle(\{1,a\},1)\rangle,\ I_2 = \langle(\{1,a^2\},1)\rangle, \cdots, I_{\frac{p-1}{2}} = \langle(\{1,a^{\frac{p-1}{2}}\},1)\rangle,$$

be all the different ideals generated by idempotent elements of $E_3(C_p)$ with 2-sets as supports. Using Corollary 1.5, it is seen that by taking unions of those ideals we form $2^{\frac{p-1}{2}} - 1$ different ideals of $E_3(C_p)$.
With respect to the 3-sets, observe that there are $\binom{p-1}{2}$ sets of the form $\{1\,, a^m,\, a^n\}$ with $1 \leq m < n < p$ and by (3) $J_{a^m,\,a^n} = J_{a^{p-m},\,a^{n-m}} = J_{a^{p+m-n},\,a^{p-n}}$. Since p is a prime number greater than 5, the sets

$$\{1,\, a^m,\, a^n\}, \quad \{1,\, a^{p-m},\, a^{n-m}\}, \quad \{1,\, a^{p+m-n},\, a^{p-n}\}$$

are all different. Thus there are $\frac{(p-1)(p-2)}{6}$ different $\mathcal{J}$-classes of the 3-sets. Now consider the ideals $I_{m,n} = \langle(\{1,a^m,a^n\},1)\rangle$. By Lemma 1.1 $I_{m,n} = J_{a^m,\,a^n} \cup 0$. Therefore using the $\mathcal{J}$-classes of the 3-sets, we construct $2^{\frac{(p-1)(p-2)}{6}} - 1$ different ideals in $E_3(C_p)$, which are unions of the ideals $I_{m,n}$, $(0 < m < n < p)$. Note that none of these ideals contains an element of $E_3(C_p)$ with a 2-set as support. Then they are all different from the ideals which correspond to the $\mathcal{J}$-classes of the 2-sets. Since 0 is also an ideal, there are at least $2^{\frac{(p-1)(p-2)}{6}} + 2^{\frac{p-1}{2}} - 1$ nontrivial ideals in $E_3(C_p)$. Finally, by Lemma 3.1, we conclude that C_p has non-elementary domains. □

Now we prove the next.

Proposition 3.3. *Let H be a proper subgroup of a finite group G. If H contains non-elementary domains, then G contains non-elementary domains.*

Proof. Let D be a non-elementary domain of H. By Theorem 1.9, D is a $\mathcal{T}$-set of $H \times H$, where $\mathcal{T}$ is the semigroup generated by the symbols g, h, t with defining relations given in (7). Since $D \subseteq G \times G$ is a $\mathcal{T}$-set (considering $\mathcal{T}$ acting on $G \times G$), we have $D \in C(G)$.
Suppose that D is an elementary domain of G and take an elementary partial representation $\Gamma\colon G \to M_l(KH')$, where $H' = \operatorname{St} A$ for some $A \in \mathcal{P}_1(G)$ such that $D = \{(x, y) \in G \times G \mid \Gamma(x)\Gamma(y) \neq 0\}$. If there exists $x \in A \setminus H$, by (14) $\Gamma(x) \neq 0$. On the other hand, the pair $(x, 1)$ is not in D which implies $\Gamma(x) = 0$. We conclude that $A \subseteq H$ and $H' = \operatorname{St} A$ is a subgroup of H. Thus the map $\Gamma' = \Gamma_{|H}\colon H \to M_l(KH')$ is an elementary partial representation of H and

$$\{(x, y) \in H \times H \mid \Gamma'(x)\Gamma'(y) \neq 0\} = \{(x, y) \in G \times G \mid \Gamma(x)\Gamma(y) \neq 0\} = D.$$

Therefore D is an elementary domain of H, contradicting our hypothesis. Hence D is not an elementary domain of G. □

From Theorem 3.2 and Proposition 3.3 we obtain the next.

Proposition 3.4. *Let G be a finite group such that there exists a prime number $p \geq 5$ dividing the order of G. Then G has non-elementary domains. Equivalently, if a finite group G contains only elementary domains, there exist $m, n \in \mathbb{N}$ such that $|G| = 2^m 3^n$.*

4. Elementary domains and abelian groups

The purpose of this section is to prove that the finite abelian groups containing only elementary domains are C_1, C_2 and C_3. That is, the only possibilities for m and n in Proposition 3.4 are $(m, n) = (0, 0), (1, 0)$ or $(m, n) = (0, 1)$. We give the next.

