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Generalised triangle groups of type $(3, q, 2)$

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Abstract. If G is a group with a presentation of the form $\langle x, y | x^3 = y^q = W(x, y)^2 = 1\rangle$, then either G is virtually soluble or G contains a free subgroup of rank 2. This provides additional evidence in favour of a conjecture of Rosenberger.

1. Introduction

A *generalised triangle group* is a group G with a presentation of the form

$$\langle x, y | x^p = y^q = W(x, y)^r = 1\rangle$$

where $p, q, r \geq 2$ are integers and $W(x, y)$ is a word of the form

$$x^{\alpha(1)}y^{\beta(1)} \cdots x^{\alpha(k)}y^{\beta(k)}$$

$(0 < \alpha(i) < p,\ 0 < \beta(i) < q)$. We say that G is of *type* (p, q, r). The parameter k is called the *length-parameter*. (The *syllable-length*, or *free-product length*, of W regarded as a word in $\mathbb{Z}_p * \mathbb{Z}_q$ is $2k$.) Without loss of generality, we assume that $p \leq q$.

A conjecture of Rosenberger [20] asserts that a Tits alternative holds for generalised triangle groups:

Conjecture A (Rosenberger)**.** *Let G be a generalised triangle group. Then either G is soluble-by-finite or G contains a non-abelian free subgroup.*

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This conjecture has been verified in a large number of special cases. (See for example the survey in [10].) In particular it is now known:

- when $r \geq 3$ [9];
- when $\frac{1}{p} + \frac{1}{q} \geq \frac{1}{2}$ [3, 14];
- when $q \geq 6$ [19, 22, 4, 5, 1, 7, 17];
- when $k \leq 6$ [20, 19, 21];
- for $(p, q, r) = (3, 4, 2)$ [2, 18];
- for $(p, q, r) = (2, 4, 2)$ and k odd [6].

In the present article we describe a proof of the Rosenberger Conjecture for the cases $(p, q, r) = (3, 3, 2)$ and $(p, q, r) = (3, 5, 2)$, hence completing the proof of the following

Theorem B. *Let G be a generalised triangle group of type (p, q, r) with $p, q \geq 3$ and $r \geq 2$. Then either G is soluble-by-finite or G contains a non-abelian free subgroup.*

Thus the Rosenberger Conjecture is now reduced to three cases, where $p = r = 2$ and $q \in \{3, 4, 5\}$.

The results presented here have been posted online at [15, 16], where the arguments are given in more detail. In particular, the proof in the case $q = 3$ requires a certain amount of computer calculation using GAP [11]: [15] provides full details of the computations involved, including code and output. Also included in [15] are some partial results on the case $(p, q, r) = (2, 3, 2)$ of the Rosenberger Conjecture.

Our strategy of proof is essentially the same for the two cases $q = 3$ and $q = 5$, but the details differ substantially. A theoretical analysis of the *trace polynomial* (see § 2.2 for details) reduces the problem to a finite set of candidate words W by finding an upper bound for the length-parameter k. In the case $q = 5$ the analysis is more detailed and yields the bound $k \leq 4$; the conjecture has already been proved when $k \leq 4$ by Levin and Rosenberger [19].

In the case $q = 3$ the analysis yields only the bound $k \leq 20$. This however is sufficient for a computer-based attack on the problem: a computer search using GAP [11] refines the set of candidates to a list of 19 words. The conjecture is known for the 8 shortest words in the list, by work of Levin and Rosenberger [19] and Williams [21]. The remainder of the words satisfy a small cancellation condition, which ensures the existence of nonabelian free subgroups.

Section 2 below contains some preliminary results on trace polynomials and equivalence of words. The proof of the main result for the case $q = 3$ is contained in Section 3, and for the case $q = 5$ in Section 4.

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2. Preliminaries

2.1. Equivalence of words

Our object of study is a group

$$G = \langle x, y | x^p = y^q = W(x, y)^r = 1\rangle$$

where

$$W(x, y) = x^{\alpha(1)} y^{\beta(1)} \cdots x^{\alpha(k)} y^{\beta(k)},$$

and $0 < \alpha(i) < p$, $0 < \beta(i) < q$ for each i.

We think of the word W as a cyclically reduced word in the free product

$$\mathbb{Z}_p * \mathbb{Z}_q = \langle x, y | x^p = y^q = 1\rangle.$$

We regard two such words W, W' as *equivalent* if one can be transformed to the other by moves of the following types:

- cyclic permutation of W,
- inversion of W,
- automorphism of $\mathbb{Z}_p$ or of $\mathbb{Z}_q$, and
- (if $p = q$) interchange of x, y.

It is clear that, if W, W' are equivalent words, then the resulting groups

$$G = \langle x, y | x^p = y^q = W(x, y)^r = 1\rangle$$

and

$$G' = \langle x, y | x^p = y^q = W'(x, y)^r = 1\rangle$$

are isomorphic. Hence for the purposes of studying the Rosenberger Conjecture (Conjecture A) it is enough to consider words up to equivalence.

2.2. Trace Polynomials

Suppose that $X, Y \in SL(2,\mathbb{C})$ are matrices, and $W = W(X,Y)$ is a word in X, Y. Then the trace of W can be calculated as the value of a 3-variable polynomial, where the variables are the traces of X, Y and XY [12]. We can use this to find and analyse *essential representations* from G to $PSL(2,\mathbb{C})$. (A representation of G is *essential* if the images of $x, y, W(x,y)$ have orders p, q, r respectively.)

We can force the images x, y to have orders p, q in $PSL(2,\mathbb{C})$ by mapping them to matrices $X, Y \in SL(2,\mathbb{C})$ of trace $2\cos(\pi/p)$ and $2\cos(\pi/q)$ respectively. Then the trace of $W(X,Y) \in SL(2,\mathbb{C})$ is given by a one-variable polynomial $\tau_W(\lambda)$, where λ denotes the trace of XY. We will refer to τ_W as the *trace polynomial* of W. Since we are in practice interested in the case where $r = 2$, we obtain an essential representation by choosing λ to be a root of τ_W.

We recall here some properties of τ_W. Details can be found, for example, in [10]. (Complete formulae for the coefficients of τ_W are given in [17, Appendix].)

