Algebra and Discrete Mathematics
Volume 14 **(2012)**. Number 2. pp. 217 – 229

RESEARCH ARTICLE

Orthoscalar representations of the partially ordered set $(N, 4)$

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Communicated by Yu. A. Drozd

Abstract. We obtain a one-parameter series of orthoscalar representations of the partially ordered set $(N, 4)$. This proves that the classification of such representations is a problem of infinite type.

1. Introduction

Many problems of functional analysis can be formulated and solved in terms of the theory of representations of $*$-quivers and $*$-algebras. Representations, in Hilbert spaces, of $*$-algebras with self-adjoint generators whose sum is a multiple of the identity and whose spectra are fixed were studied in numerous works (see, e.g., [1–3]). They are naturally associated with orthoscalar representations of certain $*$-quivers (or graphs) investigated in [4–7].

Collections of operators with special fixed spectra and the sum equal to the identity operator that are associated with the extended Dynkin graphs $\widetilde{D}_4, \widetilde{E}_6$, and $\widetilde{E}_7$ were studied in [1, 3, 8, 9]. Some results on representations of algebras associated with $\widetilde{E}_8$ are presented in [10]. In [11–12] infinite two-parameter series of irreducible representations of graphs $\widetilde{E}_6, \widetilde{E}_7$, and $\widetilde{E}_8$ with above-mentioned special characters were constructed explicitly (canonical forms of such representations were presented).

Representations of partially ordered sets (posets) were introduced by L. A. Nazarova and A. V. Roiter in [13], where and algorithm was

2010 MSC: 16G20.

Key words and phrases: partially ordered set, orthoscalar representation, infinite type.

constructed allowing to find out whether a certain poset has finitely or infinitely many indecomposable representations. Kleiner, in his paper [14], proved using this algorithm that a poset is of finite type if and only if it does not contain “critical” subsets: (1, 1, 1, 1), (2, 2, 2), (1, 3, 3), (1, 2, 5) and $(N, 4) = \{a_1 < a_2 > b_1 < b_2; \;\; c_1 < c_2 < c_3 < c_4\}$ (here $(l_1, l_2, ..., l_m)$ denotes the cardinal sum of chains of lengths $l_1, l_2, ..., l_m$).

Results on finite representation type of quivers were translated to finite dimensional Hilbert (unitary) spaces in [4]; the analogue of Gabriel's theorem was proved for quivers and their orthoscalar representations.

For the proof of Kleiner's theorem analogue for orthoscalar representations of posets it should be proved, in particular, that the classification of critical posets is a problem of infinite type. For primitive posets this problem reduces (see [15]) to the similar problem regarding extended Dynkin graphs.

In the present paper it is proved that the classification of orthoscalar representations of the last critical poset $(N, 4)$ is a problem of infinite type. For another definition of orthoscalar representations of posets and Kleiner's theorem in this treatment, see [17–18].

2. Notation and auxiliary facts

Recall some notation and facts related to orthoscalar representations of quivers [4–6]. A quiver Q with a set of vertices Q_v, $|Q_v| = N$ and a set of arrows Q_a is called divided if $Q_v = \overset{\circ}{Q} \bigsqcup \overset{\bullet}{Q}$ and, for any $\alpha \in Q_a$, its origin t_α belongs to $\overset{\circ}{Q}$ and the end h_α belongs to $\overset{\bullet}{Q}$. One says that the quiver Q is of multiplicity one if, for $\alpha \neq \beta$, one has either $t_\alpha \neq t_\beta$ or $h_\alpha \neq h_\beta$. The vertices from $\overset{\circ}{Q}$ and $\overset{\bullet}{Q}$ are called even and odd respectively.

Let $m = |\overset{\bullet}{Q}|, \; n = |\overset{\circ}{Q}|, \; \overset{\bullet}{Q} = \{i_1, i_2, ..., i_m\}, \; \overset{\circ}{Q} = \{j_1, j_2, ..., j_n\}$.

