Algebra and Discrete Mathematics RESEARCH ARTICLE
Volume 14 **(2012)**. Number 2. pp. 323 – 336

Projective resolution of irreducible modules over tiled order

Viktor Zhuravlev, Dmytro Zhuravlyov

Communicated by V. V. Kirichenko

Abstract. We indicate the method for computing the kernels of projective resolution of irreducible module over tiled order. On the base of this method we construct projective resolution of irreducible module and calculate the global dimension of tiled order. The evident view of kernels of projective resolution allows to check easily the regularity of tiled order.

1. Tiled orders over discrete valuation rings

Recall [2] that a *semimaximal ring* is a semiperfect semiprime right Noetherian ring A such that for each primitive idempotent $e \in A$ the ring eAe is a discrete valuation ring (not necessarily commutative).

Denote by $M_n(B)$ the ring of all $n \times n$ matrices over a ring B.

Theorem 1 (see [2]). *Each semimaximal ring is isomorphic to a finite direct product of prime rings of the following form:*

$$\Lambda = \begin{pmatrix} \mathcal{O} & \pi^{\alpha_{12}}\mathcal{O} & \dots & \pi^{\alpha_{1n}}\mathcal{O} \\ \pi^{\alpha_{21}}\mathcal{O} & \mathcal{O} & \dots & \pi^{\alpha_{2n}}\mathcal{O} \\ \dots & \dots & \dots & \dots \\ \pi^{\alpha_{n1}}\mathcal{O} & \pi^{\alpha_{n2}}\mathcal{O} & \dots & \mathcal{O} \end{pmatrix}, \tag{1}$$

2010 MSC: 16E05, 16G20, 16G10, 16D40.
Key words and phrases: tiled order, projective resolution, distributive module, kernel of epimorphism.

where $n \geq 1$, $\mathcal{O}$ *is a discrete valuation ring with a prime element* π, *and* α_{ij} *are integers such that*

$$\alpha_{ij} + \alpha_{jk} \geq \alpha_{ik}, \qquad \alpha_{ii} = 0$$

for all i, j, k.

The ring $\mathcal{O}$ is embedded into its classical division ring of fractions $\mathcal{D}$, and (1) is the set of all matrices $(a_{ij}) \in M_n(\mathcal{D})$ such that

$$a_{ij} \in \pi^{\alpha_{ij}}\mathcal{O} = e_{ii}\Lambda e_{jj},$$

where $e_{11}, \dots, e_{nn}$ are the matrix units of $M_n(\mathcal{D})$. It is clear that $Q = M_n(\mathcal{D})$ is the classical ring of fractions of Λ.

Obviously, the ring A is right and left Noetherian.

Definition 1. A module M is *distributive* if its lattice of submodules is distributive, i.e.,

$$K \cap (L + N) = K \cap L + K \cap N$$

for all submodules K, L, and N.

Clearly, any submodule and any factormodule of a distributive module are distributive modules.

A *semidistributive module* is a direct sum of distributive modules. A ring A is *right* (*left*) *semidistributive* if it is semidistributive as the right (left) module over itself. A ring A is *semidistributive* if it is both left and right semidistributive (see [9]).

Theorem 2 (see [8]). *The following conditions for a semiperfect semiprime right Noetherian ring* A *are equivalent:*

- A *is semidistributive;*
- A *is a direct product of a semisimple artinian ring and a semimaximal ring.*

By a *tiled order* over a discrete valuation ring, we mean a Noetherian prime semiperfect semidistributive ring Λ with nonzero Jacobson radical. In this case, $\mathcal{O} = e\Lambda e$ is a discrete valuation ring with a primitive idempotent $e \in \Lambda$.

Definition 2. An integer matrix $\mathcal{E} = (\alpha_{ij}) \in M_n(\mathbb{Z})$ is called

- an *exponent matrix* if $\alpha_{ij} + \alpha_{jk} \geq \alpha_{ik}$ and $\alpha_{ii} = 0$ for all i, j, k;
- a *reduced exponent matrix* if $\alpha_{ij} + \alpha_{ji} > 0$ for all i, j, $i \neq j$.

We use the following notation: $\Lambda = \{\mathcal{O}, \mathcal{E}(\Lambda)\}$, where $\mathcal{E}(\Lambda) = (\alpha_{ij})$ is the exponent matrix of the ring Λ, i.e.

$$\Lambda = \sum_{i,j=1}^{n} e_{ij} \pi^{\alpha_{ij}} \mathcal{O},$$

in which e_{ij} are the matrix units. If a tiled order is *reduced*, i.e., $\Lambda/R(\Lambda)$ is the direct product of division rings, then $\alpha_{ij} + \alpha_{ji} > 0$ if $i \neq j$, i.e., $\mathcal{E}(\Lambda)$ is reduced.

We denote by $\mathcal{M}(\Lambda)$ the poset (ordered by inclusion) of all projective right Λ-modules that are contained in a fixed simple Q-module U. All simple Q-modules are isomorphic, so we can choose one of them. Note that the partially ordered sets $\mathcal{M}_l(\Lambda)$ and $\mathcal{M}_r(\Lambda)$ corresponding to the left and the right modules are anti-isomorphic.

