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Prethick subsets in partitions of groups

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Abstract. A subset S of a group G is called *thick* if, for any finite subset F of G, there exists $g \in G$ such that $Fg \subseteq S$, and *k-prethick*, $k \in \mathbb{N}$ if there exists a subset K of G such that $|K| = k$ and KS is thick. For every finite partition $\mathcal{P}$ of G, at least one cell of $\mathcal{P}$ is k-prethick for some $k \in \mathbb{N}$. We show that if an infinite group G is either Abelian, or countable locally finite, or countable residually finite then, for each $k \in \mathbb{N}$, G can be partitioned in two not k-prethick subsets.

Introduction

For a group G and a natural number k, we use the standard notations $[G]^k$ and $[G]^{<\omega}$ for the set of all k-subsets of G and the set of all finite subsets of G.

A subset S of G is called

- *large* if $G = KS$ for some $K \in [G]^{<\omega}$;
- *thick* if $G \setminus S$ is not large;
- *k-prethick* if there exists $K \in [G]^k$ such that KS is thick;
- *prethick* if S is k-prethick for some $k \in \mathbb{N}$;
- *small* if $L \setminus S$ is large for each large subset L of G;
- *P-small* if there exists an injective sequence $(g_n)_{n\in\omega}$ in G such that the subsets $\{g_n S : n \in \omega\}$ are pairwise disjoint;
- *thin* if $S \cap gS$ is finite for each $g \in G \setminus \{e\}$, e is the identity of G.

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To be precise we should add the adjective ”left” to each of above definitions because each of them has the "right" counterpart, for example, S is right large if $G = SF$ for some $F \in [G]^{<\omega}$. But in this paper we deal only with left-side versions, so we omit the adjective "left". In the dynamical terminology [6, p. 85], a large subset is called syndetic. A subset S is prethick if and only if there exists $K \in [G]^{<\omega}$ such that, for each $F \in [G]^{<\omega}$, $Fg \subseteq KS$ for some $g \in G$, so a prethick subset is exactly a piecewise syndetic set in the terminology of [6, p. 85]. We note also that large, small, thick and thin subsets can be defined in much more general context of balleans [14], [16], [17].

Every infinite group G can be partitioned in $\aleph_0$ large subsets [11] and in $\aleph_0$ small subsets [12]. If G is amenable then G can not be partitioned in $> \aleph_0$ large subsets. If H is a countable subgroup of G and $G = HR$ is a decomposition of G into right cosets then $\{hR : h \in H\}$ is a partition of G in $\aleph_0$ P-small subsets. P-small subsets were introduced by I. Prodanov [10] and studied systematically by T. Banakh and N. Lyaskovska [1], [2], [8].

Every infinite group G can be partitioned in $|G|$ thick subsets [9]. For generalizations and applications of this statement see [4], [13]. For an infinite group G, $\mu(G)$ denotes the minimal cardinal k such that G can be partitioned in k thin subsets. By [15], $\mu(G) = |G|$ if $|G|$ is a limit cardinal and $\mu(G) = \kappa$ if $|G| = \kappa^+$.

Let G be a group and let $A_1, \cup \ldots \cup A_n$ be a partition of G. By [6, Corollary 4.41], at least one cell of the partition is prethick, for an elementary proof of much more general statement see [16, Theorem 11.2]. By [7, Theorem 12.7], there exists a cell A_i and $K \in [G]^{<\omega}$ such that $G = KA_iA_i^{-1}$ and $|K| \leqslant 2^{2^{n-1}-1}$. It is an open problem [7, Problem 13.4.4] whether K can be chosen so that $|K| \leqslant n$. This is so if G is amenable [16, Theorem 12.8]. Comparing these results, we run into the following question.

Given an infinite group G, does there exist a natural number $k = k(G)$ such that, for any partition $G = A_1 \cup A_2$, at least one cell of the partition is k-prethick?

We give a negative answer to this question if G is either Abelian, or countable locally finite, or countable residually finite.

