

On radical square zero rings

Claus Michael Ringel and Bao-Lin Xiong

Communicated by Yu. A. Drozd

Dedicated to the memory of Andrei Roiter

ABSTRACT. Let Λ be a connected left artinian ring with radical square zero and with n simple modules. If Λ is not self-injective, then we show that any module M with $\text{Ext}^i(M, \Lambda) = 0$ for $1 \leq i \leq n + 1$ is projective. We also determine the structure of the artin algebras with radical square zero and n simple modules which have a non-projective module M such that $\text{Ext}^i(M, \Lambda) = 0$ for $1 \leq i \leq n$.

Xiao-Wu Chen [C] has recently shown: given a connected artin algebra Λ with radical square zero then either Λ is self-injective or else any CM module is projective. Here we extend this result by showing: If Λ is a connected artin algebra with radical square zero and n simple modules then either Λ is self-injective or else any module M with $\text{Ext}^i(M, \Lambda) = 0$ for $1 \leq i \leq n + 1$ is projective. Actually, we will not need the assumption on Λ to be an artin algebra; it is sufficient to assume that Λ is a left artinian ring. And we show that for artin algebras the bound $n + 1$ is optimal by determining the structure of those artin algebras with radical square zero and n simple modules which have a non-projective module M such that $\text{Ext}^i(M, \Lambda) = 0$ for $1 \leq i \leq n$.

From now on, let Λ be a left artinian ring with radical square zero, this means that Λ has an ideal I with $I^2 = 0$ (the radical) such that

2010 MSC: 16D90, 16G10; 16G70.

Key words and phrases: Artin algebras; left artinian rings; representations, modules; Gorenstein modules, CM modules; self-injective algebras; radical square zero algebras.

Λ/I is semisimple artinian. We also assume that Λ is connected (the only central idempotents are 0 and 1). The modules to be considered are usually finitely generated left Λ -modules. Let n be the number of (isomorphism classes of) simple modules.

Given a module M , we denote by PM a projective cover, by QM an injective envelope of M . Also, we denote by ΩM a syzygy module for M , this is the kernel of a projective cover $PM \rightarrow M$. Since Λ is a ring with radical square zero, all the syzygy modules are semisimple. Inductively, we define $\Omega_0 M = M$, and $\Omega_{i+1} M = \Omega(\Omega_i M)$ for $i \geq 0$.

Lemma 1. *If M is a non-projective module with $\text{Ext}^i(M, \Lambda) = 0$ for $1 \leq i \leq d + 1$ (and $d \geq 1$), then there exists a simple non-projective module S with $\text{Ext}^i(S, \Lambda) = 0$ for $1 \leq i \leq d$.*

Proof. The proof is obvious: We have $\text{Ext}^i(M, \Lambda) \simeq \text{Ext}^{i-1}(\Omega M, \Lambda)$, for all $i \geq 2$. Since M is not projective, $\Omega M \neq 0$. Now ΩM is semisimple. If all simple direct summands of ΩM are projective, then also ΩM is projective, but then the condition $\text{Ext}^1(M, \Lambda) = 0$ implies that $\text{Ext}^1(M, \Omega M) = 0$ in contrast to the existence of the exact sequence $0 \rightarrow \Omega M \rightarrow PM \rightarrow M \rightarrow 0$. Thus, let S be a non-projective simple direct summand of ΩM . \square

Lemma 2. *If S is a non-projective simple module with $\text{Ext}^1(S, \Lambda) = 0$, then PS is injective and ΩS is simple and not projective.*

Proof. First, we show that PS has length 2. Otherwise, ΩS is of length at least 2, thus there is a proper decomposition $\Omega S = U \oplus U'$ and then there is a canonical exact sequence

$$0 \rightarrow PS \rightarrow PS/U \oplus PS/U' \rightarrow S \rightarrow 0,$$

which of course does not split. But since $\text{Ext}^1(S, \Lambda) = 0$, we have $\text{Ext}^1(S, P) = 0$, for any projective module P . Thus, we obtain a contradiction.

This shows also that ΩS is simple. Of course, ΩS cannot be projective, again according to the assumption that $\text{Ext}^1(S, P) = 0$, for any projective module P .