The set $\mathcal{M}(\Lambda)$ is completely determined by the exponent matrix $\mathcal{E}(\Lambda) = (\alpha_{ij})$. Namely, if Λ is reduced, then

$$\mathcal{M}(\Lambda) = \{p_i^z \,|\, i = 1, \dots n, \text{and } z \in \mathbb{Z}\},$$

where

$$p_i^z \leq p_j^{z'} \quad \Longleftrightarrow \quad \begin{cases} z - z' \geq \alpha_{ij} & \text{if } \mathcal{M}(\Lambda) = \mathcal{M}_l(\Lambda), \\ z - z' \geq \alpha_{ji} & \text{if } \mathcal{M}(\Lambda) = \mathcal{M}_r(\Lambda). \end{cases}$$

Obviously, $\mathcal{M}(\Lambda)$ is an infinite periodic set.

Let P be an arbitrary poset. A subset of P is called a chain if any two of its elements are related. A subset of P is called a antichain if no two distinct elements of the subset are related.

Definition 3. A right (resp. left) Λ-module M (resp. N) is called a right (resp. left) Λ-lattice if M (resp. N) is a finitely generated free $\mathcal{O}$-module.

Given a tiled order Λ we denote $Lat_r(\Lambda)$ (resp. $Lat_l(\Lambda)$) the category of right (resp. left) Λ-lattices. We denote by $S_r(\Lambda)$ (resp. $S_l(\Lambda)$) the partially ordered by inclusion set, formed by all Λ-lattices contained in a fixed simple $M_n(\mathcal{D})$-module W (resp. in a left simple $M_n(\mathcal{D})$-module V). Such Λ-lattices are called irreducible.

Let $\Lambda = \{\mathcal{O}, \mathcal{E}(\Lambda)\}$ be a tiled order, W (resp. V) is a simple right (resp. left) $M_n(\mathcal{D})$-module with $\mathcal{D}$-basis $e_1, \dots, e_n$ such that $e_i e_{jk} = \delta_{ij} e_k$ ($e_{ij} e_k = \delta_{jk} e_i$).

Then any right (resp. left) irreducible Λ-lattice M (resp. N), lying in W (resp. in V) is a Λ-module with $\mathcal{O}$-basis $(\pi^{\alpha_1} e_1, \dots, \pi^{\alpha_n} e_n)$, while

$$\begin{cases} \alpha_i + \alpha_{ij} \geq \alpha_j, & \text{for the right case;} \\ \alpha_{ij} + \alpha_j \geq \alpha_i, & \text{for the left case.} \end{cases} \tag{2}$$

Thus, irreducible Λ-lattices M can be identified with integer-valued vector $(\alpha_1, \dots, \alpha_n)$ satisfying (2). We shall write $\mathcal{E}(M) = (\alpha_1, \dots, \alpha_n)$ or $M = (\alpha_1, \dots, \alpha_n)$.

The order relation on the set of such vectors and the operations on them corresponding to sum and intersection of irreducible lattices are obvious.

Remark 1. Obviously, irreducible Λ-lattices $M_1 = (\alpha_1, \dots, \alpha_n)$ and $M_2 = (\beta_1, \dots, \beta_n)$ are isomorphic if and only if $\alpha_i = \beta_i + z$ for $i = 1, \dots, n$ and $z \in \mathbb{Z}$.

2. Kernel of epimorphism from direct sum of modules to their sum

Let Λ be the reduced tiled order with the exponent matrix $\mathcal{E}(\Lambda) = (\alpha_{ij})$, M is irreducible right Λ-module and $P(M)$ is its projective cover.

The following statement holds.

Proposition 1 ([10]). *Let $X_1, \dots, X_s$ be the set of all maximal submodules of irreducible and non-projective Λ-module M with $\mathcal{E}(M) = (\alpha_1, \dots, \alpha_n)$ and $\mathcal{E}(X_i) = \mathcal{E}(M) + e_{j_i}$, where $e_k = (\underbrace{0, \dots, 0}_{k-1}, 1, 0, \dots, 0)$.*

Then

$$P(M) = \bigoplus_{i=1}^{s} \pi^{\alpha_{j_i}} P_{j_i} \qquad \text{and} \qquad M = \sum_{i=1}^{s} \pi^{\alpha_{j_i}} P_{j_i}.$$

This statement allows to find easily the projective cover of irreducible module over tiled order.

Theorem 3 ([10]). *Let $M_1, \dots, M_n$ be submodules of distributive module $M = \sum_{i=1}^{n} M_i$ and epimorphism $\varphi\colon \bigoplus_{i=1}^{n} M_i \mapsto M$ operates by the rule*

$\varphi(m_1, \ldots, m_n) = m_1 + \ldots + m_n$. *Then* ker $\varphi = \{(y_1, \ldots, y_n) \mid y_i = \sum\limits_{j \neq i} m_{ij},\ m_{ij} = -m_{ji} \in M_i \cap M_j\}$.

Since the tiled order is a semidistributive ring we have the following corollary.