Recall that a group G is *locally finite* if every finite subset of G generates a finite subgroup and *residually finite* if for every$g \in G \setminus \{e\}$ there is a normal subgroup N of finite index such that $g \notin N$.

For convenience of formulations, we say that a partition $\mathcal{P}$ of a group G is *k-meager* if each cell of $\mathcal{P}$ is not k-prethick, equivalently, $G \setminus KP$ is large for all $P \in \mathcal{P}$ and $K \in [G]^k$.

1. Results

Theorem 1. *For every countable residually finite group G and every $k \in \mathbb{N}$, there exists a k-meager 2-partition of G.*

Proof. We enumerate the family $[G]^k$ as $\{K_n : n \in \omega\}$ and choose a decreasing chain $\{N_n : n \in \omega\}$ of subgroups of finite index of G such that $\bigcap_{n\in\omega} N_n = \{e\}$, e is the identity of G. Suppose that there exist two injective sequences $\langle a_n\rangle_{n\in\omega}$, $(b_n)_{n\in\omega}$ in G such that

$$K_i a_i N_i \cap K_j b_j N_j = \varnothing$$

for all $i, j \in \omega$. We put

$$A = \bigcup_{i\in\omega} K_i a_i N_i, \quad B = G \setminus A,$$

and show that A is not k-prethick. On the contrary, assume that KA is thick for some $K \in [G]^k$ and pick $n \in \omega$ such that $K = K_n^{-1}$. Let L_n be a set of representatives of left cosets of G by N_n. Since $K_n^{-1}A$ is thick and L_n is finite, there exists $g \in G$ such that $L_n g \subset K_n^{-1}A$. Clearly, $L_n g N_n = L_n N_n = G$ so $b_n \in K_n^{-1} A N_n$ and $K_n b_n N_n \cap A = \varnothing$, a contradiction. The same arguments show that B is not k-prethick.

To construct the sequences $\langle a_n\rangle_{n\in\omega}$, $(b_n)_{n\in\omega}$ we need some special choice of $\{N_n : n \in \omega\}$:

$$(2k)^2 \sum_{i=0}^{n} \frac{1}{|G : N_i|} < 1.$$

Since $|G : N_0| > (2k)^2 > k^2$, there is $g_0 \in G \setminus K_0^{-1} K_0 N_0$. We put $a_0 = e$, $b_0 = g_0$, so $K_0 a_0 N_0 \cap K_0 b_0 N_0 = \varnothing$. Suppose we have chosen $a_0, \dots, a_n$ and $b_0, \dots, b_n$ such that

$$K_i a_i N_i \cap K_j b_j N_j = \varnothing,\ i, j \in \{0, \dots, n\}.$$

Since $|G : N_{n+1}| > k^2$, there is g_{n+1} such that

$$K_{n+1} N_{n+1} \cap K_{n+1} g_{n+1} N_{n+1} = \varnothing.$$

Let us consider the set

$$S = \bigcup_{i=0}^{n} (K_i a_i N_i \cup K_i b_i N_i),$$

denote by pr the canonical projection $G \to G/N_{n+1}$ and observe that

$$|pr(K_{n+1} \cup K_{n+1}g_{n+1})^{-1}S| \leqslant (2k)^2 \sum_{i=0}^{n} |N_i : N_{n+1}| =$$

$$(2k)^2 \sum_{i=0}^{n} \frac{|G : N_{n+1}|}{|G : N_i|} < |G : N_{n+1}|.$$

We take $h \in G$ such that $pr(h) \notin pr(K_{n+1} \cup K_{n+1}g_{n+1})^{-1}S$. Then $(K_{n+1}hN_{n+1}) \cup K_{n+1}g_{n+1}hN_{n+1}) \cap S = \varnothing$. We put $a_{n+1} = h$, $b_{n+1} = g_{n+1}h$.