Now let us consider the injective envelope Q of ΩS . It contains PS as a submodule (since PS has ΩS as socle). Assume that Q is of length at least 3. Take a submodule I of Q of length 2 which is different from PS

and let $V = PS + I$, this is a submodule of Q of length 3. Thus, there are the following inclusion maps u_1, u_2, v_1, v_2 :

$$\begin{array}{ccc} \Omega S & \xrightarrow{u_1} & PS \\ v_1 \downarrow & & \downarrow u_2 \\ I & \xrightarrow{v_2} & V \end{array}$$

The projective cover $p : PI \rightarrow I$ has as restriction a surjective map $p' : \text{rad } PI \rightarrow \Omega S$. But $\text{rad } PI$ is semisimple, thus p' is a split epimorphism, thus we obtain a map $w : \Omega S \rightarrow PI$ such that $pw = v_1$. We consider the exact sequence induced from the sequence $0 \rightarrow \Omega S \rightarrow PS \rightarrow S \rightarrow 0$ by the map w :

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega S & \xrightarrow{u_1} & PS & \xrightarrow{e_1} & S \longrightarrow 0 \\ & & w \downarrow & & \downarrow w' & & \parallel \\ 0 & \longrightarrow & PI & \xrightarrow{u'_1} & N & \xrightarrow{e'_1} & S \longrightarrow 0 \end{array}$$

Here, N is the pushout of the two maps u_1 and w . Since we know that $u_2u_1 = v_2v_1 = v_2pw$, there is a map $f : N \rightarrow V$ such that $fu'_1 = v_2p$ and $fw' = u_2$. Since the map $\begin{bmatrix} v_2p & u_2 \end{bmatrix} : PI \oplus PS \rightarrow V$ is surjective, also f is surjective.

But recall that we assume that $\text{Ext}^1(S, \Lambda) = 0$, thus $\text{Ext}^1(S, PI) = 0$. This means that the lower exact sequence splits and therefore the socle of $N = PI \oplus S$ is a maximal submodule of N (since I is a local module, also PI is a local module). Now f maps the socle of N into the socle of V , thus it maps a maximal submodule of N into a simple submodule of V . This implies that the image of f has length at most 2, thus f cannot be surjective. This contradiction shows that Q has to be of length 2, thus $Q = PS$ and therefore PS is injective. □

Lemma 3. *If S is a non-projective simple module with $\text{Ext}^i(S, \Lambda) = 0$ for $1 \leq i \leq d$, then the modules $S_i = \Omega_i S$ with $0 \leq i \leq d$ are simple and not projective, and the modules $P(S_i)$ are injective for $0 \leq i < d$.*

Proof. The proof is again obvious, we use induction. If $d \geq 2$, we know by induction that the modules S_i with $0 \leq i \leq d - 1$ are simple and not projective, and that the modules $P(S_i)$ are injective for $0 \leq i < d - 1$.

But $\text{Ext}^1(\Omega_{d-1}S, \Lambda) \simeq \text{Ext}^d(S, \Lambda) = 0$, thus Lemma 2 asserts that also S_d is simple and not projective and that $P(S_{d-1})$ is injective. \square

Lemma 4. *Let S_0, S_1, \dots, S_b be simple modules with $S_i = \Omega_i(S_0)$ for $1 \leq i \leq b$. Assume that there is an integer $0 \leq a < b$ such that the modules S_i with $a \leq i < b$ are pairwise non-isomorphic, whereas S_b is isomorphic to S_a . In addition, we assume that the modules $P(S_i)$ for $a \leq i < b$ are injective. Then S_a, \dots, S_{b-1} is the list of all the simple modules and Λ is self-injective.*

Proof. Let \mathcal{S} be the subcategory of all modules with composition factors of the form S_i , where $a \leq i < b$. We claim that this subcategory is closed under projective covers and injective envelopes. Indeed, the projective cover of S_i for $a \leq i < b$ has the composition factors S_i and S_{i+1} (and $S_b = S_a$), thus is in \mathcal{S} . Similarly, the injective envelope for S_i with $a < i < b$ is $Q(S_i) = P(S_{i-1})$, thus it has the composition factors S_{i-1} and S_i , and $Q(S_a) = Q(S_b) = P(S_{b-1})$ has the composition factors S_{b-1} and S_a . Since we assume that Λ is connected, we know that the only non-trivial subcategory closed under composition factors, extensions, projective covers and injective envelopes is the module category itself. This shows that S_a, \dots, S_{b-1} are all the simple modules. Since the projective cover of any simple module is injective, Λ is self-injective. \square