Lemma 2.1.
- *τ_W has degree k;*
- *when $p, q \leq 3$, $\tau_W(\lambda)$ is monic and has integer coefficients;*
- *in general, the coefficients of τ_W are real algebraic integers.*

We also note a few more elementary properties.

Lemma 2.2. 1) *Let J denote the interval*

$$J = [2\cos(\pi/p + \pi/q), 2\cos(\pi/p - \pi/q)] \subset \mathbb{R}.$$

Then $\tau_W(J) \subset [-2, 2]$.

2) *If $p = q = 3$ and G does not contain a non-abelian free subgroup, then the roots of τ_W belong to $\{0, 1, \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\}$.*

3) *If $p = 3$, $q = 5$ and G does not contain a non-abelian free subgroup, then the roots of τ_W belong to $\{0, 1, \frac{1+\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$.*

Proof. The space $\mathcal{M}_q$ of matrices in $SU(2) \subset SL_2(\mathbb{C})$ with trace $2\cos(\pi/q)$ is path-connected. (Indeed, it is homeomorphic to the 2-sphere S^2.) Writing D_p for the diagonal matrix with diagonal entries $e^{i\pi/p}, e^{-i\pi/p}$, fix $X = D_p \in M_p$ and let Y vary continuously in M_q from D_q to D_q^{-1}. Then $\lambda = tr(XY)$ will vary continuously from $tr(D_pD_q) = 2\cos(\pi/p + \pi/q)$ to $tr(D_pD_q^{-1}) = 2\cos(\pi/p - \pi/q)$. By the Intermediate Value Theorem every $j \in J$ arises as $tr(XY)$ for some $Y \in M_q$ and $X = D_p \in M_p$, so $\tau_W(j) = tr(W(X,Y)) \in [-2,2]$ since $W(X,Y) \in SU(2)$.

Any root λ of τ_W corresponds to an essential representation $\rho : G \to PSL(2,\mathbb{C})$. When $p = q = 3$, the image of ρ is a subgroup of $PSL(2,\mathbb{C})$ generated by two elements of order 3 and containing an element of order 2. Any such subgroup contains a free non-abelian subgroup unless it is isomorphic to A_4 or A_5, in which case each of $\rho(xy), \rho(xy^{-1})$ has order $2, 3$ or 5. so $\lambda = tr(\rho(xy))$ and $1 - \lambda = tr(\rho(x))tr(\rho(y)) - tr(\rho(xy)) = tr(\rho(xy^{-1}))$ both belong to $\{0, \pm 1, \frac{\pm 1 \pm \sqrt{5}}{2}\}$. This is possible only for $\lambda \in \{0, 1, \frac{1 \pm \sqrt{5}}{2}\}$, as claimed.

A similar argument applies when $p = 3$ and $q = 5$. Here the image of ρ is generated by an element of order 3 and an element of order 5, and it contains an element of order 2. Such a subgroup of $PSL(2,\mathbb{C})$ contains a non-abelian free subgroup unless it is isomorphic to A_5, in which case each of $\rho(xy), \rho(xy^{\pm 1})$ has order $2, 3$ or 5. In this case $tr(\rho(x))tr(\rho(y)) = 2\cos(\pi/5) = \frac{1+\sqrt{5}}{2}$, so $\lambda, \frac{1+\sqrt{5}}{2} - \lambda \in \{0, \pm 1, \frac{\pm 1 \pm \sqrt{5}}{2}\}$, which is possible only if $\lambda \in \{0, 1, \frac{\pm 1 + \sqrt{5}}{2}\}$, as claimed. □

Lemma 2.3. *If $p = q = 3$ and W, W' are equivalent with length-parameter k, then either $\tau_W(\lambda) = \tau_{W'}(\lambda)$ or $\tau_W(\lambda) = (-1)^k \tau_{W'}(1 - \lambda)$.*

Proof. Since the trace of a matrix is a conjugacy invariant, it follows that the trace polynomial is unchanged by cyclically permuting W. Moreover, if $X \in SL(2,\mathbb{C})$ then the traces of X, X^{-1} are equal, so the trace polynomial is unchanged by inverting W.

If $tr(X) = 1 = tr(Y)$, then $tr(Y^{-1}) = 1$ also. Interchanging x, y in W has the effect on $\tau_W(\lambda) = tr(W(X,Y))$ of replacing $\lambda = tr(XY)$ by $tr(YX) = \lambda$ – in other words, no change.

Finally,

$$tr(XY^{-1}) + tr(XY) = tr(X)tr(Y) = 1.$$

Hence replacing y by y^2 has the effect of replacing $\tau_W(\lambda) = tr(W(X,Y)$ by

$$\begin{aligned} tr(W(X,Y^2)) = tr(W(X,-Y^{-1})) = \\ = (-1)^k tr(W(X,Y^{-1})) = (-1)^k \tau_W(1-\lambda), \end{aligned}$$

as claimed. □

Theorem 2.4. *Let* $G = \langle x, y | x^3 = y^3 = W(x,y)^2 = 1 \rangle$ *where* $W = x^{\alpha(1)}y^{\beta(1)} \cdots x^{\alpha(k)}y^{\beta(k)}$ *with* $\alpha(i), \beta(i) \in \{1,2\}$ *for each* i*. If* G *does not contain a free subgroup of rank* 2*, then* $\tau_W(\lambda)$ *has the form*

$$\tau_W(\lambda) = \lambda^a(\lambda-1)^b(\lambda^2-\lambda-1)^c$$

with $a, b \le 1$ *and* $c \le 3(a+b+1)$*. In particular* $k = a+b+2c \le 20$*.*

Proof. By Lemma 2.2 we may assume that the roots of τ_W all lie in $\{0, 1, \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\}$. Moreover, τ_W is monic with integer coefficients by Lemma 2.1, so the the two potential roots $\frac{1\pm\sqrt{5}}{2}$ occur with equal multiplicity c say, and τ_W has the form

$$\tau_W(\lambda) = \lambda^a(\lambda-1)^b(\lambda^2-\lambda-1)^c$$

for some non-negative integers a, b, c.