Corollary 1 ([10]). *Let M be irreducible Λ-module and $P(M) = \bigoplus\limits_{i=1}^{s} \pi^{\alpha_{j_i}} P_{j_i}$, $M = \sum\limits_{i=1}^{s} \pi^{\alpha_{j_i}} P_{j_i}$. Then the kernel of epimorphism $\varphi\colon P(M) \mapsto M$ equals to* ker $\varphi = \{(y_1, \ldots, y_n) \mid y_i = \sum\limits_{k \neq i} m_{ik},\ m_{ik} = -m_{ki} \in \pi^{\alpha_{j_i}} P_{j_i} \cap \pi^{\alpha_{j_k}} P_{j_k}\}$.

The kernel K as the submodule in $\bigoplus\limits_{i=1}^{n} M_i$ can be formally written as

$$K = \sum_{i<j} (M_i \cap M_j)(e_i - e_j), \text{where } e_k = (\underbrace{0, \ldots, 0}_{k-1}, 1, 0, \ldots, 0).$$

3. Distributive equations and the system of distributive equations

Let $\mathcal{O}$ be a discrete valuation ring with unique maximal ideal $\mathfrak{m} = \pi\mathcal{O}$, where π is a prime element of a ring. $\mathcal{O}$, $F = \mathcal{O}/\pi\mathcal{O}$ — is a skew field, $\Lambda = \{\mathcal{O}, \mathcal{E}(\Lambda) = (\alpha_{ij})\}$ — is a reduced tiled order over a discrete valuation ring $\mathcal{O}$ with exponent matrix $\mathcal{E}(\Lambda) = (\alpha_{ij}) \in M_n(Z)$.

The equation of the form $\sum\limits_{i=1}^{s} a_i m_i = 0$, where $a_i \in F$, $m_i \in M_i$, $M_1, \ldots, M_s$ are the submodules of the distributive module M, we will call distributive.

Let $M_1, \ldots, M_s$ be the submodules of distributive module $M = \sum\limits_{i=1}^{s} M_i$ and epimorphism $\varphi\colon \bigoplus\limits_{i=1}^{s} M_i \to M$ operates by the rule $\varphi(m_1, \ldots, m_s) = m_1 + \cdots + m_s$. Then by the theorem 3 ker $\varphi = \{(y_1, \ldots, y_s) \in \bigoplus\limits_{i=1}^{s} M_i \mid y_i = -m_{1i} - \cdots - m_{i-1i} + m_{ii+1} + \cdots + m_{is}$, where $m_{ij} \in M_i \cap M_j, i < j\}$.

In other words, distributive equation

$$m_1 + \cdots + m_s = 0, \tag{3}$$

where $m_i \in M_i$ $i = 1, \dots, s$, $M_1, \dots, M_s$ are the submodules of the distributive module M, has solution

$$
\begin{aligned}
&m_1 = m_{12} + \cdots + m_{1s},\\
&m_2 = -m_{12} + m_{23} + \cdots + m_{2s},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&m_i = -m_{1i} - \cdots - m_{i-1i} + m_{ii+1} + \cdots + m_{is},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&m_s = -m_{1s} - \cdots - m_{s-1s}.
\end{aligned}
\tag{4}
$$

Distributive equation

$$a_1 m_1 + \cdots + a_s m_s = 0 \tag{5}$$

where $a_i \in F$, $m_i \in M_i$, $a_i \neq 0$ for all $i = 1, \dots s$ by replacement $a_i m_i = z_i \in M_i$ reduces to the equation (3) and then

$$
\begin{aligned}
&a_1 m_1 = m_{12} + \cdots + m_{1s},\\
&a_2 m_2 = -m_{12} + m_{23} + \cdots + m_{2s},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&a_i m_i = -m_{1i} - \cdots - m_{i-1i} + m_{ii+1} + \cdots + m_{is},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&a_s m_s = -m_{1s} - \cdots - m_{s-1s}.
\end{aligned}
\tag{6}
$$

where $m_{ij} \in M_i \cap M_j$ for all i, j $i \neq j$. Hence

$$
\begin{aligned}
&m_1 = a_1^{-1}(m_{12} + \cdots + m_{1s}),\\
&m_2 = a_2^{-1}(-m_{12} + m_{23} + \cdots + m_{2s}),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&m_i = a_i^{-1}(-m_{1i} - \cdots - m_{i-1i} + m_{ii+1} + \cdots + m_{is}),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&m_s = a_s^{-1}(-m_{1s} - \cdots - m_{s-1s}).
\end{aligned}
\tag{7}
$$

Let now consider the system of distributive equations.

$$
\begin{aligned}
&a_{11} m_1 + \cdots + a_{1s} m_s = 0,\\
&a_{21} m_1 + \cdots + a_{2s} m_s = 0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&a_{t1} m_1 + \cdots + a_{ts} m_s = 0,
\end{aligned}
\tag{8}
$$

where $m_i \in M_i$ for all i, $M_1, \dots, M_s$ are the submodules of the distributive module M, $a_{ij} \in F$ for all i, j.