Lemma 4.1. *The groups C_9 and $C_3 \times C_3$ contain non-elementary domains.*

Proof. We first check that there are non-elementary domains in C_9. By (ii) of Corollary 1.7 and Theorem 2.4 we only need to verify that the

number of non-trivial ideals of $E_3(C_9)$ is greater than 59.
The $\mathcal{J}$-classes of $E_3(C_9)$ induced by the 3-sets are: $J_{a,\,a^2},\ J_{a,\,a^3},\ J_{a,\,a^4},\ J_{a,\,a^5},\ J_{a,\,a^6},\ J_{a,\,a^7},\ J_{a^2,\,a^4},\ J_{a^2,\,a^5},\ J_{a^2,\,a^6}$ and $J_{a^3,\,a^6}$.
Consequently, by taking unions of the ideals induced by the 3-sets, we can form $2^{10}-1$ non-trivial ideals of $E_3(C_9)$, and there exist non-elementary domains in C_9.
With respect to $C_3 \times C_3 = \langle a, b \,|\, a^3 = b^3 = [a,\, b] = 1\rangle$, using (iii) of Corollary 1.7 and Theorem 2.4, it is enough to prove that $|Y^*(E_3)| > 63$. We note that the semigroup $E_3(C_3 \times C_3)$ has 12 $\mathcal{J}$-classes induced by the 3-sets, which are: $J_{a,\,a^2},\ J_{b,\,b^2},\ J_{a^2b,\,b^2a},\ J_{ab,\,a^2b^2},\ J_{a,\,b},\ J_{a,\,b^2},\ J_{a,\,ab},\ J_{a,\,a^2b},\ J_{a,\,ab^2},\ J_{a,\,a^2b^2},\ J_{b,\,ab^2}$, and $J_{b,\,a^2b^2}$. Then the number of nontrivial ideals of $E_3(C_3 \times C_3)$ is greater than 63. □

By Lemma 4.1 and Proposition 3.3, an abelian group containing only elementary domains has order 2^m3. We already know that C_4 has non-elementary domains. Then in order to prove that $m \leq 1$ and $(m, n) \neq (1, 1)$, we give the next.

Lemma 4.2. *The groups $C_2 \times C_2$ and C_6 contain non-elementary domains.*

The proof of Lemma 4.2 is similar to that of Lemma 4.1. Finally we obtain the next.

Theorem 4.3. *The finite abelian groups which contain only elementary domains are C_1, C_2 and C_3.*

5. Elementary domains and non-abelian groups

In this section we prove that any finite non-abelian group contains non-elementary domains.

Let G be non-abelian group and $m, n \in \mathbb{N}$ such that the order of G is 2^m3^n. Suppose that G contains only elementary domains.
If $m \geq 2$ or $n \geq 2$, G would contain an abelian subgroup of order 4 or 9. By Proposition 3.3 and Theorem 4.3, there would be non-elementary domains for G, which contradicts our assumption. Consequently $m \leq 1$ and $n \leq 1$ and since G is non-abelian, it must be isomorphic to S_3.

We shall verify that $S_3 = \langle g, h \,|\, g^2 = h^2 = (gh)^3 = 1\rangle$ contains non-elementary domains.
Indeed, by (v) of Corollary 1.7, we see that S_3 contains at most 15 elementary domains. Note that the $\mathcal{J}$-classes induced by the 3-element sets

are $J_{g,h}$, $J_{g,(gh)^2}$, $J_{h,gh}$ and $J_{gh,(gh)^2}$. From these $\mathcal{J}$-classes we form 15 different ideals of $E_3(S_3)$, and since 0 is also an ideal, we get that S_3 contains non-elementary domains.

Then Theorem 4.3 can be extended to finite arbitrary groups as follows.

Theorem 5.1. *The finite groups containing only elementary domains are C_1, C_2 and C_3.*

6. Total factor sets

A partial factor set σ of a group G is called *total* if $\operatorname{dom}\sigma = G \times G$. When the field K is algebraically closed, there is an epimorphism from $pM_{G\times G}(G)$ to any other component of $pM(G)$ (see [4]). Hence in some sense we can determine the structure of $pM(G)$ if we know the structure of $pM_{G\times G}(G)$.