We deduce the desired bounds on a, b, c from Lemma 2.2 as follows. In this case the interval J in Lemma 2.2 is $J = [-1, 2]$. Hence $2^a = |\tau_W(2)| \le 2$ and $2^b = |\tau_W(-1)| \le 2$, so $a \le 1$ and $b \le 1$. In addition,

$$\left(\frac{5}{4}\right)^c \left(\frac{1}{2}\right)^{a+b} = \left|\tau_W\left(\frac{1}{2}\right)\right| \le 2.$$

It follows that

$$c\ln(5) \le (a+b+2c+1)\ln(2),$$

which implies the desired conclusion

$$c \le 3(a+b+1)$$

given that $a+b \in \{0,1,2\}$. □

3. The case $q = 3$

3.1. Small Cancellation

In this section we prove a result on one-relator products of groups where the relator satisfies a certain small cancellation condition. We will apply this specifically to generalised triangle groups of types $(3,3,2)$, but as the result seems of independent interest, we prove it in the widest generality available.

Suppose that Γ_1, Γ_2 are groups, and $U \in \Gamma_1 * \Gamma_2$ is a cyclically reduced word of length at least 2. (Here and throughout this section, *length* means length in the free product sense.) A word $V \in \Gamma_1 * \Gamma_2$ is called a *piece* if there are words V', V'' with $V' \neq V''$, such that each of $V \cdot V'$, $V \cdot V''$ is cyclically reduced as written, and each is equal to a cyclic conjugate of U or of U^{-1}. A cyclic subword of U is a *non-piece* if it is not a piece.

By a *one-relator product* $(\Gamma_1 * \Gamma_2)/U$ of groups Γ_1, Γ_2 we mean the quotient of their free product $\Gamma_1 * \Gamma_2$ by the normal closure of a cyclically reduced word U of positive length. Recall [13] that a *picture* over the one-relator product $G = (\Gamma_1 * \Gamma_2)/U$ is a graph $\mathcal{P}$ on a surface Σ (which for our purposes will always be a disc) whose corners are labelled by elements of $\Gamma_1 \cup \Gamma_2$, such that

1) the labels around any vertex, read in clockwise order, spell out a cyclic permutation of U or U^{-1};
2) the labels in any region of $\Sigma \setminus \mathcal{P}$ either all belong to Γ_1 or all belong to Γ_2;
3) if a region has k boundary components labelled by words $W_1, \ldots, W_k \in \Gamma_i$ (read in anti-clockwise order; with $i = 1, 2$), then the quadratic equation
$$\prod_{j=1}^{k} X_j W_j X_j^{-1} = 1$$
is solvable for $X_1, \ldots, X_k$ in Γ_i. (In particular, if $k = 1$ then $W_1 = 1$ in Γ_i).

Note that edges of $\mathcal{P}$ may join vertices to vertices, or vertices to the boundary $\partial\Sigma$, or $\partial\Sigma$ to itself, or may be simple closed curves disjoint from the rest of $\mathcal{P}$ and from $\partial\Sigma$.

The *boundary label* of $\mathcal{P}$ is the product of the labels around $\partial\Sigma$. By a version of van Kampen's Lemma, there is a picture with boundary label $W \in \Gamma_1 * \Gamma_2$ if and only if W belongs to the normal closure of U.

A picture is *minimal* if it has the fewest possible vertices among all pictures with the same (or conjugate) boundary labels. In particular every minimal picture is *reduced*: no edge e joins two distinct vertices in such a way that the labels of these two vertices that start and finish at the endpoints of e are mutually inverse.

In a reduced picture, any collection of parallel edges between two vertices (or from one vertex to itself) corresponds to a collection of consecutive 2-gonal regions, and the labels within these 2-gonal regions spell out a piece.

Since U is cyclically reduced, no corner of an interior vertex is contained in a 1-gonal region.

Theorem 3.1. *Let ℓ be an even positive integer. Suppose that* $U \equiv U_1 \cdot U_2 \cdot U_3 \cdot U_4 \cdot U_5 \cdot U_6 \in \Gamma_1 * \Gamma_2$ *with each* U_i *a non-piece of length at least* ℓ*. Suppose also that* $A, B \in \Gamma_1 * \Gamma_2$ *are reduced words of length* ℓ *such that* A *is not equal to any cyclic conjugate of* $B^{\pm 1}$ *and such that no* U_i *is equal to a subword of a power of* A*. Then* $G := (\Gamma_1 * \Gamma_2)/\langle\langle U \rangle\rangle$ *contains a non-abelian free subgroup.*

Proof. Since ℓ is even and positive, any reduced word of length ℓ in $\Gamma_1 * \Gamma_2$ is cyclically reduced. Replacing A by A^{-1} and/or B by B^{-1} if necessary, we may assume that each of A, B begins with a letter from Γ_1 and ends with a letter from Γ_2. Choose a large positive integer $N > 20K\ell$, where K is the length of U, and define $X := A^N B^N$, $Y := B^N A^N$. We claim that X, Y freely generate a free subgroup of G.

We prove this claim by contradiction. Suppose that $Z(X, Y)$ is a non-trivial reduced word in X, Y such that $Z(X, Y) = 1$ in G. Then there exists a picture $\mathcal{P}$ on the disc D^2 over the one-relator product G with boundary label $Z(X, Y)$. Without loss of generality, we may assume that $\mathcal{P}$ is minimal, hence reduced.

Suppose that v is an interior vertex of $\mathcal{P}$. The vertex label of v is U or U^{-1} – by symmetry we can assume it is U. The subword U_1 of U corresponds to a sequence of consecutive corners of v; at least one of these corners does not belong to a 2-gonal region of $\mathcal{P}$, since U_1 is a non-piece. It follows that at least one of the corners of v within the subword U_1 of the vertex label does not belong to a 2-gonal region. The same follows for the subwords $U_2, \ldots, U_6$, so v has at least 6 non-2-gonal corners.

Now consider the (cyclic) sequence of *boundary* (that is, non-interior) vertices of $\mathcal{P}$, $v_1, \ldots, v_n$ say. This is intended to mean that the closed path ∂D^2, with an appropriate choice of starting point, meets a sequence of arcs that go to v_1, separated by 2-gons, then a sequence of arcs that go to v_2, separated by 2-gons, and so on, finishing with a sequence of arcs that go to v_n, separated by 2-gons, before returning to its starting point. Note that it is possible that an arc of $\mathcal{P}$ joins two points on ∂D^2; any such arc is ignored here. Note also that we do not insist that $v_i \neq v_j$ for $i \neq j$ in general. It is possible for the sequence of boundary vertices to visit a vertex v several times. Nevertheless it is important to regard such visits as pairwise distinct, so the notation $v_1, v_2, \ldots$ is convenient. We say that a boundary vertex is *simple* if it appears only once in this sequence.