A representation T of a quiver Q associates a vertex $i \in Q_v$ with a vector space $T(i)$ and an arrow $\alpha : \; j \to i, \alpha \in Q_a$, with a linear mapping $T_{ij} : \; T(j) \to T(i)$. A representation T of a divided quiver of multiplicity one with fixed bases of spaces $T(i), \; i \in Q_v$ can be associated with a matrix divided into m horizontal and n vertical strips, i.e., with a matrix

$$T = [T_{i_l, j_k}]_{l=\overline{1,m}}^{k=\overline{1,n}}.$$

We assume that $T_{i_l, j_k} = 0$ if there does not exist $\alpha \in Q_a$ such that $t_\alpha = j_k, \; h_\alpha = i_l$. Let $\overrightarrow{T_i} = [T_{i,j_1} \,|\, T_{i,j_2} \,|\, ... \,|\, T_{i,j_n}]$,

$$T_j^{\downarrow} = \begin{bmatrix} T_{i_1,j} \\ \hline \vdots \\ \hline T_{i_m,j} \end{bmatrix}, \qquad \begin{array}{l} \overrightarrow{T_i} : \bigoplus_{k=1}^{n} T(j_k) \to T(i), \\ T_j^{\downarrow} : T(j) \to \bigoplus_{l=1}^{m} T(i_l), \end{array}$$

A divided quiver of multiplicity one is called ordered, if $\overset{\bullet}{Q}$ and $\overset{\circ}{Q}$ are posets.

A representation T of an ordered divided quiver Q of multiplicity one is called orthoscalar [1] if the spaces $T(i),\ i \in Q_v$ are finite dimensional Hilbert (unitary) spaces (over the field of complex numbers $\mathbb{C}$), every $i \in Q_v$ is associated with a positive real number χ_i, and the following conditions are satisfied:

1) $\overrightarrow{T_i} \cdot \overrightarrow{T_i}^* = \chi_i I_i$ for $i \in \overset{\bullet}{Q}$;
   $T_j^{\downarrow *} \cdot T_j^{\downarrow} = \chi_j I_j$ for $j \in \overset{\circ}{Q}$;

2) if $i' < i''$, $\quad i', i'' \in \overset{\bullet}{Q}$, then $\chi_{i'} > \chi_{i''}$ and $\overrightarrow{T_{i'}} \cdot \overrightarrow{T_{i''}}^* = 0$;
   if $j' < j''$, $\quad j', j'' \in \overset{\circ}{Q}$, then $\chi_{j'} > \chi_{j''}$ and $T_{j'}^{\downarrow *} \cdot T_{j''}^{\downarrow} = 0$.

If $m = 1$, a representation T of an ordered quiver Q is called an orthoscalar representation of a poset.

With an orthoscalar representation T of an ordered divided quiver of multiplicity one we associate two N-dimensional vectors ($N = m + n$): the dimension d of the representation T, $d = \{d(j)\}_{j \in Q_v}$, where $d(j) = \dim T(j)$, and the character χ of the representation T, $\chi = \{\chi(j)\}_{j \in Q_v}$, $\chi(j) = \chi_j$ is defined above. It is easy to see that

$$\sum_{l=1}^{m} d(i_l)\chi(i_l) = \sum_{k=1}^{n} d(j_k)\chi(j_k). \tag{1}$$

Indeed, the space of rows $\overrightarrow{x} = (x_1, x_2, ..., x_s)$ over the field of complex numbers with the dot product $(\overrightarrow{x}, \overrightarrow{y}) = (x_1\overline{y}_1 + ... + x_s\overline{y}_s)$ is a unitary space. The norm $||\overrightarrow{x}||$ of a row $\overrightarrow{x}$ is defined as $\sqrt{(\overrightarrow{x}, \overrightarrow{x})}$, two rows are orthogonal if $(\overrightarrow{x}, \overrightarrow{y}) = 0$. The unitary space of columns (and the column norm $||y^{\downarrow}||$) are defined similarly. Then, equality (1) for the matrix of representation $T = [z_{ij}]$ means that the sum of squares of row norms for the matrix T is equal to the sum of squares of column norms for the matrix T, and is equal to $\sum_{ij} z_{ij}\overline{z}_{ij}$.

[1] Definition belongs to A. V. Roiter.