Theorem 1. *Let Λ be a connected left artinian ring with radical square zero. Assume that Λ is not self-injective. If S is a non-projective simple module such that $\text{Ext}^i(S, \Lambda) = 0$ for $1 \leq i \leq d$, then the modules $S_i = \Omega_i S$ with $0 \leq i \leq d$ are pairwise non-isomorphic simple and non-projective modules and the modules $P(S_i)$ are injective for $0 \leq i < d$.*

Proof. According to Lemma 3, the modules S_i (with $0 \leq i \leq d$) are simple and non-projective, and the modules $P(S_i)$ are injective for $0 \leq i < d$. If at least two of the modules S_0, \dots, S_d are isomorphic, then Lemma 4 asserts that Λ is self-injective, but this we have excluded. \square

Theorem 2. *Let Λ be a connected left artinian ring with radical square zero and with n simple modules. The following conditions are equivalent:*

- (i) Λ is self-injective, but not a simple ring.

- (ii) *There exists a non-projective module M with $\text{Ext}^i(M, \Lambda) = 0$ for $1 \leq i \leq n + 1$.*
- (iii) *There exists a non-projective simple module S with $\text{Ext}^i(S, \Lambda) = 0$ for $1 \leq i \leq n$.*

Proof. First, assume that Λ is self-injective, but not simple. Since Λ is not semisimple, there is a non-projective module M . Since Λ is self-injective, $\text{Ext}^i(M, \Lambda) = 0$ for all $i \geq 1$. This shows the implication (i) \implies (ii). The implication (ii) \implies (iii) follows from Lemma 1. Finally, for the implication (iii) \implies (i) we use Theorem 1. Namely, if Λ is not self-injective, then Theorem 1 asserts that the simple modules $S_i = \Omega_i S$ with $0 \leq i \leq n$ are pairwise non-isomorphic. However, these are $n + 1$ simple modules, and we assume that the number of isomorphism classes of simple modules is n . This completes the proof of Theorem 2. \square

Note that the implication (ii) \implies (i) in Theorem 2 asserts in particular that either Λ is self-injective or else that any CM module is projective, as shown by Chen [C]. Let us recall that a module M is said to be a CM module provided $\text{Ext}^i(M, \Lambda) = 0$ and $\text{Ext}^i(\text{Tr } M, \Lambda) = 0$, for all $i \geq 1$ (here Tr denotes the transpose of the module); these modules are also called Gorenstein-projective modules, or totally reflexive modules, or modules of G-dimension equal to 0. Note that in general there do exist modules M with $\text{Ext}^i(M, \Lambda) = 0$ for all $i \geq 1$ which are not CM modules, see [JS].

We also draw the attention to the generalized Nakayama conjecture formulated by Auslander-Reiten [AR]. It asserts that for any artin algebra Λ and any simple Λ -module S there should exist an integer $i \geq 0$ such that $\text{Ext}^i(S, \Lambda) \neq 0$. It is known that this conjecture holds true for algebras with radical square zero. The implication (iii) \implies (i) of Theorem 2 provides an effective bound: *If n is the number of simple Λ -modules, and S is simple, then $\text{Ext}^i(S, \Lambda) \neq 0$ for some $0 \leq i \leq n$.* Namely, in case S is projective or Λ is self-injective, then $\text{Ext}^0(S, \Lambda) \neq 0$. Now assume that S is simple and not projective and that Λ is not self-injective. Then there must exist some integer $1 \leq i \leq n$ with $\text{Ext}^i(S, \Lambda) \neq 0$, since otherwise the condition (iii) would be satisfied and therefore condition (i).

Theorem 1 may be interpreted as a statement concerning the Ext-quiver of Λ . Recall that the Ext-quiver $\Gamma(R)$ of a left artinian ring R has as vertices the (isomorphism classes of the) simple R -modules, and if S, T

Note that the Ext-quiver of a connected self-injective left artinian ring with radical square zero and n vertices is just $\Delta(n, 1)$. Our further interest lies in the cases $t > 1$.