If v_j is connected to ∂D^2 by k arcs separated by $k-1$ 2-gons, then this corresponds to a common (cyclic) subword W_j of $Z(X, Y)$ and U, of length $k-1$. Let $\kappa(j) \leq 6$ be the maximum integer t such that, for some $s \in \{1, \dots, 6\}$, W_j contains a subword equal to $(U_s \cdot U_{s+1} \cdots U_{s+t})^{\pm 1}$ (indices modulo 6). If no such t exsits, we define $\kappa(j) = -1$.

If v_j is a simple boundary vertex with only $r \leq 4$ corners not belonging to 2-gons, then it is easy to see that $\kappa(j) \geq 5 - r$:

There are more complex rules for non-simple boundary vertices. Nevertheless, it is an easy consequence of Euler's formula, together with the fact that interior vertices have 6 or more non-2-gonal corners, that

$$\sum_{j=1}^{n} \kappa(j) \geq 6.$$

Now consider the word $Z(X, Y)$ as a cyclic word in $\Gamma_1 * \Gamma_2$. Where a letter $X = A^N B^N$ or $Y = B^N A^N$ is followed by another letter X or Y, then there is no cancellation in $\Gamma_1 * \Gamma_2$. Similarly there is no cancellation where X^{-1} or Y^{-1} is followed by X^{-1} or Y^{-1}. Where X is followed by Y^{-1} or *vice versa*, or where Y is followed by X^{-1} or *vice versa*, then there is possible cancellation, but since $A \neq B$ the amount of cancellation is limited to at most ℓ letters from either side.

If Z has length L as a word in $\{X^{\pm 1}, Y^{\pm 1}\}$, then after cyclic reduction in $\Gamma_1 * \Gamma_2$ it consists of L subwords of the form $A^{\pm(N-1)}$, L subwords of the form $B^{\pm(N-1)}$, and L subwords $V_1, \dots, V_L$, each of length at most 2ℓ.

Now suppose that v_j is a boundary vertex of $\mathcal{P}$ with $\kappa(j) \geq 0$. Then $U_i^{\pm 1}$ is equal to a subword of W_j for some i. Since U_i cannot be a subword of a power of A, W_j is not entirely contained within one of the segments labelled $A^{\pm(N-1)}$.

If, in addition, $\kappa(j) > 0$, then W_j has a subword of the form $(U_i U_{i+1})^{\pm 1}$ (subscripts modulo 6) As above, W_j cannot be contained in one of the subwords $A^{\pm(N-1)}$. If it is contained in a subword of $B^{\pm(N-1)}$, then it is a periodic word of period ℓ (that is, its i-th letter is equal to its $(i + \ell)$-th letter for all i for which this makes sense). Since U_{i+1} has length at least ℓ, there are at least two distinct subwords of $U_i U_{i+1}$ equal to U_i, contradicting the fact that U_i is a non-piece in U.

Thus we see that the subwords W_j of $Z(X, Y)$ corresponding to boundary vertices v_j with $\kappa(j) > 0$ can occur only at certain points of $Z(X, Y)$: where an $A^{\pm(N-1)}$-segment meets a $B^{\pm(N-1)}$-segment; or at part of one of the words V_i.

In particular, the number of boundary vertices v_j with $\kappa(j) > 0$ is bounded above by $L(2\ell + 1)$. It follows that

$$\kappa := \sum_{j} \kappa(j) \leq 5L(2\ell + 1),$$

where the sum is taken over those boundary vertices v_j with $\kappa(j) \geq 0$.

The goal is to show that the total positive contribution to the sum κ from those v_j with $\kappa(j) > 0$ is cancelled out by negative contributions to κ from other boundary vertices. This will show that $\kappa \leq 0$, contradicting the assertion above that $\kappa \geq 6$.

Recall that K is the length of U. Thus each $A^{\pm(N-1)}$-segment of $\partial\mathcal{P}$ is joined to at least $(N-1)\ell/K$ boundary vertices, at most 2 of which (those at the ends of the segment) can make non-negative contributions to κ. The remaining vertices each contribute at most -1 to κ. Since $N > 20K\ell$, it follows that the negative contributions outweigh the positive contributions, as required.

This gives the desired contradiction, which proves the theorem. □

Corollary 3.2. *Let Γ_1 and Γ_2 be groups, and suppose $x \in \Gamma_1$ and $y \in \Gamma_2$ are elements of order greater than 2. Suppose that $W \equiv U_1 \cdot U_2 \cdot U_3 \in \Gamma_1 * \Gamma_2$ with each U_i a non-piece of length at least 4. Then $G = (\Gamma_1 * \Gamma_2)/\langle\langle W^2 \rangle\rangle$ contains a non-abelian free subgroup.*

Proof. Let $A_1 = xyxy$, $A_2 = xy^{-1}xy^{-1}$, $A_3 = xyxy^{-1}$ and $A_4 = xyx^{-1}y^{-1}$. Then for $i \neq j$, A_i is not equal to a cyclic conjugate of $A_j^{\pm 1}$. Hence if (say) U_1 is equal to a subword of a power of A_i, it cannot be equal to a subword of a power of A_j. Hence there is at least one $A \in \{A_i,\ 1 \leq i \leq 4\}$ with the property that no U_i is equal to a subword of a power of A. Now choose $B \in \{A_i,\ 1 \leq i \leq 4\} \smallsetminus \{A\}$ and apply the theorem, with $U_4 = U_1$, $U_5 = U_2$ and $U_6 = U_3$. □

3.2. Conclusion

Theorem 3.3. *Let $G = \langle x, y | x^3 = y^3 = W(x,y)^2 = 1 \rangle$ be a generalised triangle group of type $(3,3,2)$. Then the Rosenberger Conjecture holds for G: either G is soluble-by-finite, or G contains a non-abelian free subgroup.*

Proof. Write

$$W = x^{\alpha(1)} y^{\beta(1)} \cdots x^{\alpha(k)} y^{\beta(k)}.$$

A computer search using GAP [11] (see [15] for details) produces a list of all words W, up to equivalence, for which the trace polynomial τ_W has the form indicated in Theorem 2.4: see Table 1. If W is not equivalent to a word in the list, then G has a nonabelian free subgroup by Theorem 2.4, so we may restrict our attention to the words W in Table 1.