The conditions of orthoscalarity, in particular, mean that the rows in a single (ith) horizontal strip of a block matrix T are orthogonal and have constant norm ($\sqrt{\chi_i}$), and if $i < j$ in $\overset{\bullet}{Q}$ then the rows of the ith and jth horizontal strips are orthogonal. Similar properties hold for the columns of T.

A non-ordered divided quiver of multiplicity one can be considered as a special case of an ordered quiver (in $\overset{\circ}{Q}$ and $\overset{\bullet}{Q}$ all elements assigned to be incomparable).

Let $\operatorname{Rep} Q$ be the category of representations of a non-ordered quiver Q whose objects are representations T and a morphism of a representation T to a representation $\widetilde{T}$ is defined as a family of linear mappings $C = \{C_i\}_{i \in Q_v}$, $C_i : \; T(i) \to \widetilde{T}(i)$, such that, for every $\alpha \in Q_v$ with $t_\alpha = j$, $h_\alpha = i$, the diagram

$$\begin{array}{ccc} T(j) & \xrightarrow{\;T(\alpha)\;} & T(i) \\ {\scriptstyle C_j}\downarrow & & \downarrow{\scriptstyle C_i} \\ \widetilde{T}(j) & \xrightarrow{\;\widetilde{T}(\alpha)\;} & \widetilde{T}(j) \end{array} \qquad (2)$$

is commutative, i.e., $C_i T_{ij} = \widetilde{T}_{ij} C_j$.

Define the matrices $A = \operatorname{diag}\{C_{i_1}, ..., C_{i_m}\}$, $B = \operatorname{diag}\{C_{j_1}, ..., C_{j_n}\}$. Then the commutativity of the diagram (2) implies

$$AT = \widetilde{T}B. \qquad (3)$$

In what follows, we also use the notation $C = (A, B)$. Two representations T and $\widetilde{T}$ are equivalent if there exists an invertible morphism from T to $\widetilde{T}$ (with the matrices A and B being invertible).

Define the category $\operatorname{Rep}_{os} Q$ of orthoscalar representations of a non-ordered divided quiver Q of multiplicity one as a subcategory of $\operatorname{Rep} Q$, whose objects are orthoscalar representations of Q and whose morphisms are morphisms $C = \{C_i\}_{i \in Q_v}$ from $\operatorname{Rep} Q$, such that in addition to the commutativity of diagrams (2), the diagram

$$\begin{array}{ccc} T(j) & \xleftarrow{\;T(\alpha)^*\;} & T(i) \\ {\scriptstyle C_j}\downarrow & & \downarrow{\scriptstyle C_i} \\ \widetilde{T}(j) & \xleftarrow{\;\widetilde{T}(\alpha)^*\;} & \widetilde{T}(j) \end{array} \qquad (4)$$

is also commutative, i.e.,

$$AT = \widetilde{T}B \quad \text{and} \quad BT^* = \widetilde{T}^*A. \tag{5}$$

Let S be a poset, i.e., for some ordered quiver Q we have $m = 1$, $\overset{\bullet}{Q} = S$.

With a representation T we associate a matrix

$$\begin{aligned} &T = [T_{j_1}|T_{j_2}| \ \dots \ |T_{j_n}], \\ &T_{j_k} \equiv T_{i_1, j_k} : \ T(j_k) \to T(i_1). \end{aligned} \tag{6}$$

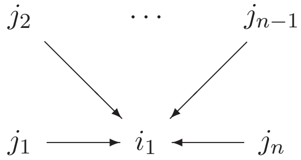
The orthoscalarity of the representation T of the partially ordered set S means that

$$\text{a) } T^*_{j_k} T_{j_k} = \chi_{j_k} I_{j_k}, \quad k = 1, \dots, n, \tag{7}$$

$$\text{b) } T^*_{j_k} T_{j_l} = 0 \text{ for } j_k < j_l, \text{ and } \chi_{j_k} > \chi_{j_l}, \tag{8}$$

$$\text{c) } \sum_{k=1}^{n} T_{j_k} T^*_{j_k} = \chi_{i_1} I. \tag{9}$$

The representation T could be considered also as an orthoscalar representation of the quiver Q