Denote $A = (a_{ij}) \in M_{t\times s}(F)$. We can consider that vectors which are the rows of the matrix A are linearly independent. Indeed, if

$$(a_{t1}, \dots, a_{ts}) = a_1(a_{11}, \dots, a_{1s}) + \dots + a_{t-1}(a_{t-11}, \dots, a_{t-1s}),$$

then

$$a_{t1}m_1 + \dots + a_{ts}m_s = \\ = a_1(a_{11}m_1 + \cdots a_{1s}m_s) + \dots + a_{t-1}(a_{t-11}m_1 + \dots + a_{t-1s}m_s).$$

Thus the last equation is a corollary of previous equations. Therefore $t \leq s$. When $t = s$ are the vectors which are the rows of the matrix A and construct the basis of space F^s. Then

$$\begin{array}{l} (1, 0, \dots, 0) = c_{11}(a_{11}, \dots, a_{1s}) + \dots + c_{1s}(a_{s1}, \dots, a_{ss}), \\ (0, 1, \dots, 0) = c_{21}(a_{11}, \dots, a_{1s}) + \dots + c_{2s}(a_{s1}, \dots, a_{ss}), \\ \dots\dots\dots\dots\dots\dots\dots \\ (0, 0, \dots, 1) = c_{s1}(a_{11}, \dots, a_{1s}) + \dots + c_{ss}(a_{s1}, \dots, a_{ss}). \end{array} \tag{9}$$

where $c_{ij} \in F$ for all i, j. Hence

$$\begin{array}{l} m_1 = c_{11}(a_{11}m_1 + \ldots + a_{1s}m_s) + \dots + c_{1s}(a_{s1}m_1 + \ldots + a_{ss}m_s), \\ m_2 = c_{21}(a_{11}m_1 + \ldots + a_{1s}m_s) + \dots + c_{2s}(a_{s1}m_1 + \ldots + a_{ss}m_s), \\ \dots\dots\dots\dots\dots\dots\dots \\ m_s = c_{s1}(a_{11}m_1 + \ldots + a_{1s}m_s) + \dots + c_{ss}(a_{s1}m_1 + \ldots + a_{ss}m_s). \end{array} \tag{10}$$

Thus, for existence of nonzero solution of the system (8) it is needed rank $A < s$, that is $t < s$.

Obviously, the solution of the system (8) depends on the set $M_1, \dots, M_s$. Denote $\overline{M} = (M_1, \dots, M_s)$. Thus the system (8) uniquely is determined by the matrix A and the set of the irreducible modules $M_1, \dots, M_s$ and we will denote it by $(A, \overline{M})$.

Consider the method of solving the system of distributive equations $(A, \overline{M})$. Determine the sequence of systems $(A_1, \overline{M}_1)$, $(A_2, \overline{M}_2)$, …, $(A_t, \overline{M}_t)$ inductively. Notably, let $A_1 = A$, $\overline{M}_1 = \overline{M}$. Let we have the system $(A_k, \overline{M}_k)$, where $A_k = (a_{ij}^{(k)})$ — is rectangular matrix of size $(t + 1 - k) \times v_k$, $\overline{M}_k = (M_1^{(k)}, \dots, M_{v_k}^{(k)})$.

Let in the first row of the matrix A_k the elements $a_{1j_1}^{(k)}, a_{1j_2}^{(k)}, \dots, a_{1j_{l_k}}^{(k)}$ are not equal to 0, and all other elements are equal to 0. That is the first equality of the system is of the form

$$a_{1j_1}^{(k)} m_{j_1}^{(k)} + \dots + a_{1j_{l_k}}^{(k)} m_{j_{l_k}}^{(k)} = 0.$$

Then by (7)

$$\begin{aligned} m_{j_1}^{(k)} &= (a_{1j_1}^{(k)})^{-1}(m_{j_1j_2}^{(k)} + \cdots + m_{j_1j_{l_k}}^{(k)}), \\ m_{j_2}^{(k)} &= (a_{1j_2}^{(k)})^{-1}(-m_{j_1j_2}^{(k)} + \cdots + m_{j_2j_{l_k}}^{(k)}), \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ m_{j_{l_k}}^{(k)} &= (a_{1j_{l_k}}^{(k)})^{-1}(-m_{j_1j_2}^{(k)} - \cdots - m_{j_{l_k-1}j_{l_k}}^{(k)}). \end{aligned} \tag{11}$$

Substituting these expressions for $m_{j_l}^{(k)}$ when $l = 1, \ldots, l_k$ in other equalities of the system $(A_k, \overline{M}_k)$, we will have the system of $t-k$ equalities

$$\sum_{r=1}^{l_k} a_{ij_r}^{(k)}(a_{1j_r}^{(k)})^{-1}(-m_{j_1j_r}^{(k)} - \cdots - m_{j_{r-1}j_r}^{(k)} + m_{j_rj_{r+1}}^{(k)} + \cdots + m_{j_rj_{l_k}}^{(k)}) + \\ + \sum_{j \neq j_1, \ldots, j_l} a_{ij}^{(k)} m_j^{(k)} = 0$$

$i = 2, \ldots, t+1-k$. This is the system of $C_{l_k}^2 + (v_k - l_k) = v_{k+1}$ unknowns: $m_{j_xj_y}^{(k)}$ $(x \neq y,\ x, y = 1, \ldots, l_k)$ and $m_j^{(k)}$ $(j \neq j_r,\ r = 1, \ldots, l_k)$, where $m_{j_xj_y}^{(k)} \in M_{j_x}^{(k)} \cap M_{j_y}^{(k)}$.