After ω steps, we get the required sequences $\langle a_n\rangle_{n\in\omega}$, $(b_n)_{n\in\omega}$. □

Theorem 2. *For every countable locally finite group G and every $k \in \mathbb{N}$, there exists a k-meager 2-partition of G.*

Proof. We enumerate the family $[G]^k$ as $\{K_n : n \in \omega\}$ and write G as a union of an increasing chain of finite subgroups $\{G_n : n \in \omega\}$ and, for each $n \in \omega$, pick a system R_n of representatives of right cosets of G by G_n and note that $R_n \cap G_n = \{e\}$. Suppose there exist two injective sequences in G such that

$$a_i \in G_i, b_i \in G_i, \quad K_i a_i R_i \cap K_j b_j R_j = \varnothing$$

for all $i, j \in \omega$. We put

$$A = \bigcup_{i\in\omega} K_i a_i R_i,\ B = G \setminus A$$

and show that A is not k-prethick. On the contrary, assume that KA is thick for some $K \in [G]^k$ and pick $n \in \omega$ such that $K = K_n^{-1}$. Since $K_n^{-1}A$ is thick and G_n is finite, there exists $g \in R_n$ such $G_n g \subset K_n^{-1}A$. Then $b_n g \in K_n^{-1}A$ but $K_n b_n R_n \cap A = \varnothing$, a contradiction. The same arguments show that B is not k-prethick.

To construct the sequences $\langle a_n\rangle_{n\in\omega}$, $(b_n)_{n\in\omega}$ we need a special choice of $\{G_n : n \in \omega\}$ and $\{R_n : n \in \omega\}$. For each $n \in \omega$, we pick $g_n \in G$ such that $K_n \cap K_n g_n = \varnothing$. We choose $\{G_n : n \in \omega\}$ so that, for each $n \in \omega$:

$$K_n \cup K_n g_n \subset G_n, \quad K_n \cap K_n g_n = \varnothing, \quad (2k)^2 \sum_{i=0}^{n} \frac{1}{|G_i|} < 1.$$

For each $n \in \omega$, we take an arbitrary system X_n of representatives of right cosets of G_{n+1} by G_n, $X_n \cap G_n = \{e\}$ and put

$$R_{n,m} = X_n X_{n+1} \dots X_m, \quad R_n = \bigcup_{m\geqslant n} R_{n,m}.$$

We put $a_0 = e$, $b_0 = g_0$, so $a_0, b_0 \in G_0$, $K_0a_0 \cup K_0b_0 \subset G_0$, $K_0a_0R_0 \cap K_0b_0R_0 = \varnothing$. Suppose we have chosen $a_0, \dots, a_n$ and $b_0, \dots, b_n$ such that $a_i \in G_i$, $b_i \in G_i$ and

$$K_ia_i \cup K_ib_i \subset G_i, \quad K_ia_iR_i \cap K_jb_jR_j = \varnothing$$

for all $i, j \in \{0, \dots, n\}$. We denote

$$S = \bigcup_{i=0}^{n}(K_ia_iR_{i,n+1} \cup K_ib_iR_{i,n+1}),$$

observe that $S \subset G_{n+1}$ and

$$|(K_{n+1} \cup K_{n+1}g_{n+1})^{-1}S| \leqslant (2k)^2 \sum_{i=0}^{n} \frac{|G_{n+1}|}{|G_n|} < |G_{n+1}|.$$

We take $h \in G_{n+1} \setminus (K_{n+1} \cup K_{n+1}g_{n+1})^{-1}S$, put $a_{n+1} = h$, $b_{n+1} = g_{n+1}h$. Then $(K_{n+1}a_{n+1} \cup K_{n+1}b_{n+1}) \cap S = \varnothing$. It follows that $K_ia_iR_i \cap K_jb_jR_j = \varnothing$ for all $i, j \in \{0, \dots, n+1\}$.