	$W(x,y)$	SCC
1	xy	NO
2	$xyxy^2$	NO
3	xyx^2y^2	NO
4	$xyxyx^2y^2$	NO
5	$xyxyx^2yx^2y^2$	NO
6	$xyxy^2x^2yx^2y^2$	NO
7	$xyxyx^2y^2x^2yxy^2$	NO
8	$xyxyx^2y^2x^2yx^2yxy^2$	NO
9	$(xyxyx)(y^2x^2y^2x)(yx^2yx^2y^2)$	YES
10	$(xyxy)(x^2y^2x^2yx)(y^2x^2yx^2y^2xy^2)$	YES
11	$(xyxy)(x^2y^2x^2yx^2)(y^2xy^2xyx^2y^2)$	YES
12	$(xyxy)(x^2y^2xy^2x^2y^2)(xyx^2yx^2y^2)$	YES
13	$(xyxy)(x^2y^2x^2y^2)(xy^2x^2y^2xyx^2yx^2y^2)$	YES
14	$(xyxy)(x^2y^2xy^2x^2yxy)(x^2y^2x^2yx^2y^2xy^2)$	YES
15	$(xyxy)(x^2y^2x^2y^2)(xy^2x^2yx^2y^2x^2yxyx^2y^2xy^2)$	YES
16	$(xyxyx^2y^2)(x^2yxy^2xy^2x^2y^2x^2)(yxy^2xyx^2yx^2y^2x^2yxy^2)$	YES
17	$(xyxyx^2y^2x^2)(yxy^2xyx^2yx^2y^2x^2yxy^2x)(y^2x^2y^2x^2yxy^2)$	YES
18	$(xyxyx^2y^2x^2yx^2)(yxy^2xyx^2)(y^2xyxy^2x^2y^2x^2yx^2y^2xy^2)$	YES
19	$(xyx^2y^2x^2yx^2)(y^2xy^2xyxy^2x^2)(y^2x^2yxy^2x^2yx^2yxy^2xy)$	YES

TABLE 1. Words in $\mathbb{Z}_3 * \mathbb{Z}_3$ with trace polynomial as in Theorem 2.4. The final column indicates whether or not W satisfies the small-cancellation hypotheses of Corollary 3.2. In those cases where it does, the bracketing indicates a subdivision of W into three non-pieces of length ≥ 4: $W \equiv U_1 \cdot U_2 \cdot U_3$.

For those W in Table 1 for which $k \geq 7$ (namely, numbers 9-19) the small cancellation hypotheses of Corollary 3.2 are satisfied, and so G contains a nonabelian free subgroup.

For $k \leq 6$ (words 1-8) in the table, the result is known. Specifically, groups 1-3 are well-known to be finite of orders $12, 180$ and 288 respectively; groups 4-6 were proved to have nonabelian free subgroups in [19]; and finally groups 7 and 8 were shown in [21] (see also [15]) to be *large*. (That is, each contains a subgroup of finite index which admits an epimorphism onto a non-abelian free group.)

This completes the proof. □

4. The case $q = 5$

To prove the result in the case $q = 5$, we first prove a number of preliminary results.

Lemma 4.1. *Let $p : \overline{K} \to K$ be a regular covering of connected 2-complexes with K finite, with covering transformation group abelian of torsion-free rank at least 2. Let F be a field. If*

$$H_2(\overline{K}, F) = 0 \neq H_1(\overline{K}, F),$$

then

$$\dim_F\ H_1(\overline{K}, F) = \infty.$$

Proof. Let $\{a, b\}$ be a basis for a free abelian subgroup A of the group of covering transformations of $p : \overline{K} \to K$, and let α be a cellular 1-cycle of $\overline{K}$ over F that represents a non-zero element of $H_1(\overline{K}, F)$. If the $F[a]$-submodule of $H_1(\overline{K}, F)$ generated by α is free, then $H_1(\overline{K}, F)$ is infinite-dimensional over F, as claimed. So we may assume that there is a cellular 2-chain β of $\overline{K}$ with $d(\beta) = f(a)\alpha$ for some non-zero polynomial $f(a) \in F[a]$.

For similar reasons, we may also assume that $d(\gamma) = g(b)\alpha$ for some cellular 2-chain γ of $\overline{K}$ and some non-zero polynomial $g(b) \in F[b]$.

Now $f(a)\gamma - g(b)\beta \in H_2(\overline{K}, F) = 0$. In other words $f(a)\gamma = g(b)\beta$ in the group $C_2(\overline{K}, F)$ of cellular 2-chains of $\overline{K}$, which is a free module over the unique factorisation domain $FA \cong F[a^{\pm 1}, b^{\pm 1}]$. Since $f(a), g(b)$ are coprime in $F[a^{\pm 1}, b^{\pm 1}]$, it follows that there is a 2-chain δ with $f(a)\delta = \beta$ and $g(b)\delta = \gamma$. Hence $f(a)(d(\delta) - \alpha) = d(\beta) - f(a)\alpha = 0$, in the group $C_1(\overline{K}, F)$ of cellular 1-chains of $\overline{K}$. But $C_1(\overline{K}, F)$ is also a free module over the domain $F[a^{\pm 1}, b^{\pm 1}]$, and $f(a) \neq 0$, so $d(\delta) = \alpha$, contradicting the hypothesis that α represents a non-zero element of $H_1(\overline{K}, F)$.

This contradiction completes the proof. □

Lemma 4.2. *Let E be the set of midpoints of edges of a regular icosahedron $\mathcal{I} \subset \mathbb{R}^3$ centred at the origin, and let $M = \mathbb{Z}E$ its $\mathbb{Z}$-span in $\mathbb{R}^3$. Let $V = \{1, a, b, c\} \subset Isom^+(\mathcal{I}) \subset SO(3)$ be the Klein 4-group, and let $C = \{1, c\} \subset V$. Then, regarding M as a $\mathbb{Z}V$-module via the action of V by isometries of $\mathcal{I}$, we have the following.*

1) *$M \cong \mathbb{Z}^6$ as an abelian group.*
2) *$H_0(C, M) = \mathbb{Z} \otimes_{\mathbb{Z}C} M \cong \mathbb{Z}_2^4 \oplus \mathbb{Z}^2$.*

3) *The induced action of* V/C *on* $H_0(C,M)/(\text{torsion})$ *is mutliplication by* -1.