Let

$$\overline{M}_{k+1} = \{M_{j_x}^{(k)} \cap M_{j_y}^{(k)}, x \neq y, x, y = 1, \ldots, l_k\} \cup \\ \cup \{M_j^{(k)}, 1 \leq j \leq V_k, j \neq j_1, \ldots, j_{l_k}\}.$$

Obtained rectangular matrix of the size $(t-k) \times v_{k+1}$ from unknowns $m_{j_xj_y}^{(k)}$, $x \neq y$, $x, y = 1, \ldots, l_k$, $m_j^{(k)}$ and $m_j^{(k)}$, $j \neq j_r$, $r = 1, \ldots, l_k$ we denote by A_{k+1}. Thus, we have obtained the system of distributive equations $(A_{k+1}, \overline{M}_{k+1})$.

Solving only one equation from each system $(A_1, \overline{M}_1)$, $(A_2, \overline{M}_2)$, …, $(A_{t-1}, \overline{M}_{t-1})$, we will obtain the system $(A_t, \overline{M}_t)$ of one distributive equation, which we will obtain also by the following formulas (7). We will have the $(t-1)$-th union of relations (11) for $k = 1, \ldots, t-1$ between old unknowns of the system $(A_k, \overline{M}_k)$ and innovated unknowns of the system $(A_{k+1}, \overline{M}_{k+1})$. Using these relations in inverse order, firstly when $k = t-1$, then when $k = t-2$ and so forth, we will obtain the solution of the system $(A_1, \overline{M}_1) = (A, \overline{M})$.

Note that since module $\overline{M}_{k+1}$ either coincides with the module $\overline{M}_k$ or is the intersection of two modules from $\overline{M}_k$, and the solution of the system of distributive equations (8) is of the form

$$\begin{aligned} &m_1 = b_{11}x_1 + b_{12}x_2 + \cdots + b_{1r}x_r,\\ &m_2 = b_{21}x_1 + b_{22}x_2 + \cdots + b_{2r}x_r,\\ &\dots\dots\dots\dots\dots\dots\dots\\ &m_s = b_{s1}x_1 + b_{s2}x_2 + \cdots + b_{sr}x_r. \end{aligned} \tag{12}$$

where $b_{ij} \in F$, $x_i \in X_i$, X_i — is the intersection of not more than the $(t+1)$'s module from the set $M_1, M_2, \dots, M_s$.

Proposition 2. *The solution (12) of the system of distributive equations (8), which is obtained by mentioned method, is the general solution of this system.*

Proof. The proof will be made with the help of induction by the number of equations of the system(8). The base of the induction is $t = 1$. The solution of distributive equation (5) is of the form (7) and is the general solution of the equation by the theorem 3.

Let we have the system (8) of t equations and let the solution of the system of $(t-1)$ distributive equations, which is obtained by the mentioned method, be the general solution of the system of distributive equations.

Let $a_{11}, \dots, a_{1l}$ in the system (8) be the elements of the first row, which are not equal to 0, $a_{1l+1} = \cdots = a_{1s} = 0$. The the system (8) subject to the solution of the first equation from the system is equivalent to the system

$$\begin{aligned} &m_1 = a_{11}^{-1}(m_{12} + \cdots + m_{1l}),\\ &m_2 = a_{12}^{-1}(-m_{12} + m_{23} + \cdots + m_{2l}),\\ &\dots\dots\dots\dots\dots\dots\dots\\ &m_i = a_{1i}^{-1}(-m_{1i} - \cdots - m_{i-1i} + m_{ii+1} + \cdots + m_{il}),\\ &\dots\dots\dots\dots\dots\dots\dots\\ &m_l = a_{1l}^{-1}(-m_{1l} - \cdots - m_{l-1l}). \end{aligned} \tag{13}$$

$$\begin{aligned} &a_{21}a_{11}^{-1}\sum_{k=2}^{l} m_{1k} + \cdots + a_{2l}a_{1l}^{-1}\sum_{k=1}^{l-1} m_{kl} + \sum_{k=l+1}^{s} a_{2k}m_k = 0,\\ &\dots\dots\dots\dots\dots\dots\dots\\ &a_{t1}a_{11}^{-1}\sum_{k=2}^{l} m_{1k} + \cdots + a_{tl}a_{1l}^{-1}\sum_{k=1}^{l-1} m_{kl} + \sum_{k=l+1}^{s} a_{tk}m_k = 0. \end{aligned} \tag{14}$$

From (13) it follows that $m_1 + \cdots + m_l = 0$, and it is the first equation of the system (8). By assumption of induction the solution of the system (14) is a general

$$\begin{array}{l} m_{12} = b_{11}x_1 + b_{12}x_2 + \cdots + b_{1r}x_r, \\ \dots\dots\dots\dots\dots\dots\dots \\ m_{l-1l} = b_{u1}x_1 + b_{u2}x_2 + \cdots + b_{ur}x_r, \\ m_{l+1} = b_{u+11}x_1 + b_{u+12}x_2 + \cdots + b_{u+1r}x_r, \\ \dots\dots\dots\dots\dots\dots\dots \\ m_s = b_{u+s-l1}x_1 + b_{u+s-l2}x_2 + \cdots + b_{u+s-lr}x_r. \end{array} \tag{15}$$

where $u = C_l^2$, $b_{ij} \in F$, $x_k \in X_k$, $k = 1, \dots, r$, X_k — is the intersection of not more than t modules from the set $M_1, \dots, M_s$.