After ω steps, we get the required sequences $\langle a_n \rangle_{n\in\omega}$, $(b_n)_{n\in\omega}$. □

Lemma 1. *Let G_1, G_2 be groups, G be a direct product of G_1 and G_2, $k \in \mathbb{N}$. If there exists a k-meager 2-partition of G_1 then G also admits such a partition.*

Proof. If $A \cup B$ is a k-meager partition of G_1 then $(A \otimes G_2) \cup (B \otimes G_2)$ is a k-meager partition of G. □

Lemma 2. *Let an infinite group G be a subgroup of a direct product $H = \otimes_{\alpha<\kappa} H_\alpha$ of countable groups, S be a countable subset of G. Then there exists a countable subgroup S' of G and a subgroup T of G such that $S \subseteq S'$ and $G = S' \otimes T$.*

Proof. We denote by S_0 the subgroup of G generated by S and choose a countabe subset $I_0 \subseteq \kappa$ such that $S \subseteq \otimes_{\alpha\in I_0} H_\alpha$. If $pr_{I_0}G = S_0$ then $G = S_0 \otimes pr_{\kappa\setminus I_0}G$. Otherwise, we choose a countable subgroup S_1 of G such that $pr_{I_0}G = S_1$ and a countable subset $I_1 \subseteq \kappa$ such that $S_1 \subseteq \otimes_{\alpha\in I_1} H_\alpha$. If $pr_{I_1}G = S_1$ then S_1 is a direct factor of G. Otherwise, we choose a countable subgroup S_2 of G such that $pr_{I_1}G = pr_{I_1}S_2$ and a countable subset $I_2 \subseteq \kappa$ such that $S_2 \subseteq \otimes_{\alpha\in I_2} H_\alpha$. Proceeding by this way, we either get a direct factor S_n on some step $n \in \omega$ or a direct factor $S' = \bigcup_{n\in\omega} S_n$. □

Lemma 3. *Each countable subset S of an Abelian group G is contained in some countable direct factor S' of G.*

Proof. Apply Lemma 2 and Theorems 23.1 and 24.1 from [5]. □

Theorem 3. *For every infinite Abelian group G and every $k \in \mathbb{N}$, there exists a k-meager 2-partition of G.*

Proof. Applying Lemma 1 and Lemma 3, we may suppose that G is countable. We use [5, Theorem 21.3] to write G as a direct sum $G = D \oplus R$ of the divisible part D of G and some reduced group R. Since $\bigcap_{n\in\mathbb{N}} nR = \{0\}$ and R/nR is a direct sum of cyclic groups, R is residually finite. If R is infinite, we apply Theorem 1 and Lemma 1, so we may suppose that D is infinite. If D contains a Prüffer p-group, we apply Theorem 2 and Lemma 1. In view of [5, Theorem 23.1] and Lemma 1, it remains to prove theorem for the group $\mathbb{Q}$ of rational numbers.

We put $I = \{x \in \mathbb{Q} : 0 \leqslant x < 1\}$ and write $\mathbb{Q}$ as a sum $\mathbb{Z} + I$. By Theorem 1, there exists a $3k$-meager partition $\mathbb{Z} = A_0 \cup B_0$. We put

$$A = A_0 + I,\ B = B_0 + I$$

and show that A, B are not k-prethick in $\mathbb{Q}$. On the contrary, assume that one cell, say A, is k-prethick and choose $K \in [\mathbb{Q}]^k$ such that $K + A$ is thick. Take an arbitrary $C \in [\mathbb{Z}]^k$ and pick $q \in \mathbb{Q}$ such that $q+C \subset K+A$. We write $q = \lfloor q \rfloor + x$, $x \in I$, $\lfloor K \rfloor = \{\lfloor x \rfloor : x \in K\}$. Then

$$\lfloor q \rfloor + x + C \subset \lfloor K \rfloor + I + A_0 + I,$$

so $\lfloor q \rfloor + C \subset \lfloor K \rfloor + A_0 + I + I - I$ and

$$\lfloor q \rfloor + C \subseteq (\{-1, 0, 1\} + \lfloor K \rfloor) + A_0,$$

which is impossible because A_0 is not $3k$-thick. □

2. Comments

We do not know whether every infinite group G admits a k-meager 2-partition for each $k \in \mathbb{N}$, so we formulate some partial questions in this direction.

Question 1. *Does an infinite group G admit a k-meager 2-partition, $k \in \mathbb{N}$ provided that G is finitely generated? G is amenable? G is a free group of uncountable rank? G is the group of all permutations of ω?*

By [16, Theorem 3.9], an infinite group G can be partitioned in two large subsets $G = A_1 \cup A_2$. Clearly, A_1, A_2 are not thick, so $A_1 \cup A_2$ is a a 1-meager 2-partition.