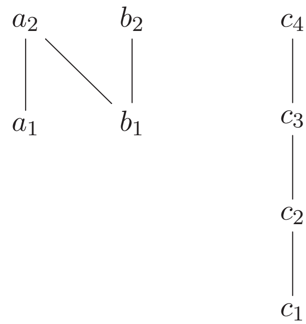
Define the category $\operatorname{Rep}_{os} S$ as a full subcategory of $\operatorname{Rep}_{os} Q$, whose objects are orthoscalar representations of a partially ordered set S (i.e., a morphism $C' : \ T \to \widetilde{T}$ in the category $\operatorname{Rep}_{os} S$ is defined as a pair of matrices (A, B), where $A = C_i$, $B = \operatorname{diag}\{C_{j_1}, \dots, C_{j_n}\}$, such that equalities (5) hold).

It was proved (see, e.g., [16]) that T and $\widetilde{T}$ are equivalent in $\operatorname{Rep}_{os} Q$, $\operatorname{Rep}_{os} S$ if and only if they are unitarily equivalent, i.e., an invertible morphism C consists of unitary matrices C_i, C_j. Decomposable representations are defined in a natural way; if $T = T_1 \oplus T_2$ in the category $\operatorname{Rep}_{os} Q$ then $T_1(i) \oplus T_2(i)$ is the orthogonal sum of unitary spaces.

A representation T is called a Schur (brick) representation in the category $\operatorname{Rep}_{os} Q$ if its endomorphism ring in this category is one-dimensional (isomorphic to $\mathbb{C}$). As is known, a representation T is indecomposable in the category $\operatorname{Rep}_{os} Q$ if and only if it is a Schur representation (see, e.g., [6], Note 4).

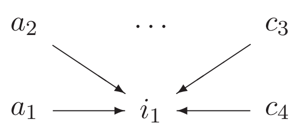
3. Orthoscalar representations of the partially ordered set $(N, 4)$

Hence, let S be a partially ordered set with a Hasse diagram

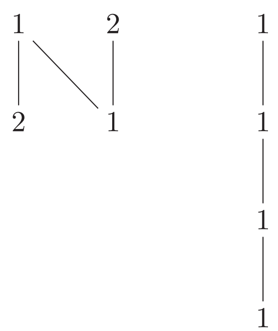


(10)

i.e., the set $(N, 4)$. Let Q be a quiver corresponding to S, $\overset{\circ}{Q} = \{a_1, a_2, b_1, b_2, c_1, c_2, c_3, c_4\} = S$, $\overset{\bullet}{Q} = \{i_1\}$:



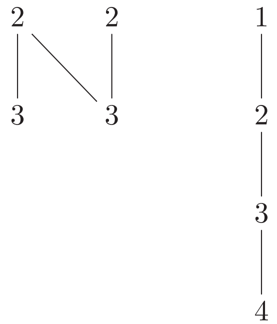
Let T be an indecomposable orthoscalar representation of a poset S in the dimension $d = \{d_{i_1}; d_{a_1}, ..., d_{c_4}\}$. Assume that the dimension is the following:



$$, \quad d_{i_1} = \dim T(i_1) = 5. \tag{11}$$

(we arrange the dimensions of representation spaces in accordance with the location of the vertices of the Hasse diagram for visibility).

Fix the character $\chi = \{\chi_{i_1}; \chi_{a_1}, \ldots, \chi_{c_4}\}$ of the representation T:



$$, \quad \chi_{i_1} = 5. \tag{12}$$

Prior to the calculation of matrix elements of the representation T, with the use of relations (7) – (9) we reduce the representation to the "canonical" form by using admissible unitary transformations ($\widetilde{T} = U_i T_{ij} V_j^*$).