Then the expression for $m_1, \dots, m_l$ we obtain from (13) and (15). Moreover, from the generality of the solution (15) and the unambiguity of expressions for $m_1, \dots, m_l$ in (13) we will obtain the generality of the system's solution (13) – (14). From the equivalence of the systems (13) – (14) and (8) we obtain that the solution of the system (8), which is obtained by the mentioned method, is the general. The proposition is proved. □

The solution (12) of the system of distributive equations (8) we will write as

$$X_1 \begin{pmatrix} b_{11} \\ \vdots \\ b_{s1} \end{pmatrix} + X_r \begin{pmatrix} b_{1r} \\ \vdots \\ b_{sr} \end{pmatrix}.$$

Remark 2. The form of the solution (12) depends on the set of equations which are solved from each system $(A_k, \overline{M}_k)$.

If $Y_1 \begin{pmatrix} c_{11} \\ \vdots \\ c_{s1} \end{pmatrix} + Y_p \begin{pmatrix} c_{1p} \\ \vdots \\ c_{sp} \end{pmatrix}$ is the another solution of the system (12), then from the generality of the solutions we have

$$X_1 \begin{pmatrix} b_{11} \\ \vdots \\ b_{s1} \end{pmatrix} + X_r \begin{pmatrix} b_{1r} \\ \vdots \\ b_{sr} \end{pmatrix} = Y_1 \begin{pmatrix} c_{11} \\ \vdots \\ c_{s1} \end{pmatrix} + Y_p \begin{pmatrix} c_{1p} \\ \vdots \\ c_{sp} \end{pmatrix}.$$

The set of irreducible modules $\{\overline{M}_1, \dots, \overline{M}_v\}$ constructs the partially ordered set by inclusion.

Let us consider the cases when the expression for the set of system's solutions can be simplified.

- Let the module $\overline{M}_1$ be the submodule of modules $\overline{M}_{i_1},\dots,\overline{M}_{i_z}$ and vector $\bar{b}_1$ is linearly expressed over F through vectors $\bar{b}_{i_1},\dots,\bar{b}_{i_z}$: $\bar{b}_1=\alpha_1\bar{b}_{i_1}+\dots+\alpha_z\bar{b}_{i_z}$, where $\alpha_j\in F$. Then

$$\begin{aligned}&\bar{b}_1\overline{M}_1+\bar{b}_{i_1}\overline{M}_{i_1}+\dots+\bar{b}_{i_z}\overline{M}_{i_z}=\\&\quad=(\alpha_1\bar{b}_{i_1}+\dots+\alpha_z\bar{b}_{i_z})\overline{M}_1+\bar{b}_{i_1}\overline{M}_{i_1}+\dots+\bar{b}_{i_z}\overline{M}_{i_z}=\\&=\bar{b}_{i_1}(\alpha_1\overline{M}_1+\overline{M}_{i_1})+\bar{b}_{i_2}(\alpha_2\overline{M}_1+\overline{M}_{i_2})+\dots+\bar{b}_{i_z}(\alpha_z\overline{M}_1+\overline{M}_{i_z})=\\&\qquad=\bar{b}_{i_1}\overline{M}_{i_1}+\bar{b}_{i_2}\overline{M}_{i_2}+\dots+\bar{b}_{i_z}\overline{M}_{i_z}.\end{aligned}$$

- Let the vectors $\bar{b}_i$ and $\bar{b}_j$ be collinear and not equal to zero. Then

$$\bar{b}_i\overline{M}_i+\bar{b}_j\overline{M}_j=\bar{b}_i\overline{M}_i+\lambda\bar{b}_i\overline{M}_j=\bar{b}_i(\overline{M}_i+\lambda\overline{M}_j)=\bar{b}_i(\overline{M}_i+\overline{M}_j).$$

4. The construction of projective resolution of irreducible modules over tiled order

Let $X_1,\dots,X_s$ be the set of all maximal submodules of irreducible and non-projective Λ-module M with $\mathcal{E}(M)=(\alpha_1,\dots,\alpha_n)$ and $\mathcal{E}(X_i)=\mathcal{E}(M)+e_{j_i}$, where $e_k=(\underbrace{0,\dots,0}_{k-1},1,0,\dots,0)$. Then by the proposition 1

$$P(M)=\bigoplus_{i=1}^{s}\pi^{\alpha_{j_i}}P_{j_i}\qquad\text{and}\qquad M=\sum_{i=1}^{s}\pi^{\alpha_{j_i}}P_{j_i}.$$

By the corollary 1 the kernel of epimorphism $\varphi\colon P(M)\mapsto M$ equals to $\ker\varphi=\{(y_1,\dots,y_n)\mid y_i=\sum\limits_{k\neq i}m_{ik},\ m_{ik}=-m_{ki}\in P_{j_i}\cap P_{j_k}\}$.