Proof. If e is the midpoint of the edge joining two vertices u,v of $\mathcal{I}$, then $e=(u+v)/2$. Thus E is contained in the $\mathbb{Q}$-span W of the set of vertices of $\mathcal{I}$. Since the vertices occur in 6 antipodal pairs, the $\mathbb{Q}$-span $\mathbb{Q}M$ of E has dimension at most 6 over $\mathbb{Q}$.

On the other hand, for any vertex v, $\sqrt{5}\cdot v$ is the sum of the 5 vertices adjacent to v in $\mathcal{I}$. Thus $\sqrt{5}\cdot v\in W$. It also follows that $\sqrt{5}\cdot e\in M$ for any $e\in E$: specifically, $(\sqrt{5}+3)\cdot e$ is the sum of the midpoints of the eight edges of $\mathcal{I}$ that share a vertex with the edge containing e. If $e_1,e_2,e_3\in E$ are chosen to be linearly independent over $\mathbb{R}$ – and hence over $\mathbb{Q}[\sqrt{5}]$ – then $e_1,e_2,e_3,\sqrt{5}\cdot e_1,\sqrt{5}\cdot e_2,\sqrt{5}\cdot e_3\in M$ are linearly independent over $\mathbb{Q}$. Thus $\mathbb{Q}M=\mathbb{Q}\otimes_{\mathbb{Z}}M$ has dimension exactly 6 over $\mathbb{Q}$. Since $M\subset\mathbb{Q}M$ is torsion-free and finitely generated, it follows that $M\cong\mathbb{Z}^6$, as claimed.

If, in the above, we choose e_1,e_2,e_3 to lie on the axes of the rotations $a,b,c\in V$ respectively, then we obtain a decomposition

$$\mathbb{Q}M=\mathbb{Q}[\sqrt{5}]e_1\oplus\mathbb{Q}[\sqrt{5}]e_2\oplus\mathbb{Q}[\sqrt{5}]e_3$$

of $\mathbb{Q}M$ as a $\mathbb{Q}[\sqrt{5}]$-vector space, with respect to which a,b,c act as the diagonal matrices $diag(1,-1,-1)$, $diag(-1,1,-1)$ and $diag(-1,-1,1)$ respectively. Let

$$M_+:=M\cap\mathbb{Q}[\sqrt{5}]e_3\quad\text{and}\quad M_-:=M\cap(\mathbb{Q}[\sqrt{5}]e_1\oplus\mathbb{Q}[\sqrt{5}]e_2).$$

Then $M_-\cap M_+=\{0\}$, while $e_1,e_2,\sqrt{5}e_1,\sqrt{5}e_2\in M_-$ and $e_3,\sqrt{5}e_3\in M_+$, so M_-, M_+ are free abelian of ranks 4 and 2 respectively.

Moreover, M/M_- is naturally embedded in the vector space $\mathbb{Q}M/\mathbb{Q}M_-$, so is also free abelian – necessarily of rank 2. Note that M_- is closed under the action of V on M. Under the induced action on M/M_-, each of a,b acts as the antipodal map, multiplication by -1, and c acts as the identity. Clearly also c acts on M_- as the antipodal map.

Hence $(1-c)M=2M_-$, so

$$H_0(C,M)=M/(1-c)M=M/2M_-\cong\mathbb{Z}_2^4\oplus\mathbb{Z}^2,$$

as claimed.

Finally, the quotient of $H_0(C,M)$ by its torsion subgroup is naturally isomorphic to M/M_-, and the induced action of V/C on this quotient is via the antipodal map. $\square$

Lemma 4.3. *Let $G = \langle x, y | x^3 = y^5 = W(x,y)^2 = 1\rangle$ and suppose that $(\lambda - \alpha)^2$ divides the trace polynomial $\tau_W(\lambda)$ of W, for some $\alpha \in \{0, 1, (1+\sqrt{5})/2, (-1+\sqrt{5})/2\}$. Let $\rho : G \to A_5$ be the natural epimorphism corresponding to the root α of $\tau_W(\lambda)$. Let $C \subset A_5$ be a subgroup of order 2 and $V \subset A_5$ its centraliser of order 4. Then G has subgroups $N_1 \lhd N_2 \lhd \rho^{-1}(V)$ such that*

1) $\rho(N_2) = \{1\}$;
2) $\rho^{-1}(C)/N_2 \cong \mathbb{Z}^2$;
3) $\rho^{-1}(V)/N_2 \cong \mathbb{Z}^2 \rtimes_{(-1)} \mathbb{Z}_2$;
4) N_2/N_1 *is a non-zero vector space over* $\mathbb{Z}_2$.

Proof. Let $\Lambda = \mathbb{C}[\lambda]/\langle(\lambda - \alpha)^2\rangle$, and choose matrices

$$X = \begin{pmatrix} e^{i\pi/3} & 0 \\ 1 & e^{-i\pi/3} \end{pmatrix}, Y = \begin{pmatrix} e^{i\pi/5} & \lambda - \alpha - 2\cos(8\pi/15) \\ 0 & e^{-i\pi/5} \end{pmatrix} \in SL_2(\Lambda)$$

so that

$$tr(X) = 1, \quad tr(Y) = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad tr(XY) = \lambda - \alpha.$$

Then X, Y determine a representation $\widehat{\rho} : G \to PSL_2(\Lambda)$, since $tr(W(X,Y)) = \tau_W(\lambda) = 0$ in Λ. If $\phi : PSL_2(\Lambda) \to PSL_2(\mathbb{C})$ is the natural epimorphism obtained by setting $\lambda = \alpha$, then the image of $\rho = \phi \circ \widehat{\rho}$ is isomorphic to A_5. Let K denote the kernel of ρ and let L denote the kernel of $\widehat{\rho}$.