Question 2. *Does an infinite group G admit a 2-meager 2-partition?*

Let G be a finite group, A be a non-empty subset of G, $|G| = n$, $|A| = m$. By [18], there exists a subset B of G such that $G = BA$ and $|B| < \frac{n}{m}(\log m + 2)$, so A is k-prethick for $k \geqslant \frac{n}{m}(\log m + 2)$. Hence, any 2-partition of G is not k-meager for $k \geqslant 2(\log n + 2)$.

For $k, m \in \mathbb{N}$, we say that a subset S of G is

- *m-thick* if, for every $F \in [G]^m$, there exists $g \in G$ such that $Fg \subseteq S$;
- *(k, m)-prethick* if there exists $K \in [G]^k$ such that KS is m-thick.

Question 3. *Given a group G, does there exist $k = k(G, m)$ such that, for every 2-partition of G, at least one cell is (k, m)-prethick? For $m = 2$, this is so: $k = 2$.*

In what follows all group topologies are supposed to be Hausdorff.

Recall that a topological group G is *totally bounded* if each neighbourhood of e is large (equivalently, G is a subgroup of some compact topological group). If A is a thick subset of G then $A \cap gU \neq \varnothing$ for every $g \in G$ and every neighbourhood U of e, so A is dense in G. The converse statement does not hold: every countable totally bounded group has a small dense subset [3].

Question 4. *Let G be an infinite totally bounded group, $k \in \mathbb{N}$. Does there exist a partition $G = A_1 \cup A_2$ such that KA_1 and KA_2 are not dense for each $K \in [G]^k$?*

If G is countable, this is so. We take a sequence $(U_n)_{n\in\omega}$ of compact neighbourhoods of the identity in the completion H of G such that, for each $n \in \omega$,

$$(2k)^2 \sum_{i=0}^{n} \mu(U_i) < 1,$$

where μ is the Haar measure on H. Following the proof of Theorem 1 with U_n instead of N_n, we can choose two injective sequences $\langle a_n \rangle_{n\in\omega}$, $(b_n)_{n\in\omega}$ and a sequence of compact neighbourhoods $(V_n)_{n\in\omega}$ of the identity in H such that $V_i \subset U_i$ and $K_i a_i V_i \cap K_j b_j V_j = \varnothing$ for all $i, j \in \omega$. We put

$$A = \bigcup_{i\in\omega} K_i a_i (V_i \cap G), \quad B = G \setminus A,$$

and note that KA, KB are not dense in G for each $K \in [G]^k$.

Thus, Theorem 1 remains true if a countable group G is a subgroup of a compact topological group. Since each Abelian group admits a totally bounded topology, we get a proof of Theorem 3 with usage of Lemmas 1 and 3 but no reference to Theorem 2.

If a countable topological group G is not totally bounded then G can be easily partitioned $G = A \cup B$ so that KA, KB are not dense for each $K \in [G]^{<\omega}$. We choose a neighbourhood U of e such that $G \neq FU$ for each $F \in [G]^{<\omega}$, enumerate $\{K_n : n \in \omega\}$ the family $[G]^{<\omega}$ and choose inductively two injective sequences $\langle a_n \rangle_{n\in\omega}$, $(b_n)_{n\in\omega}$ in G such that

$$K_i a_i W \cap K_j b_j W = \varnothing$$

for all $i, j \in \omega$. Put $A = \bigcup_{i\in\omega} K_i a_i W$, $B = G \setminus A$.

Given a countable non-discrete topological group with countable base of topology, it is easy to find a thin dense subset.

Question 5. *Let G be a countable totally bounded group. Has G a thin dense subset? What about $G = \mathbb{Z}^\#$, the group $\mathbb{Z}$ endowed with the maximal totally bounded topology?*

Question 6. *How can one detect whether a given subset A of $\mathbb{Z}$ is dense in $\mathbb{Z}^\#$?*

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