We reduce some matrix elements to zero elements, and some nonzero elements to positive or negative elements (by the multiplication of a row or a column of the matrix by a certain number $e^{i\varphi}$; this is a unitary matrix transformation[2]). In this reduction, for simplicity, we use the following notation:

The symbol $\overline{0}|_k$ at any place of the matrix T means that one can obtain a zero element at this place in the kth step with the use of unitary transformations of the rows of the horizontal strip and the columns of the vertical strip that correspond to this place. The symbol $\overline{0}_k$ means that a zero element is obtained solely with the use of unitary transformations of columns. The symbol $0|_k$ means that a zero element is obtained solely with the use of unitary transformations of rows. The symbol $\overrightarrow{0_k}$ means that a zero element is obtained due to the orthogonality of columns of the vertical strip (or of two distinct strips, comparable in the sense of partial order), and the symbol $0{\downarrow_k}$ means that a zero element is obtained due to the orthogonality of rows of the horizonal strip (or of two distinct strips, comparable in the sense of partial order). Moreover, while obtaining a zero element on the kth step, we do not "spoil" the zero elements obtained earlier. The symbol a_{ij}^{+} (a_{ij}^{-}) means that an element at the indicated place is made positive (negative). We hope that the step-by-step reduction process can be easily reproduced.

Furthermore, embed our representation to another matrix problem for which it is easier to obtain the "canonical" form and calculate matrix elements.

[2]The reduction technique of representation matrices for an orthoscalar representation construction of a fixed dimension belongs to L. A. Nazarova.

Consider the orthoscalar representation Γ of the quiver P (two numbers located at every vertex are the dimension of the representation space corresponding to it and the corresponding character value in parentheses):

$$\begin{array}{ccccccccccccc}
 & & & & & & & & s_4\ 1(3) & & & & \\
 & & & & & & & & \downarrow & & & & \\
t_1 & & s_1 & & t_2 & & s_2 & & t_3 & & s_5 & & t_4 \\
1(1) & \longleftarrow & 2(2) & \longrightarrow & 3(3) & \longleftarrow & 4(4) & \longrightarrow & 5(5) & \longleftarrow & 3(3) & \longrightarrow & 1(1) \\
 & & & & & & & & \uparrow & & & & \\
 & & & & & & & & s_3\ 2(2) & & & &
\end{array}$$

$$\overset{\circ}{P} = \{s_1, s_2, s_3, s_4, s_5\}, \quad \overset{\bullet}{P} = \{t_1, t_2, t_3, t_4\}.$$

Denote by $A_{s_i,t_i} \equiv A_{ij}$ the matrix blocks of the representation Γ. Then

$$\Gamma = \left[\begin{array}{c|c|c|c|c}
A_{11} & & & & \\ \hline
A_{21} & A_{22} & & & \\ \hline
 & A_{32} & A_{33} & A_{34} & A_{35} \\ \hline
 & & & & A_{45}
\end{array}\right]$$

(empty cells contain zero elements). Denote the matrix elements of Γ by a_{ij}; therefore, $\Gamma = [a_{ij}]$ is a matrix of a "general" dimension 10×12.

Reduce the representation T of the poset S,

$$T = [T_{c_1} | T_{c_2} | T_{c_3} | T_{c_4} | T_{b_2} | T_{b_1} | T_{a_2} | T_{a_1}]$$

to

$$T = \left[\begin{array}{c|c|c|c|cc|c|c|cc}
a_{53} & a_{54} & a_{55} & a_{56} & \overleftarrow{0_2} & \overleftarrow{0_2} & \overset{+}{a}_{59} & \overrightarrow{0_2} & \overline{0}_2 & a_{5,12} \\
a_{63} & a_{64} & a_{65} & a_{66} & a_{67} & a_{68} & 0|_1 & a_{6,10} & a_{6,11} & a_{6,12} \\
a_{73} & a_{74} & a_{75} & a_{76} & a_{77} & a_{78} & 0|_1 & a_{7,10} & a_{7,11} & a_{7,12} \\
a_{83} & a_{84} & a_{85} & a_{86} & a_{87} & a_{88} & 0|_1 & a_{8,10} & a_{8,11} & a_{8,12} \\
a_{93} & a_{94} & a_{95} & a_{96} & a_{97} & a_{98} & 0|_1 & a_{9,10} & a_{9,11} & a_{9,12}
\end{array}\right]$$

and embed it into Γ in the following way (the indices at the matrix elements of T correspond to their future location in the matrix Γ): $A_{32} = [T_{c_1}\ T_{c_2}\ T_{c_3}\ T_{c_4}]$, ($T_{c_i}$ are united in a single block), $A_{33} = T_{b_2}$, $A_{34} = T_{b_1}$, $A_{35} = [T_{a_2}\ T_{a_1}]$ (T_{a_i} are united in a single block) $A_{45} = [a_{10,10}\ a_{10,11}\ 0]$.