For any $A \in \mathcal{P}_1(G)$, using (13) and (14), we produce an elementary partial representation $\Gamma : G \to M_l(KH)$, where $H = \mathrm{St}A$ and $l = \frac{|A|}{|H|}$. Our interest is to find the elements of $\mathcal{P}_1(G)$ that determine elementary partial representations whose (idempotent) factor sets are total. Such sets will be also called *total.*

The next theorem gives us a condition for an element A of $\mathcal{P}_1(G)$ to be total.

Theorem 6.1. *Let G be a group of order n and $A \in \mathcal{P}_1(G)$. Suppose that $|A| = n - k$ for some $0 < k < n$ and that the stabilizer H of A has order $|H| = m$. If $n > k(2m + 1)$ then A is total.*

Proof. Let $\Gamma\colon G \to M_{\frac{n-k}{m}}(KH)$ be the elementary partial representation corresponding to A. Write $A = \bigcup_{i=1}^{\frac{n-k}{m}} Hg_i$, as a disjoint union of cosets, where $1 = g_1, g_2, \ldots, g_{\frac{n-k}{m}} \in A$. Further, for $x \in G$ set $A(x) = \{x, g_2x, \ldots, g_{\frac{n-k}{m}}x\}$. Note that the function $f\colon I_x \ni i \mapsto g_ix \in A \cap A(x)$ is bijective, where I_x is as in (12).
Therefore $|I_x| = |A\cap A(x)| = |A|+|A(x)|-|A\cup A(x)| \geq n-k+\frac{n-k}{m}-n = \frac{n-k(m+1)}{m}$, and by hypothesis $n > k(2m + 1)$ we obtain $|I_x| > \frac{n-k}{2m}$, for all $x \in G$. In particular, $\Gamma(x) \neq 0$ for any $x \in G$.
Thus, $\Gamma(x) = \sum_{i\in I_x} e_{i,j}(h)$, where $i \in I_x$, $j = j_{i,x} \in \{1, \ldots, \frac{n-k}{m}\}$ and $h = h_{i,x} \in H$ satisfy $g_ix = hg_j$. Since $g_jx^{-1} = h^{-1}g_i$, we get that $i \in I_x$ is equivalent to $j = j_{i,x} \in I_{x^{-1}}$. Now take $y \in G$ and write

$\Gamma(y) = \sum_{i\in I_y} e_{s,t}(h')$. We have $\Gamma(x)\Gamma(y) = 0 \Leftrightarrow I_{x^{-1}} \cap I_y = \emptyset \Leftrightarrow |I_{x^{-1}}| + |I_y| = |I_{x^{-1}} \cup I_y|$. Since $I_{x^{-1}} \cup I_y \subseteq \{1, \ldots, \frac{n-k}{m}\}$ we obtain $|I_{x^{-1}}| + |I_y| \leq \frac{n-k}{m}$, which contradicts $|I_a| > \frac{n-k}{2m}$, for all $a \in G$. So we have $\Gamma(x)\Gamma(y) \neq 0$ for all $x, y \in G$ and A is total. $\square$

When we calculated the elementary partial representations of the cyclic group G of order 2 or 3, it was verified that each $A \in \mathcal{P}_1(G)$ with $|A| < |G|$ was not total. Therefore by Theorem 6.1 we obtain the next.

Corollary 6.2. *Let G be a finite group and $A \in \mathcal{P}_1(G)$ such that $|A| = |G| - 1$, then A is total if and only if $n > 3$.*

Unfortunately the converse of Theorem 6.1 is not true. We show this with the next example.

Example 2. *A counterexample for the converse of Theorem 6.1.*

Consider the cyclic group $\langle a \mid a^8 = 1\rangle$, and let $A = \{1, a, a^3, a^4, a^5, a^7\}$, then $A \in \mathcal{P}_1(G)$ and $H = \operatorname{St} A = \langle a^4\rangle$.
Then $A = \langle a^4\rangle \cup \langle a^4\rangle\, a \cup \langle a^4\rangle\, a^3$, using (13) and (14) we see that the elementary partial representation induced by A is $\Gamma\colon C_8 \to M_3(KH)$ given by:

$$\begin{aligned}
&\Gamma(1) = e_{11}(1) + e_{22}(1) + e_{33}(1), && \Gamma(a) = e_{12}(1) + e_{31}(a^4),\\
&\Gamma(a^2) = e_{23}(1) + e_{32}(a^4), && \Gamma(a^3) = e_{13}(1) + e_{21}(a^4),\\
&\Gamma(a^4) = e_{11}(a^4) + e_{22}(a^4) + e_{33}(a^4), && \Gamma(a^5) = e_{31}(1) + e_{12}(a^4),\\
&\Gamma(a^6) = e_{32}(1) + e_{23}(a^4), && \Gamma(a^7) = e_{21}(1) + e_{13}(a^4).
\end{aligned}$$

It is readily seen that the factor set of Γ is total.