Clearly $G/K \cong A_5$. Now $K/L \cong \widehat{\rho}(K)$ is the normal closure of $(xy)^2.L$, so it is isomorphic to the subgroup of $PSL(2,\Lambda)$ generated by

$$(XY)^2 = -I + (\lambda - \alpha)(XY)$$

together with its conjugates by elements of $\widehat{\rho}(G)$. Let $Z = \phi(XY) \in A_5 \subset SU(2)$ denote the matrix obtained from XY by subsituting $\lambda = \alpha$. Note that $tr(Z) = 0$, in other words, $Z \in sl_2(\mathbb{C})$. Since $(\lambda - \alpha)^2 = 0$ in Λ, we also have

$$(XY)^2 = -I + (\lambda - \alpha)Z.$$

For similar reasons, for any $M \in \widehat{\rho}(G)$ we have

$$M(XY)^2M^{-1} = -I + \phi(M)Z\phi(M)^{-1}.$$

Moreover, since $(\lambda-\alpha)^2=0$ in Λ we have, for any $A,B\in sl_2(\mathbb{C})$,

$$(I-(\lambda-\alpha)A)(I-(\lambda-\alpha)B)=I-(\lambda-\alpha)(A+B).$$

Thus $K/L\cong\rho(K)$ is isomorphic to the additive subgroup of $sl_2(\mathbb{C})$ generated by MZM^{-1} for all $M\in\widehat{A_5}\subset SU(2)$. There are precisely 30 such conjugates of Z; geometrically they correspond to the midpoints of the edges of a regular icosahedron centred at the origin in $\mathbb{R}^3$, where we identify $SU(2)$ with the 3-sphere of unit-norm quaternions, and $\mathbb{R}^3$ with the space of purely imaginary quaternions. As an abelian group, therefore, $K/L\cong\rho(K)\cong\mathbb{Z}^6$ by Lemma 4.2.

Now K/L is also an A_5-module. Its structure as an A_5-module does not need to concern us, but Lemma 4.2 gives us some information about its structure as a C-module and as a V-module. This in turn gives information on the structure of $\Delta:=(\rho)^{-1}(C)$.

Specifically, $H_0(C,K/L)=H_0(\Delta/K,K/L)\cong\mathbb{Z}_2^4\oplus\mathbb{Z}^2$. It follows from the 5-term exact sequence

$$H_2(\Delta/L)\to H_2(\Delta/K)\to H_0(\Delta/K,K/L)\to H_1(\Delta/L)\to H_1(\Delta/K)\to 0$$

and the fact that $\Delta/K\cong\mathbb{Z}_2$ that $H_1(\Delta/L)$ has torsion-free rank 2, and that the torsion subgroup of $H_1(\Delta/L)$ is a non-zero finite abelian 2-group.

Now let $N_0=[\Delta,\Delta].L$ and define $N_2\supset N_1\supset N_0$ such that N_2/N_0 is the torsion-subgroup of $\Delta/N_0=H_1(\Delta/L)$ and $N_1/N_0=2(N_2/N_0)$. Then $N_0\triangleleft\rho^{-1}(V)$ since $[\Delta,\Delta]$ and L are both normal in $\rho^{-1}(V)$. Hence also $N_1,N_2\triangleleft\rho^{-1}(V)$ since N_1 and N_2 are characteristic in Δ which in turn is normal in $\rho^{-1}(V)$.

By construction, $\Delta/N_2\cong\mathbb{Z}^2$, while N_2/N_1 is a non-zero $\mathbb{Z}_2$-vector space.

Finally, since V/C acts on $\mathbb{Z}^2\cong\Delta/N_2$ by the antipodal map, it follows that $\rho^{-1}(V)/N_2\cong\mathbb{Z}^2\rtimes_{(-1)}\mathbb{Z}_2$, as required. □

5. Conclusion

Theorem 5.1. *Let $G=\langle x,y|x^3=y^5=W(x,y)^2=1\rangle$. If the trace polynomial $\tau_W(\lambda)$ of W has a multiple root, then G contains a nonabelian free subgroup.*

Proof. We may assume that the root α is one of 0, 1, $(\pm1+\sqrt{5})/2$, for otherwise the result is immediate from Lemma 2.2. Let $\rho:G\to A_5$ be the essential representation corresponding to α, let $c=\rho(W)\in A_5$,

$C = \{1, c\} \subset A_5$ the subgroup generated by c, and $V = \{1, a, b, c\} \subset A_5$ its centraliser in A_5.

Let $N_1 \lhd N_2 \lhd \rho^{-1}(V) < G$ be the subgroups promised by Lemma 4.3. Let $\Gamma = \rho^{-1}(C) < \rho^{-1}(V)$ be the unique index 2 subgroup such that $N_2 \subset \Gamma$ and $\Gamma/N_2 \cong \mathbb{Z}^2$. Then Γ has index 30 in G and contains no conjugate of x or of y.

Applying the Reidemeister-Scheier process to the presentation of G in the statement of the Theorem, we obtain a presentation of Γ of the form

$$\Gamma = \langle k_1, \cdots, k_{31} | r_1, \ldots, r_{30}, s_1^2, s_2^2 \rangle,$$

where $r_1, \ldots, r_{10}$ are rewrites of conjugates of x^3; $r_{11}, \ldots, r_{16}$ are rewrites of conjugates of y^5; and $r_{17}, \ldots, r_{30}$ and $s_1^2 = W^2, s_2^2 = \hat{a}W^2\hat{a}^{-1}$ are rewrites of conjugates of W^2, with $\rho(\hat{a}) = a$ and so $s_1 = W, s_2 = \hat{a}W\hat{a}^{-1} \in \Gamma$.

Let K be the 2-complex model of this presentation, $F = \mathbb{Z}_2$, and $p : \overline{K} \to K$ the regular cover correspdonding to the normal subgroup $N_2 \lhd \Gamma$. Let $L \subset K$ be the subcomplex obtained by omitting the 2-cells corresponding to the relators s_1^2, s_2^2, and let $\overline{L} := p^{-1}(L) \subset \overline{K}$.

Now, since Γ/N_2 is torsion-free, and since $s_1^2 = 1 = s_2^2$ in Γ, $s_1, s_2 \in N_2$. Hence each lift of each 2-cell s_i^2 $(i = 1, 2)$ to $\overline{K}$ is bounded by the square of some path in $\overline{K}^{(1)}$. As a consequence, the 2-cells in $\overline{K} \setminus \overline{L}$ represent elements of $H_2(\overline{K}, F)$, and it follows that the inclusion-induced map $H_1(\overline{L}, F) \to H_1(\overline{K}, F)$ is an isomorphism.