The kernel K_1 as submodule in $\bigoplus\limits_{i=1}^{n}M_i$ can be formally written as

$$K_1=\sum_{i<j}(M_i\cap M_j)(e_i-e_j),\text{where } e_k=(\underbrace{0,\dots,0}_{k-1},1,0,\dots,0).$$

Let the n-th kernel of projective resolution of irreducible module be of the form $K_n=\sum\limits_{i=1}^{s}M_i\overline{f}_i$, where M_i — are irreducible modules and $P(M_i)=\bigoplus\limits_{k=1}^{l_i}P_{j_k}^{(i)}$.

Obviously, there is the epimorphism $\psi : \bigoplus_{k=1}^{l_i} P(M_i) \to K_n$, which operates by the formula

$$\psi(m_1^1, \ldots, m_{l_1}^1, m_1^2, \ldots, m_{l_2}^2, \ldots, m_1^s, \ldots, m_{l_s}^s) = \\ = (m_1^1 + \ldots + m_{l_1}^1)\overline{f}_1 + \ldots + (m_1^s + \ldots + m_{l_s}^s)\overline{f}_s.$$

Proposition 3. *Let $M_1, \ldots, M_s$ be Λ-modules, $P(M_i) = \bigoplus_{k=1}^{l_i} P_{j_k}^{(i)}$ be the projective cover of module M_i and the epimorphism $\psi : \bigoplus_{k=1}^{l_i} P(M_i) \to K_n$ operates by the rule*

$$\psi(m_1^1, \ldots, m_{l_1}^1, m_1^2, \ldots, m_{l_2}^2, \ldots, m_1^s, \ldots, m_{l_s}^s) = \\ = (m_1^1 + \ldots + m_{l_1}^1)\overline{f}_1 + \ldots + (m_1^s + \ldots + m_{l_s}^s)\overline{f}_s.$$

If projective module $P_1^{(1)}$ is the submodule of projective modules $P_1^{(2)}, P_1^{(3)}, \ldots, P_1^{(t)}$ and $\overline{f}_1 = \alpha_2\overline{f}_2 + \alpha_3\overline{f}_3 + \cdots + \alpha_t\overline{f}_t$, where $\alpha_i \in F$, then there exists the epimorphism $\overline{\psi} : \left(\bigoplus_{k=2}^{l_1} P_k^{(1)}\right) \oplus \left(\bigoplus_{i=2}^{s} P(M_i)\right) \to K_n$, which operates by the rule

$$\overline{\psi}(m_2^1, \ldots, m_{l_1}^1, m_1^2, \ldots, m_{l_2}^2, \ldots, m_1^s, \ldots, m_{l_s}^s) = \\ = (m_2^1 + \ldots + m_{l_1}^1)\overline{f}_1 + \ldots + (m_1^s + \ldots + m_{l_s}^s)\overline{f}_s.$$

Proof. Let write down the operation of the epimorphism ψ in the form

$$\psi(m_1^1, \ldots, m_{l_s}^s) = \\ = m_1^1\overline{f}_1 + (m_2^1 + \ldots + m_{l_1}^1)\overline{f}_1 + (m_1^2 + \ldots + m_{l_2}^s)\overline{f}_2 \ldots + (m_1^s + \ldots + m_{l_s}^s)\overline{f}_s = \\ = m_1^1(\alpha_2\overline{f}_2 + \ldots + \alpha_t\overline{f}_t) + (m_2^1 + \ldots + m_{l_1}^1)\overline{f}_1 + \ldots + (m_1^s + \ldots + m_{l_s}^s)\overline{f}_s = \\ = (m_2^1 + \ldots + m_{l_1}^1)\overline{f}_1 + \left((m_1^2 + \alpha_2 m_1^1) + m_2^2 + \ldots + m_{l_2}^2\right)\overline{f}_2 + \ldots \\ + \left((m_1^t + \alpha_t m_1^1) + m_2^t + \ldots + m_{l_t}^t\right)\overline{f}_t + (m_1^{t+1} + \ldots + m_{l_{t+1}}^{t+1})\overline{f}_{t+1} + \ldots \\ + (m_1^s + \ldots + m_{l_s}^s)\overline{f}_s.$$

Since $P_1^{(1)} \subseteq P_1^{(k)}$ for all $k = 2, \ldots, t$, then $m_1^k + \alpha_k m_1^1 = \overline{m}_1^k$ for all $k = 2, \ldots, t$, where $\overline{m}_1^k \in P_1^{(k)}$.