7. Small degree elementary partial representations

Throughout this section G will denote a finite abelian group.

In [6] the authors gave a description of the elementary partial representations of abelian groups of degrees ≤ 4. We will use that description to identify the sets $A \in \mathcal{P}_1(G)$ such that the induced elementary partial representations $\Gamma\colon G \to M_m(KH)$, where $m \leq 4$, have total factor set.

1×1 *elementary partial representations.* Here $\Gamma\colon G \to KH$ and $A = H$. Then:

$$\Gamma\colon h \mapsto h, \quad g \mapsto 0, \quad \text{for each } h \in H, g \in G \setminus H.$$

Hence A is total if and only if $A = G$.

2×2 *elementary partial representations.* Write $A = H \cup aH$ as a disjoint union of cosets. Suppose that A is total, then $I_x \neq \emptyset$ for all $x \in G$. The latter implies

$$G = H \cup aH \cup a^{-1}H.$$

Now we use (13) and (14) to determine Γ. For $h \in H$, $\Gamma(h) = e_{11}(h) + e_{22}(h)$. If $ah = a^{-1}h'$ for some $h' \in H$, then $\Gamma(ah) = e_{12}(h) + e_{21}(h')$ and in the case in which $ah \notin a^{-1}H$, we obtain $\Gamma(ah) = e_{12}(h)$.
Analogously, if $a^{-1}h = ah'$, then $\Gamma(a^{-1}h) = e_{12}(h') + e_{21}(h)$, and if $a^{-1}h \notin aH$, we get $\Gamma(a^{-1}h) = e_{21}(h)$.

Thus for A being total we also need the condition $aH = a^{-1}H$, but this implies $G = H \cup aH = A$ which leads to $H = \operatorname{St} A = \operatorname{St} G = G$, and this contradicts $\frac{|A|}{|H|} = 2$. Summarizing we conclude:

Proposition 7.1. *If $A \in \mathcal{P}_1(G)$ is such that its induced elementary partial representation has degree 2, then A is not total.*

3×3 *elementary partial representations.* Write $A = H \cup aH \cup bH$, as a disjoint union of cosets. By [6, Theorem 3.2], there are 5 non-equivalent elementary partial representations of G of degree 3. The possibly total ones are given in the next two cases:

Case 1: Let $a^2 = h_1 \in H,\ b^2 = h_2 \in H,\ ab \notin H$.
Then, Γ is equivalent to:

$$\begin{aligned}\varphi_{H,a,b,3}\colon h \mapsto e_{11}(h) + e_{22}(h) + e_{33}(h), ah \to e_{12}(h) + e_{21}(h_1h),\\ bh \mapsto e_{13}(h) + e_{31}(h_2h),\ abh \mapsto e_{23}(h_1h) + e_{32}(h_2h),\\ g \mapsto 0,\ \text{if}\, g \notin H \cup aH \cup bH \cup abH.\end{aligned}$$

Since $A = H \cup aH \cup bH$ as a disjoint union of cosets and $ab \notin H$, we have $ab \notin A$. Therefore A is total if and only if $G = A \cup abH$.

Case 2: Let $a^2 \notin H,\ a^4 = h_1 \in H,\ ab = h_2 \in H$.
Observe that $A = H \cup aH \cup bh_2{}^{-1}H$. By Lemma 2.3, we obtain an elementary partial representation equivalent to Γ choosing the elements $g_1' = 1$, $g_2' = a^{-1}$ and $g_3' = b^{-1}h_2$. Thus replacing b by $bh_2{}^{-1}$, without loss of generality we may suppose $ab = 1$.
Then, following the proof of [6, Theorem 3.2], Γ is equivalent to:

$$\begin{aligned}\varphi_{H,a,b,5}\colon h \mapsto e_{11}(h) + e_{22}(h) + e_{33}(h),\ ah \to e_{12}(h) + e_{31}(h),\\ a^{-1}h \mapsto e_{21}(h) + e_{13}(h),\ a^2h \mapsto e_{23}(h_1h) + e_{32}(h),\\ g \mapsto 0,\quad \text{if}\ g \notin H \cup aH \cup a^{-1}H \cup a^2H.\end{aligned}$$

Since $a \notin H$, $a^4 \in H$ and $ab = 1$, we get $a^2 \notin A$. Hence, A is total if and only if $G = A \cup a^2H$.

