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A commutative Bezout PM^* domain is an elementary divisor ring

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Abstract. We prove that any commutative Bezout PM^* domain is an elementary divisor ring.

The aim of this paper is to study the question of diagonalizability for matrices over a ring. It is well-known that any elementary divisor domain is a Bezout domain and it is a classical open question to determine whether the converse statement is true?

The notion of an elementary divisor ring was introduced by Kaplansky in [6]. There are a lot of researches that deal with the matrix diagonalization in different cases (the most comprehensive history of these researches can be found in [10]). It is an open question dating back at least to Helmer [5] in 1942 to decide, whether a commutative Bezout domain is always an elementary divisor domain. Helmer showed that not only does the domain of entire functions is an elementary divisor domain, it also has a property which he labeled adequate. Henriksen [4] appears to be the first person to have given an example to show that being adequate is a stronger property than that of being an elementary divisor ring. In proving this, Henriksen observed that in an adequate domain each nonzero prime ideal is contained in a unique maximal ideal [4]. It is a natural question to ask whether or not the converse holds and this question is explicitly raised in [7]. The negative answer to this question is given in [1]. Furthermore, it is shown that there exists an elementary divisor ring

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which is not adequate but which does have the property that each nonzero prime ideal is contained in a unique maximal ideal. In this paper we show that a commutative Bezout domain in which each nonzero prime ideal is contained in a unique maximal ideal is an elementary divisor ring. Note that these results are responses to open questions work [12, Questions 10, Problem 6].

We introduce the necessary definitions and facts.

All rings considered will be commutative and have identity. A ring is a *Bezout ring*, if every its finitely generated ideal is principal. A ring R is *an elementary divisor ring* if every matrix A (not necessarily square one) over R admits diagonal reduction, that is, there exist invertible square matrices P and Q such that PAQ is a diagonal matrix, say (d_{ij}), for which d_{ii} is a divisor of $d_{i+1,i+1}$ for each i. A ring R to be *right Hermite* if every 1×2 matrix over R admits diagonal reduction. Any Hermite ring is a Bezout ring. For domains, the notions of Hermite and Bezout ring are equivalent. Gillman and Henriksen showed that any commutative ring R is an Hermite ring if and only if for all $a, b \in R$ there exist $a_1, b_1, d \in R$ such that $a = a_1 d$, $b = b_1 d$ and $a_1 R + b_1 R = R$ [10]. Furthermore, they proved the following result, which we state formally.

Proposition 1. *Let R be a commutative Bezout ring. R is an elementary divisor ring if and only if R is an Hermite ring that satisfies the extra condition that for all $a, b, c \in R$ with $aR+bR+cR = R$ there exist $p, q \in R$ such that $paR + (pb + qc)R = R$.*

Definition 1. Let R be a commutative Bezout domain. A nonzero element a in R is called an adequate element if for every $b \in R$ there exist $r, s \in R$ such that $a = rs$, $rR + bR = R$, and if s' is a non-unit divisor of s, then $s'R + bR \neq R$. If every nonzero element of the ring R is adequate, then R is called an adequate ring [5, 10].

Definition 2. Let R be a commutative ring. An element $a \in R$ is called a clean element if a can be written as the sum of a unit and an idempotent. If every element of R is clean, then we say that R is a clean ring [8, 9].

Any clean ring is a Gelfand ring. Recall that a ring R is called a *Gelfand ring* if for every $a, b \in R$ such that $a + b = 1$ there are $r, s \in R$ such that $(1 + ar)(1 + bs) = 0$. A ring R is called a *PM-ring* if each prime ideal is contained in a unique maximal ideal. It had been asserted that a commutative ring is a Gelfand ring if and only if it is a PM-ring [2, 3]. A ring R is called a *PM^*-ring* if each nonzero prime ideal is contained in a unique maximal ideal [9]. A ring R is said to be *a ring of stable range 1*, if for any $a, b \in R$ such that $aR + bR = R$ there exist $t \in R$ such that $(a + bt)R = R$.

Definition 3. An element $a \in R \setminus \{0\}$ of a commutative ring R is called a PM-element if the factor ring R/aR is a PM-ring.

Proposition 2. *For a commutative ring R the following are equivalent:*

1) $a \in R$ is a PM-element;

2) for each prime ideal P such that $a \in P$ there exists a unique maximal ideal M such that $P \subset M$.

Proof. This is obvious, since $\overline{P}$ is a prime ideal of R/aR if and only if there exists a prime ideal P such that $aR \subset P$ and $\overline{P} = P/aR$. □

As a consequence of Proposition 2 we obtain the following result.