If the matrices A_{11}, A_{21}, A_{22}, A_{45} are diagonalized with the use of admissible unitary transformations, then the reduction problem of remaining matrices A_{32}, A_{33}, A_{34}, A_{35} coincides with the reduction problem of the representation T of the poset S. Moreover, two representations Γ and $\widetilde{\Gamma}$ are unitarily equivalent if and only if the embedded into them representations T and $\widetilde{T}$ of the poset S are unitarily equivalent. The result of the reduction (and the reduction process described in our notation) is the following:

$$\left[\begin{array}{c|c|c|c} A_{32} & A_{33} & A_{34} & A_{35} \\ \hline 0 & 0 & 0 & A_{45} \end{array}\right] =$$

$$= \left[\begin{array}{cccc|cc|c|ccc} a^{+}_{53} & \overline{0}_3 & \overline{0}_3 & \overline{0}_3 & \overleftarrow{0}_2 & \overleftarrow{0}_2 & a^{+}_{59} & \overrightarrow{0}_2 & \overline{0}_3 & a^{+}_{5,12} \\ a^{-}_{63} & a^{+}_{64} & \overline{0}_6 & \overline{0}_6 & a^{+}_{67} & \overline{0}_6 & 0|_1 & \overleftarrow{0}_5 & \overleftarrow{0}_5 & a^{+}_{6,12} \\ 0{\downarrow}_5 & a^{+}_{74} & a^{+}_{75} & \overline{0}_9 & a^{-}_{77} & \overrightarrow{0}_9 & 0|_1 & a^{+}_{7,10} & \overline{0}_{10} & 0|_4 \\ 0{\downarrow}_5 & 0{\downarrow}_7 & a^{+}_{85} & a^{+}_{86} & 0{\downarrow}_8 & a^{+}_{88} & 0|_1 & a^{-}_{8,10} & a^{+}_{8,11} & 0|_4 \\ 0{\downarrow}_5 & 0{\downarrow}_7 & 0|_{13} & a^{c}_{96} & 0{\downarrow}_8 & a^{c}_{98} & 0|_1 & 0{\downarrow}_{13} & a^{+}_{9,11} & 0|_4 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & a^{+}_{10,10} & a^{+}_{10,11} & 0 \end{array}\right],$$

here a^{c}_{ij} means that the element a_{ij} is a complex number;

$$\left[\begin{array}{c|c} A_{11} & 0 \\ \hline A_{21} & A_{22} \end{array}\right] = \left[\begin{array}{cc|cccc} a^{+}_{11} & a^{+}_{12} & 0 & 0 & 0 & 0 \\ \hline a^{+}_{21} & \overline{0}_{16} & a^{+}_{23} & a^{+}_{24} & \overrightarrow{0}_{12} & \overrightarrow{0}_{12} \\ a^{-}_{31} & a^{+}_{32} & 0|_{11} & a^{+}_{34} & a^{-}_{35} & \overrightarrow{0}_{15} \\ 0|_{17} & a^{+}_{42} & 0|_{11} & 0|_{14} & a^{+}_{45} & a^{-}_{46} \end{array}\right].$$

The sense of embedding the representation T to the representation Γ is in the idea that the matrix Γ in the “canonical” form is more “sparse” in the number and location of zeros; this allows to find the matrix elements by a fixed character of the representation.

We show that the present representation depends on two real parameters (t and p), i.e., all the matrix elements can be expressed with these parameters, using the condition of orthoscalarity only. At every step we will need to solve either a linear equation (obtained from the row of column orthogonality) or a quadratic equation (obtained from the equality of a row or column norm to a fixed character value).

We present the construction order of the matrix elements showing the condition near every non-zero element that allows to find it.