Theorem 3. *Let Λ be a connected left artinian ring with radical square zero and with n simple modules.*

- (a) *If there exists a non-projective simple modules S with $\text{Ext}^i(S, \Lambda) = 0$ for $1 \leq i \leq n - 1$, or if there exists a non-projective module M with $\text{Ext}^i(M, \Lambda) = 0$ for $1 \leq i \leq n$, then $\Gamma(\Lambda)$ is of the form $\Delta(n, t)$ with $t > 1$.*
- (b) *Conversely, if $\Gamma(\Lambda) = \Delta(n, t)$ and $t > 1$, then there exists a unique simple module S with $\text{Ext}^i(S, \Lambda) = 0$ for $1 \leq i \leq n - 1$, namely the module $S = S(0)$ (and it satisfies $\text{Ext}^n(S, \Lambda) \neq 0$).*
- (c) *If $\Gamma(\Lambda) = \Delta(n, t)$ and $t > 1$, and if we assume in addition that Λ is an artin algebra, then there exists a unique indecomposable module M with $\text{Ext}^i(M, \Lambda) = 0$ for $1 \leq i \leq n$, namely $M = \text{Tr } D(S(0))$ (and it satisfies $\text{Ext}^{n+1}(M, \Lambda) \neq 0$).*

Here, for Λ an artin algebra, D denotes the k -duality, where k is the center of Λ (thus $D = \text{Hom}_k(-, E)$, where E is a minimal injective cogenerator in the category of k -modules); thus $D \text{Tr}$ is the Auslander-Reiten translation and $\text{Tr } D$ the reverse.

Proof of Theorem 3. Part (a) is a direct consequence of Theorem 1, using the interpretation in terms of the Ext-quiver as outlined above. Note that we must have $t > 1$, since otherwise Λ would be self-injective.

(b) We assume that $\Gamma(\Lambda) = \Delta(n, t)$ with $t > 1$. For $0 \leq i < n$, let $S(i)$ be the simple module corresponding to the vertex i , let $P(i)$ be its projective cover, $I(i)$ its injective envelope. We see from the quiver that all the projective modules $P(i)$ with $0 \leq i \leq n - 2$ are injective, thus $\text{Ext}^j(-, \Lambda) = \text{Ext}^j(-, P(n-1))$ for all $j \geq 1$. In addition, the quiver shows that $\Omega S(i) = S(i+1)$ for $0 \leq i \leq n - 2$. Finally, we have $\Omega S(n - 1) = S(0)^a$ for some positive integer a dividing t and the injective envelope of $P(n - 1)$ yields an exact sequence

$$0 \rightarrow P(n - 1) \rightarrow I(P(n - 1)) \rightarrow S(n - 1)^{t-1} \rightarrow 0 \tag{*}$$

(namely, $I(P(n - 1)) = I(\text{soc } P(n - 1)) = I(S(0)^a) = I(S(0))^a$ and $I(S(0))/\text{soc}$ is the direct sum of b copies of $S(n - 1)$, where $ab = t$; thus

the cokernel of the inclusion map $P(n-1) \rightarrow I(P(n-1))$ consists of $t-1$ copies of $S(n-1)$.

Since $t > 1$, the exact sequence (*) shows that $\text{Ext}^1(S(n-1), P(n-1)) \neq 0$. It also implies that $\text{Ext}^1(S(i), P(n-1)) = 0$ for $0 \leq i \leq n-2$, and therefore that

$$\begin{aligned} \text{Ext}^i(S(0), P(n-1)) &= \text{Ext}^1(\Omega_{i-1}S(0), P(n-1)) \\ &= \text{Ext}^1(S(i-1), P(n-1)) \\ &= 0 \end{aligned}$$

for $1 \leq i \leq n-1$.

Since $\Omega_{n-i-1}S(i) = S(n-1)$ for $0 \leq i \leq n-1$, we see that

$$\begin{aligned} \text{Ext}^{n-i}(S(i), P(n-1)) &= \text{Ext}^1(\Omega_{n-i-1}S(i), P(n-1)) \\ &= \text{Ext}^1(S(n-1), P(n-1)) \\ &\neq 0 \end{aligned}$$

for $0 \leq i \leq n-1$. Thus, on the one hand, we have $\text{Ext}^n(S(0), \Lambda) \neq 0$, this concludes the proof that $S(0)$ has the required properties. On the other hand, we also see that $S = S(0)$ is the only simple module with $\text{Ext}^i(S, \Lambda) = 0$ for $1 \leq i \leq n-1$. This completes the proof of (b).