4×4 *elementary partial representations.* Write $A = H \cup aH \cup bH \cup cH$, as a disjoint union of cosets. As it was seen in the proof of [6, Theorem 3.3], the elementary partial representations that may have totally defined factor sets are given in the next three cases:

Case 1: Let $ac = 1$, $a^2 = b^{-1}, a^5 = h_1 \in H$. In this case Γ is equivalent to:

$$\begin{aligned}
\varphi_{H,a,b,c,2.1.1}: h &\mapsto e_{11}(h) + e_{22}(h) + e_{33}(h) + e_{44}(h),\\
ah &\mapsto e_{12}(h) + e_{31}(h) + e_{43}(h),\\
a^{-1}h &\to e_{21}(h) + e_{13}(h) + e_{34}(h),\\
a^2h &\mapsto e_{32}(h) + e_{41}(h) + e_{24}(h_1h),\\
a^{-2}h &\mapsto e_{23}(h) + e_{14}(h) + e_{42}({h_1}^{-1}h),\\
g &\mapsto 0, \text{ if } g \notin H \cup aH \cup a^{-1}H \cup a^2H \cup a^{-2}H.
\end{aligned}$$

The relations $ac = 1$, $a^2 = b^{-1}$ and $a^5 = h_1 \in H$ imply $a^2 \notin A$. Therefore Γ is total if and only if $G = A \cup a^2H$.

Case 2: Let $ac = 1$, $a^2 = b^{-1}, a^6 = h_1 \in H$. Then, Γ is equivalent to:

$$\begin{aligned}
\varphi_{H,a,b,c,2.1.2}: h &\mapsto e_{11}(h) + e_{22}(h) + e_{33}(h) + e_{44}(h),\ a^2h \mapsto e_{32}(h) + e_{41}(h),\\
a^{-1}h &\to e_{21}(h) + e_{13}(h) + e_{34}(h),\ a^3h \mapsto e_{42}(h) + e_{24}(h_1h),\\
a^{-2}h &\mapsto e_{23}(h) + e_{14}(h),\ ah \mapsto e_{12}(h) + e_{31}(h) + e_{43}(h),\\
g &\mapsto 0, \text{ if } g \notin H \cup aH \cup a^{-1}H \cup a^2H \cup a^{-2}H \cup a^3H.
\end{aligned}$$

The conditions $ac = 1$, $a^2 = b^{-1}$ and $a^6 = h_1 \in H$ imply $a^2, a^3 \notin A$. Then Γ is total if and only if $G = A \cup a^2H \cup a^3H$.

Finally, the last case is:

Case 3: Let $ac = 1$, $a^3 = h_1 \in H$, $b^2 = h_2 \in H$. Then Γ is equivalent to:

$$\begin{aligned}
\varphi_{H,a,b,c,2.2.1}: h &\mapsto e_{11}(h) + e_{22}(h) + e_{33}(h) + e_{44}(h),\ bh \mapsto e_{14}(h) + e_{41}(h_2h),\\
a^{-1}h &\to e_{21}(h) + e_{13}(h) + e_{32}({h_1}^{-1}h),\ \ abh \mapsto e_{34}(h) + e_{42}(h_2h),\\
a^{-1}bh &\mapsto e_{24}(h) + e_{43}(h_2h),\ \ ah \mapsto e_{12}(h) + e_{31}(h) + e_{23}(h_1h),\\
g &\mapsto 0, \text{ if } g \notin H \cup aH \cup a^{-1}H \cup abH \cup a^{-1}bH \cup bH.
\end{aligned}$$

Using $ac = 1$, $a^3 = h_1 \in H$ and $b^2 = h_2 \in H$, we conclude that $ab, a^{-1}b \notin A$. Therefore Γ is total if and only if $G = A \cup abH \cup a^{-1}bH$.

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