The construction order of the elements a_{ij}

1) a_{59}^+ — calculating the norm of the 9th column;
2) $a_{53}^+ = \sqrt{t}, \quad t$ — a parameter;
3) $a_{5,12}^+$ — calculating the norm of the 5th row;
4) $a_{6,12}^+$ — calculating the norm of the 12th column;
5) a_{63}^- — from the orthogonality condition of the 5th and the 6th rows;
6) a_{23}^+ — calculating the norm of the 3rd column;
7) $a_{67}^+ = \sqrt{p}, \quad p$ — another parameter;
8) a_{64}^+ — calculating the norm of the 6th row;
9) a_{24}^+ — from the orthogonality condition of the 3rd and the 4th columns;
10) a_{21}^+ — calculating the norm of the 2nd row;
11) a_{77}^- — calculating the norm of the 7th column;
12) a_{74}^+ — from the orthogonality condition of the 6th and the 7th rows;
13) a_{34}^+ — calculating the norm of the 4th column;
14) a_{31}^- — from the orthogonality condition of the 2nd and the 3rd rows;
15) a_{11}^+ — calculating the norm of the 1st column;
16) a_{12}^+ — calculating the norm of the 1st row;
17) a_{32}^+ — from the orthogonality condition of the 1st and the 2nd columns;
18) a_{42}^+ — calculating the norm of the 2nd column;
19) a_{35}^- — calculating the norm of the 3rd row;
20) a_{45}^+ — from the orthogonality condition of the 3rd and the 4th rows;
21) a_{46}^+ — calculating the norm of the 4th row;
22) a_{75}^+ — from the orthogonality condition of the 4th and the 5th columns;
23) a_{85}^+ — calculating the norm of the 5th column;
24) a_{86}^+ — from the orthogonality condition of the 5th and the 6th columns;
25) $a_{7,10}^+$ — calculating the norm of the 7th column;
26) $a_{8,10}^-$ — from the orthogonality condition of the 7th and the 8th rows;
27) $|a_{96}^c|$ — calculating the norm of the 6th column;
28) $a_{10,10}^+$ — calculating the norm of the 10th column;
29) $a_{10,11}^+$ — calculating the norm of the 10th row;
30) $a_{8,11}^+$ — from the orthogonality condition of the 10th and the 11th columns;
31) a_{88}^+ — calculating the norm of the 8th row;
32) $a_{9,11}^+$ — calculating the norm of the 11th column;
33) $|a_{98}^c|$ — calculating the norm of the 9th row;
34) $\arg a_{96}^c$ and $\arg a_{98}^c$ — from the orthogonality condition of the 8th and the 9th rows.

The next step should be the description of the formulas of the matrix elements and the range of values of independent parameters (expressions inside various radicals should be positive). However, the explicit formulas become very bulky and the description of the range of values of the parameters becomes very difficult. We simplify the problem by letting $p = 1$ and restricting the range for t. We show that the range of values for t contains an interval. Anyway, this implies that the range of values for t is infinite. As a result, the following statement is proved.

Theorem 1. *The problem of unitary classification of orthoscalar representations of the partially ordered set* $(N, 4)$ *is of infinite type.*

Proof. Let $p = 1$, and find consecutively the expressions for all matrix elements via parameter t.