Proposition 3. *A commutative domain R is a domain in which each nonzero prime ideal is contained in a unique maximal ideal of R if and only if every nonzero element of R is a PM-element.*

Proposition 4. *An element a of a commutative Bezout domain is a PM-element if and only if, for every elements $b, c \in R$ such that $aR + bR + cR = R$, an element a can be represented as $a = rs$, where $rR + bR = R$, $sR + cR = R$.*

Proof. Denote $\overline{R} = R/aR$, $\overline{b} = b + aR$, $\overline{c} = c + aR$. Since $aR + bR + cR = R$, we see that $\overline{bR} + \overline{cR} = \overline{R}$. Therefore, if $a = rs$ where $rR + bR = R$, $sR + cR = R$, then $\overline{bR} + \overline{cR} = \overline{R}$ and $\overline{0} = \overline{rs}$ where $\overline{r}\overline{R} + \overline{bR} = \overline{R}$, $\overline{s}\overline{R} + \overline{c}\overline{R} = \overline{R}$. By [2], $\overline{R}$ is a PM-ring.

If $\overline{R}$ is a PM-ring then, by [9], $\overline{0} = \overline{rs}$ where $\overline{r}\overline{R} + \overline{bR} = \overline{R}$, $\overline{s}\overline{R} + \overline{c}\overline{R} = \overline{R}$ for arbitrary $\overline{b}, \overline{c} \in \overline{R}$ such that $\overline{bR} + \overline{c}\overline{R} = \overline{R}$. Whence we obtain $aR + bR + cR = R$. Because $\overline{0} = 0 + aR = \overline{rs}$, we have $rs \in aR$, where $\overline{r} = r + aR$, $\overline{s} = s + aR$. Let $rR + aR = r_1R$, $sR + aR = s_1R$. From this $r = r_1r_0$, $a = r_1a_0$, $s = s_1s_2$, $a = s_1a_2$, where $r_0R + a_0R = R$, $s_2R + a_2R = R$. Since $r_0R + a_0R = R$, we obtain $r_0u + a_0v = 1$ for some $u, v \in R$. Since $rs \in aR$, we see that $rs = at$ for some $t \in R$. Then $r_1r_0s = r_1a_0t$, because R is a domain, and we have $a_0t = r_0s$. By the equality, $r_0u + a_0v = 1$ we have $sr_0u + sa_0v = s$, $a_0(tu + a_0v) = s$. Therefore $a = r_1a_0$, where $r_1R + bR + r_1a_0R = R$. Then $r_1R + bR = R$. Since $a_0(tu + a_0v) = s$ and $a_0R + cR + aR = R$, we obtain $a_0R + cR = R$. The proposition is proved. □

Theorem 1. *A commutative Bezout domain in which each nonzero prime ideal is contained in a unique maximal ideal is an elementary divisor ring.*

Proof. Let R be a commutative Bezout domain with the property that each nonzero prime ideal is contained in a unique maximal ideal. According to Proposition 4, let $a, b, c \in R$ be such that $aR+bR+cR = R$. According to the restrictions imposed on R, by Proposition 4, we have $b = rs$ where $rR+aR = R$, $sR+cR = R$. Let $p \in R$ be such that $sp+ck = 1$ for some $k \in R$. Hence $rsp + rck = r$ and $bp + crk = r$. Denoting $rk = q$ and we obtain $(br + cq)R + aR = R$. Let $pR + qR = dR$ and $d = pp_1 + qq_1$ with $p_1R + q_1R = R$. Hence $p_1R + (p_1b + q_1c)R = R$ and, since $pR \subset p_1R$, we obtain $p_1R + cR = R$ and $p_1R + (p_1b + q_1c)R = R$.

Since $bp + cq = d(bp_1 + cq_1)$, and $(bp + cq)R + aR = R$ we obtain $(bp_1 + cq_1)R + aR = R$. Finally, we have $ap_1R + (bp_1 + cq_1)R = R$. By Proposition 1, we obtain that R is an elementary divisor ring. The theorem is proved. □

Remark 1. Note that in order to prove this theorem, it is necessary that only the element $b \in R$ is a PM-element.

Let R be a commutative Bezout domain. We denote by $S = S(R)$ the set of all PM-elements of R. Since $1 \in R$, the set S is nonempty. Furthermore, we obtain the following result.

Proposition 5. *The set $S(R)$ of all PM-elements of a commutative domain R is a saturated multiplicatively closed set.*

Proof. Let $a, b \in S(R)$. We show that $ab \in S(R)$. Suppose the contrary. Then there exist a prime ideal P and maximal ideals M_1, M_2 such that $M_1 \neq M_2$ and $ab \in P \subset M_1 \cap M_2$. Since $ab \in P$, we obtain that $a \in P$ or $b \in R$. It is impossible because $a \in S(R)$, $b \in S(R)$ and $P \subset M_1 \cap M_2$. Therefore $S(R)$ is a multiplicatively closed set.