Since N_2/N_1 is a nonzero F-vector space, we have

$$H_1(\overline{L}, F) \cong H_1(\overline{K}, F) = H_1(N_2, F) \neq 0.$$

If $H_2(\overline{L}, F) = 0$, then by Lemma 4.1 it follows that $dim_F\ H_1(N_2, F) = \infty$. On the other hand, if $H_2(\overline{L}, F) \neq 0$ then $H_2(\overline{L}, F)$ contains a free $F(\Gamma/N_2)$-module of rank $> 0 = \chi(L)$, since $F(\Gamma/N_2)$ is an integral domain. In this case $H_1(\overline{L}, F)$ contains a non-zero free $F(\Gamma/N_2)$-submodule, by [14, Proposition 2.1 and Theorem 2.2]. Again we deduce that $dim_F\ H_1(N_2, F) = \infty$.

Thus the Bieri-Strebel invariant Σ of the $F(\Gamma/N_2)$-module N_2/N_1 is a proper subset of S^1 [8, Theorem 2.4]. But by Lemma 4.2 (3) it follows that Σ is invariant under the antipodal map: $\Sigma = -\Sigma$. Hence $\Sigma \cup -\Sigma \neq S^1$, and it follows [8, Theorem 4.1] that Γ contains a nonabelian free subgroup, as claimed. □

Corollary 5.2 (Main Theorem). *Let G be a generalised triangle group of type $(3, 5, 2)$. Then either G is virtually soluble or G contains a nonabelian free subgroup.*

Proof. By Theorem 5.1 and Lemma 2.2 the result follows unless $\tau_W(\lambda)$ has only simple roots in the set $\{0, 1, (1+\sqrt{5})/2, (-1+\sqrt{5})/2\}$, in which case the degree k of $\tau_W(\lambda)$ is at most equal to 4.

But the Rosenberger Conjecture is known for $k \leq 4$ [19]. □

References

[1] O. A. Barkovich and V. V. Benyash-Krivets, *On Tits alternative for generalized triangular groups of (2,6,2) type (Russian)* Dokl. Nat. Akad. Nauk. Belarusi **48** (2004), No. 3, 28–33.

[2] O. A. Barkovich and V. V. Benyash-Krivets, *On the Tits alternative for some generalized triangular groups of type* $(3, 4, 2)$ *(Russian)*, Dokl. Nats. Akad. Nauk Belarusi **47** (2003), no. 6, 24–27.

[3] G. Baumslag, J. W. Morgan and P. B. Shalen, *Generalized triangle groups*, Math. Proc. Cambridge Philos. Soc. **102** (1987), 25–31.

[4] V. V. Benyash-Krivets, *On free subgroups of certain generalised triangle groups (Russian)*, Dokl. Nat. Akad. Nauk. Belarusi **47** (2003), No. 3, 14–17.

[5] V. V. Benyash-Krivets, *On Rosenberger's conjecture for generalized triangle groups of types* $(2, 10, 2)$ *and* $(2, 20, 2)$ in: Proceedings of the international conference on mathematics and its applications (S. L. Kalla and M. M. Chawla, eds.), Kuwait Foundation for the Advancement of Sciences (2005), pp. 59–74.

[6] V. V. Benyash-Krivets, *On free subgroups of generalized triangular groups of type* $(2, 4, 2)$ *(Russian)* Dokl. Nats. Akad. Nauk Belarusi 51 (2007), no. 4, 29–32, 125.

[7] V. V. Benyash-Krivets and O. A. Barkovich, *On the Tits alternative for some generalized triangle groups*, Algebra Discrete Math. (2004), No. 3, 23–43.

[8] R. Bieri and R. Strebel, *Valuations and finitely presented metabelian groups*, Proc. London Math. Soc. (3) **41** (1980), 439–464.

[9] B. Fine, F. Levin and G. Rosenberger, *Free subgroups and decompositions of one-relator products of cyclics. I. The Tits alternative*, Arch. Math. (Basel) **50** (1988), 97–109.

[10] B. Fine, F. Roehl and G. Rosenberger, *The Tits alternative for generalized triangle groups.*, in: Groups - Korea '98. Proceedings of the 4th international conference, Pusan, Korea, August 10-16, 1998 (Y.-G. Baik et al, eds.), Walter de Gruyter, Berlin (2000), pp. 95–131.

[11] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.4.12; 2008. (http://www.gap-system.org)

[12] R. D. Horowitz, *Characters of free groups represented in the two-dimensional special linear group*, Comm. Pure Appl. Math. **25** (1972), 635–649.

[13] J. Howie, *The quotient of a free product of groups by a single high-powered relator. I. Pictures. Fifth and higher powers.* Proc. London Math. Soc. (3) **59** (1989), 507–540.

[14] J. Howie, *Free subgroups in groups of small deficiency*, J. Group Theory **1** (1998), 95–112.

[15] J. Howie, *Generalised triangle groups of type* $(3, 3, 2)$, arXiv:1012.2763 (2010).

[16] J. Howie, *Generalised triangle groups of type* $(3, 5, 2)$, arXiv:1102.2073 (2011).

[17] J. Howie and G. Williams, *Free subgroups in certain generalized triangle groups of type* $(2, m, 2)$, Geom. Dedicata **119** (2006), 181–197.

[18] J. Howie and G. Williams, The Tits alternative for generalized triangle groups of type (3,4,2), Algebra and Discrete Mathematics (2008) No. 4, 40-48.

[19] F. Levin and G. Rosenberger, *On free subgroups of generalized triangle groups II*, in: Group theory (Granville, OH, 1992), World Sci. Publ., River Edge, NJ, (1993), pp. 206–228.

[20] G. Rosenberger, *On free subgroups of generalized triangle groups*, Algebra i Logika **28** (1989), 227–240, 245.

[21] A. G. T. Williams, *Studies on generalised triangle groups*, Ph. D. Thesis, Heriot-Watt University (2000).

[22] A. G. T. Williams, *Generalised triangle groups of type* $(2, m, 2)$ in: Computational and Geometric Aspects of Modern Algebra, LMS Lecture Note Series 275, (M. Atkinson et al, eds.), Cambridge University Press (2000), pp. 265-279.

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