$$a^+_{59} = \sqrt{3}, \qquad a^+_{53} = \sqrt{t},$$
$$a^+_{5,12} = \sqrt{2-t}, \qquad a^+_{6,12} = \sqrt{t+1},$$
$$a^-_{63} = -\sqrt{\tfrac{(2-t)(1+t)}{t}}, \qquad a^+_{23} = \sqrt{\tfrac{3t-2}{t}},$$
$$a^+_{67} = 1, \qquad a^+_{64} = \sqrt{\tfrac{2(t-1)}{t}},$$
$$a^+_{24} = \sqrt{\tfrac{2(t^2-1)(2-t)}{t(3t-2)}}, \qquad a^+_{21} = \sqrt{\tfrac{4-4t+2t^2}{3t-2}},$$
$$a^-_{77} = -1, \qquad a^+_{74} = \sqrt{\tfrac{t}{2(t-1)}},$$
$$a^+_{34} = \sqrt{\tfrac{t(4t^2-3t-2)}{2(t-1)(3t-2)}}, \qquad a^-_{31} = -\sqrt{\tfrac{(2-t)(1+t)(4t^2-3t-2)}{(3t-2)(4-4t+2t^2)}},$$
$$a^+_{11} = \sqrt{\tfrac{14-19t+7t^2}{4-4t+2t^2}}, \qquad a^+_{12} = \sqrt{\tfrac{5(2-t)(t-1)}{4-4t+2t^2}},$$
$$a^+_{32} = \sqrt{\tfrac{5(t-1)(3t-2)(14-19t+7t^2)}{(1+t)(4-4t+2t^2)(4t^2-3t-2)}}, \qquad a^+_{42} = \sqrt{\tfrac{(9t-11)(4-4t+2t^2)}{(1+t)(4t^2-3t-2)}},$$
$$a^-_{35} = -\sqrt{\tfrac{(3t-2)(8t^3-41t^2+71t-40)}{2(t^2-1)(4t^2-3t-2)}}, \qquad a^+_{45} = \sqrt{\tfrac{10(t-1)^2(9t-11)(14-19t+7t^2)}{(1+t)(4t^2-3t-2)(8t^3-41t^2+71t-40)}},$$
$$a^-_{46} = -\sqrt{\tfrac{(5-3t)(4t^2-3t-2)}{8t^3-41t^2+71t-40}}, \qquad a^+_{75} = \sqrt{\tfrac{8t^3-41t^2+71t-40}{2(t^2-1)}},$$

$$a^+_{85} = \sqrt{\tfrac{2(t-1)(-16t^4+136t^3-455t^2+658t-335)}{(1+t)(8t^3-41t^2+71t-40)}},$$
$$a^+_{86} = \sqrt{\tfrac{5(t-1)(5-3t)(9t-11)(14-19t+7t^2)}{(8t^3-41t^2+71t-40)(-16t^4+136t^3-455t^2+658t-335)}},$$
$$a^+_{7,10} = \sqrt{\tfrac{4(4-t)(t-1)}{t+1}},$$
$$a^-_{8,10} = -\sqrt{\tfrac{-16t^4+136t^3-455t^2+658t-335}{4(4-t)(t^2-1)}},$$
$$a^c_{96} = \sqrt{\tfrac{(29-11t)(8t^3-41t^2+71t-40)}{-16t^4+136t^3-455t^2+658t-335}}\; e^{ix},$$

$a^{+}_{10,10} = \sqrt{\frac{31-37t+12t^2}{4(4-t)(t-1)}}$,
$a^{+}_{8,11} = \sqrt{\frac{(t+1)(31-37t+12t^2)(-47+57t-16t^2)}{4(4-t)(t-1)(-16t^4+136t^3-455t^2+658t-335)}}$,
$a^{+}_{88} = \sqrt{\frac{(3-t)(t-1)(89-87t+24t^2)}{-16t^4+136t^3-455t^2+658t-335}}$,
$a^{+}_{9,11} = \sqrt{\frac{4(4-t)(t^2-1)(7-4t)}{-16t^4+136t^3-455t^2+658t-335}}$,
$a^{c}_{98} = \sqrt{\frac{-8t^4+89t^3-401t^2+699t-403}{-16t^4+136t^3-455t^2+658t-335}}\, e^{iy}$,
$a^{+}_{10,11} = \sqrt{\frac{-47+57t-16t^2}{4(4-t)(t-1)}}$.

We are not trying to find exact values of the range for t. Anyway, it is not difficult to show that all the expressions inside radicals are positive for $t \in \left[\frac{7}{5}, \frac{8}{5}\right]$.

Real numbers x and y can be found from the orthogonality condition of the 10th and the 11th rows. It could be verified straight-forward that the endomorphism ring of the representation Γ (and of the representation T) is trivial; therefore, the representations Γ and T are indecomposable.

Thus, for $t \in \left[\frac{7}{5}, \frac{8}{5}\right]$ we have two (complex-conjugate) orthoscalar representations of the poset $(N, 4)$. $\square$

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Received by the editors: 14.05.2012
and in final form 25.05.2012.