(c) Assume now in addition that Λ is an artin algebra. As usual, we denote the Auslander-Reiten translation $D \text{Tr}$ by τ . Let M be a non-projective indecomposable module with $\text{Ext}^i(M, \Lambda) = 0$ for $1 \leq i \leq n$. The shape of $\Gamma(\Lambda)$ shows that $\Omega M = S^c$ for some simple module S (and we have $c \geq 1$), also it shows that no simple module is projective. Now $\text{Ext}^i(S, \Lambda) = 0$ for $1 \leq i < n$, thus according to (b) we must have $S = S(0)$. It follows that PM has to be a direct sum of copies of $P(n-1)$, say of d copies. Thus a minimal projective presentation of M is of the form

$$P(0)^c \rightarrow P(n-1)^d \rightarrow M \rightarrow 0,$$

and therefore a minimal injective copresentation of τM is of the form

$$0 \rightarrow \tau M \rightarrow I(0)^c \rightarrow I(n-1)^d.$$

In particular, $\text{soc } \tau M = S(0)^c$ and $(\tau M)/\text{soc}$ is a direct sum of copies of $S(n-1)$.

Assume that $\tau M \neq S(0)$, thus it has at least one composition factor of the form $S(n-1)$ and therefore there exists a non-zero map $f : P(n-1) \rightarrow \tau M$. Since τM is indecomposable and not injective, any map from an injective module to τM maps into the socle of τM . But the image of f is not contained in the socle of τM , therefore f cannot be factored through an injective module. It follows that

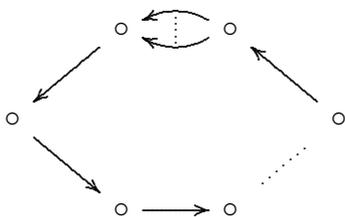
$$\text{Ext}^1(M, P(n-1)) \simeq D\overline{\text{Hom}}(P(n-1), \tau M) \neq 0,$$

which contradicts the assumption that $\text{Ext}^1(M, \Lambda) = 0$. This shows that $\tau M = S(0)$ and therefore $M = \text{Tr } DS(0)$.

Of course, conversely we see that $M = \text{Tr } DS(0)$ satisfies $\text{Ext}^i(M, P(n-1)) = 0$ for $1 \leq i \leq n$, and $\text{Ext}^{n+1}(M, P(n-1)) \neq 0$. \square

Remarks.

- (1) The module $M = \text{Tr } DS(0)$ considered in (c) has length $t^2 + t - 1$, thus the number t (and therefore $\Delta(n, t)$) is determined by M .
- (2) If Λ is an artin algebra with Ext-quiver $\Delta(n, t)$, the number t has to be the square of an integer, say $t = m^2$. A typical example of such an artin algebra is the path algebra of the following quiver



with altogether $n + m - 1$ arrows, modulo the ideal generated by all paths of length 2. Of course, if Λ is a finite-dimensional k -algebra with radical square zero and Ext-quiver $\Delta(n, m^2)$, and k is an algebraically closed field, then Λ is Morita-equivalent to such an algebra.

Also the following artin algebras with radical square zero and Ext-quiver $\Delta(1, m^2)$ may be of interest: the factor rings of the polynomial ring $\mathbb{Z}[T_1, \dots, T_{m-1}]$ modulo the square of the ideal generated by some prime number p and the variables T_1, \dots, T_{m-1} .

References

- [AR] M. Auslander, I. Reiten: On a generalized version of the Nakayama conjecture. Proc. Amer. Math. Soc. 52 (1975), 69–74.
- [ARS] M. Auslander, I. Reiten, S. Smalø: Representation Theory of Artin Algebras. 1995. Cambridge University Press.
- [C] X.-W. Chen: Algebras with radical square zero are either self-injective or CM-free. Proc. Amer. Math. Soc. Vol. 140, 93–98.
- [JS] D. A. Jorgensen, L. M. Şega: Independence of the total reflexivity conditions for modules. Algebras and Representation Theory 9 (2006), 217–226.

CONTACT INFORMATION

C. M. Ringel

Department of Mathematics, Shanghai Jiao
Tong University, Shanghai 200240, P. R. China
and King Abdulaziz University, P O Box 80200,
Jeddah, Saudi Arabia

E-Mail: ringel@math.uni-bielefeld.de

B.-L. Xiong

Department of Mathematics, Beijing Univer-
sity of Chemical Technology, Beijing 100029,
P. R. China

E-Mail: xiongbaolin@gmail.com

Received by the editors: 24.05.2012
and in final form 17.01.2013.