Let $ab \in S(R)$ for some $a, b \in R$. If $a \notin S(R)$ then there exists a prime ideal P such that $a \in P$ and $P \subset M_1 \cap M_2$ for some maximal ideals M_1, M_2 and $M_1 \neq M_2$. Therefore, $ab \in P$ and $P \subset M_1 \cap M_2, M_1 \neq M_2$. It is impossible because $ab \in S(R)$. Hence $S(R)$ is a saturated multiplicatively closed set. The Proposition is proved. □

Let R be a commutative Bezout domain and $S(R)$ be the set of all PM-elements of R. Since $S(R)$ is a saturated multiplicatively closed set, we can consider the localization of R with denominators from $S(R)$ i.e. the ring of fractious R_S. We have:

Theorem 2. *Let R be a commutative elementary divisor domain. Then a ring R_S is an elementary divisor ring.*

Proof. Suppose that R is an elementary divisor ring. We need to show that R_S is also an elementary divisor ring. Let $as^{-1}, bs^{-1}, cs^{-1}$ be any elements from R_S such that

$$as^{-1}R_S + bs^{-1}R_S + cs^{-1}R_S = R_S.$$

Then $aR+bR+cR = dR$, for some element $d \in S(R)$. Let $a = a_1d, b = b_1d$, $c = c_1d$ for some elements $a_1, b_1, c_1 \in R$ such that $a_1R + b_1R + c_1R = R$. Since R is an elementary divisor ring, there are elements $u, v, p, q \in R$ such that

$$a_1pu + (b_1p + c_1q)v = 1.$$

Then

$$apR_S + (bp + cq)R_S = R_S.$$

By [6], R_S is an elementary divisor ring. Theorem is proved. □

Let R be a commutative Bezout domain and $S = S(R)$ be the set of all PM-elements of R. Since $S(R)$ is a saturated multiplicatively closed set, we can construct by transfinite induction a natural chain

$$\{R^\alpha | \alpha \text{ is an ordinal}\}$$

of the saturated multiplicatively closed sets in R as follows. Let $R^0 = S(R)$. Let α be an ordinal greater than zero and assume R^β has been defined and is a saturated multiplicatively closed set in R, whenever $\beta < \alpha$ and let $K_\beta = R_{R_\beta}$. Then K_β is a commutative Bezout domain (see [10]) and hence $S(K_\beta)$ is a saturated multiplicatively closed set by Proposition 5.

We define R^α by $R^\alpha = \bigcup_{\beta<\alpha} R^\beta$ if α is a limit ordinal and $R^\alpha = S(K_{\alpha-1}) \cap R$ otherwise. It is obvious that R^α is a saturated multiplicatively closed set. If α, β are ordinals such that $\alpha \leqslant \beta$ then $R^\alpha \subset R^\beta \subset R$. Also $R^\alpha = R^{\alpha+1}$ for some ordinal α. In case, when $R^\alpha \neq R^{\alpha+1}$ for each ordinal α, then

$$\operatorname{card}(R^\alpha) > \operatorname{card}(\alpha).$$

Choosing β such that $\operatorname{card}(\beta) > \operatorname{card}(R)$ we obtain

$$\operatorname{card}(\beta) > \operatorname{card}(R) > \operatorname{card}(R^\beta),$$

a contradiction. We let α_0 denote the least ordinal such that

$$R^{\alpha_0} = R^{\alpha_0+1}$$

and we call

$$\{R^{\alpha}|0 \leqslant \alpha \leqslant \alpha_0\}$$

a D-chain in R. In this situation R^{-1} will denote the group of units of R.

By Theorem 2 and the fact that union of elementary divisor rings are an elementary divisor ring and using D-chain of a commutative Bezout domain we can conclude that the problem of being a commutative Bezout domain an elementary divisor ring is reduced to the case of a commutative Bezout domain where PM-elements are the only units, when $U(R) = S(R)$.

Definition 4. Let R be a commutative Bezout domain. An element $a \in R$ is called a neat element if R/aR is a clean ring.

Obvious examples of neat elements are units of a ring, and adequate elements of a ring [11]. If R is a commutative Bezout domain and a is a neat element of R, then R/aR is a clean ring [9], that is R/aR is a PM-ring. Hence we obtain the following result.

Proposition 6. *Every neat element of a commutative Bezout domain is a PM-element.*

Definition 5. A commutative ring R is said to be of the neat range 1 if for any $a, b \in R$ such that $aR + bR = R$ there exists $t \in R$ such that for the element $a + bt = c$ the ring R/cR is a clean ring [11].

Theorem 3 ([11]). *A commutative Bezout domain is an elementary divisor ring if and only if R is a ring of the neat range 1.*

From this we obtain the following result.

Theorem 4. *Let R be a commutative Bezout domain and $U(R) = S(R)$. Then R is an elementary divisor ring if and only if stable range of R is equal to 1.*

Proof. Since every neat element is a PM-element and $U(R) = S(R)$, then only units in a ring are neat elements. Then by Theorem 3, R is an elementary divisor ring if and only if R is a ring of stable range 1. Theorem is proved. □

Let R be a commutative Bezout domain and $a \in R$ is a neat element of R. By [9] the stable range of R/aR is equal to 1. Consequently by Theorem 4, we have a next result.

Theorem 5. *Let R be a commutative Bezout domain such that for every nonzero element $a \in R$ stable range of R/aR is not equal 1. Then R is not an elementary divisor ring.*

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