Thus,

$$\psi(m_1^1,\dots,m_{l_s}^s) = (m_2^1+\dots+m_{l_1}^1)\overline{f}_1+$$
$$+(\overline{m}_1^2+m_2^2+\dots+m_{l_2}^2)\overline{f}_2+\dots+(\overline{m}_1^t+m_2^t+\dots+m_{l_t}^t)\overline{f}_t+$$
$$+(m_1^{t+1}+\dots+m_{l_{t+1}}^{t+1})\overline{f}_{t+1}+\dots+(m_1^s+\dots+m_{l_s}^s)\overline{f}_s=$$
$$=\overline{\psi}\left(m_2^1,\dots,m_{l_1}^1,\overline{m}_1^2,m_2^2,\dots,m_{l_2}^2,\overline{m}_1^3,\dots,m_{l_3}^3,\dots,\right.$$
$$\left.\dots,\overline{m}_1^t,m_2^t,m_{l_t}^t,m_1^{t+1},\dots,m_{l_{t+1}}^{t+1},\dots,m_1^s,m_{l_s}^s\right). \quad \square$$

Corollary 2. *In conditions of proposition 3 projective module $P_1^{(1)}$ isn't included in projective cover of module $P(K_n)$.*

By the kernel K_n we construct the epimorphism $\psi : \bigoplus_{k=1}^{l_i} P(M_i) \to K_n$, which operates by the rule

$$\psi(m_1^1,\dots,m_{l_1}^1,m_1^2,\dots,m_{l_2}^2,\dots,m_1^s,\dots,m_{l_s}^s) =$$
$$=(m_1^1+\ldots+m_{l_1}^1)\overline{f}_1+\ldots+(m_1^s+\ldots+m_{l_s}^s)\overline{f}_s.$$

Using proposition 3 by the epimorphism ψ we construct the epimorphism $\overline{\psi}$ with the minimal number of direct summand. For the kernel K_{n+1} we obtain the system of distributive equations. This system is solved by the method given in the section 3. Hereby, we construct the projective resolution of irreducible module, indicating not only projective modules but also all intermediate kernels.

Conclusion

The results obtained in sections 3 and 4 allow to construct projective resolutions of irreducible modules over tiled order and calculate the global dimension of order. There is the program, which is written in Java programming language, on the basis of performed researches. It allows to calculate projective resolutions of irreducible modules over tiled order of any finite length.

Specifying the kernels of resolution in explicit form allows to determine easily whether the global dimension of tiled order depends on characteristic of skew field, i.e. whether the order is regular.

References

[1] Yu.A. Drozd, V.V. Kirichenko, Finite Dimensional Algebras. Springer-Verlag, Berlin-Heidelberg-New York 1994.

[2] A.G. Zavadskij and V.V. Kirichenko, *Torsion-free Modules over Prime Rings*, Zap. Nauch. Seminar. Leningrad. Otdel. Mat. Steklov. Inst. (LOMI) - 1976. - v. 57. - p. 100-116 (in Russian). English translation in J. of Soviet Math., v. 11, N 4, April 1979, p. 598-612.

[3] Zh. T. Chernousova, M. A. Dokuchaev, M. A. Khibina, V. V. Kirichenko, S. G. Miroshnichenko, and V. N. Zhuravlev, *Tiled orders over discrete valuation rings, finite Markov chains and partially ordered sets. I.*, Algebra and Discrete Math. **1** (2002) 32–63.

[4] Zh. T. Chernousova, M. A. Dokuchaev, M. A. Khibina, V. V. Kirichenko, S. G. Miroshnichenko, and V. N. Zhuravlev, *Tiled orders over discrete valuation rings, finite Markov chains and partially ordered sets. II.* Algebra and Discrete Math. **2** (no. 2) (2003) 47–86.

[5] M. Hazewinkel, N. Gubareni and V. V. Kirichenko, *Algebras, Rings and Modules.* Vol. 1, Series: Mathematics and Its Applications, **575**, Kluwer Acad. Publish., 2004. xii+380pp.

[6] M. Hazewinkel, N. Gubareni and V. V. Kirichenko, *Algebras, Rings and Modules.* Vol. 2, Series: Mathematics and Its Applications, **586**. Springer, Dordrecht, 2007. xii+400pp.

[7] V. V. Kirichenko, A. V. Zelensky, V. N. Zhuravlev, *Exponent matrices and tiled orders over discrete valuation rings*, Algebra and Computation **Vol. 15** No 5 & 6 (2005) 997–1012.

[8] V. V. Kirichenko and M. A. Khibina, *Semi-perfect semi-distributive rings*, In: Infinite Groups and Related Algebraic Topics, Institute of Mathematics NAS Ukraine, 1993, pp. 457–480 (in Russian).

[9] A. A. Tuganbaev, *Semidistributive modules and rings*, Kluwer Acad. Publ., Dordrecht, 1998.

[10] V. Zhuravlev, D. Zhuravlyov, *Tiled orders of width* 3, Algebra and Discrete Math. **1** (2009) 111–123.

Contact information

V. Zhuravlev
Department of Mechanics and Mathematics, Taras Shevchenko National University of Kyiv, Volodymyrska, 64, Kyiv 01033, Ukraine
E-Mail: vshur@univ.kiev.ua

D. Zhuravlyov
JSCB “Industrialbank”, Processing Center, Kutuzova street, 18/7, 01133 Kyiv, Ukraine
E-Mail: dzhuravlev@ukr.net

Received by the editors: 12.08.2012
and in final form 12.08.2012.