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Characterization of finite groups with some S-quasinormal subgroups of fixed order

M. Asaad and Piroska Csörgő

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Abstract. Let G be a finite group. A subgroup of G is said to be S-quasinormal in G if it permutes with every Sylow subgroup of G. We fix in every non-cyclic Sylow subgroup P of the generalized Fitting subgroup a subgroup D such that $1 < |D| < |P|$ and characterize G under the assumption that all subgroups H of P with $|H| = |D|$ are S-quasinormal in G. Some recent results are generalized.

1. Introduction

All groups considered in this paper are finite. The terminology and notations employed agree with standard usage, as in Huppert [5]. Two subgroups H and K of a group G are said to permute if $KH = HK$. It is easily seen that H and K permute if and only if the set HK is a subgroup of G. We say, following Kegel [7], that a subgroup of G is S-quasinormal in G, if it permutes with every Sylow subgroup of G.

For any group G, the generalized Fitting subgroup $F^*(G)$ is the set of all elements x of G which induce an inner automorphism on every

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chief factor of G. Clearly $F^*(G)$ is a characteristic subgroup of G and $F^*(G) \neq 1$ if $G \neq 1$ (see in [6, X. 13]). By [5, III. 4.3] $F(G) \leqslant F^*(G)$.

A number of authors have examined the structure of a finite group G under the assumption that all subgroups of G of prime order are well-situated in G. The authors [1] showed that if G is a solvable group and every subgroup of $F(G)$ of prime order or of order 4 is S-quasinormal in G, then G is supersolvable. Li and Wang [8] showed that if G is a group and every subgroup of $F^*(G)$ of prime order or of order 4 is S-quasinormal in G, then G is supersolvable.

Yao, Wang and Li in [4] gave a revised version of our earlier result in [2]:

Theorem 1.1 ([4, Theorem 1']). Let G be a group of composite order such that G is quaternion-free. Suppose G has a nontrivial normal subgroup N such that G/N is supersolvable. Then the following statements are equivalent:

(1) Every subgroup of $F^*(N)$ of prime order is S-quasinormal in G.

(2) $G = UW$, where U is a normal nilpotent Hall subgroup of odd order, W is a supersolvable Hall subgroup with $(|U|, |W|) = 1$ and every subgroup of $F(N)$ of prime order is S-quasinormal in G.

(3) N is solvable and every subgroup of $F(N)$ of prime order is S-quasinormal in G.

In this paper we generalize this theorem: instead of requiring the S-quasinormality of every subgroup of $F^*(N)$ of prime order we fix in every non-cyclic Sylow subgroup P of $F^*(N)$ a subgroup D such that $1 < |D| < |P|$ and characterize G under the assumption that all subgroups H of P with $|H| = |D|$ are S-quasinormal in G.

Theorem 1.2. Let G be a group of composite order such that G is quaternion-free. Suppose that G has a nontrivial normal subgroup N such that G/N is supersolvable. Then the following statements are equivalent:

(1) Every non-cyclic Sylow subgroup P of $F^*(N)$ has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with $|H| = |D|$ are S-quasinormal in G.

(2) $G = UW$, where U is a normal nilpotent Hall subgroup of G of odd order, W is a supersolvable Hall subgroup of G with $(|U|, |W|) = 1$, every non-cyclic Sylow subgroup P of $F(N)$ of odd order has a

subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with $|H| = |D|$ permute with R, where R is any Sylow subgroup of G with $(|R|, |U|) = 1$ and $O_2(N) \leqslant Z_\infty(G)$.

(3) N is solvable and every non-cyclic Sylow subgroup P of $F(N)$ has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with $|H| = |D|$ are S-quasinormal in G.

2. Preliminaries

Lemma 2.1 ([2, Lemma 2.1])**.** Suppose that G is a quaternion-free group and every subgroup of G of order 2 is normal in G. Then G is 2-nilpotent.

Lemma 2.2 ([4, Lemma 2])**.** Suppose that G is a quaternion-free group. If every subgroup of G of order 2 is S-quasinormal in G, then G is 2-nilpotent.

Lemma 2.3 ([3])**.** Let p be the smallest prime dividing $|G|$ and let P be a Sylow p-subgroup of G. If every maximal subgroup of P is S-quasinormal in G, then G is p-nilpotent.

Lemma 2.4 ([7])**.** Let G be a group and $H \leqslant K \leqslant G$. Then

(1) If H is S-quasinormal in G, then H is S-quasinormal in K.

(2) Suppose that H is normal in G. Then K/H is S-quasinormal in G/H if and only if K is S-quasinormal in G.

Lemma 2.5 ([9])**.** Let G be a group and let P be an S-quasinormal p-subgroup of G, where p is a prime. Then $O^p(G) \leqslant N_G(P)$.

As an immediate consequence of [10, Theorem 1.3], we have

Lemma 2.6. Let G be a group with a normal subgroup N such that G/N is supersolvable. Suppose that every non-cyclic Sylow subgroup P of $F^*(N)$ has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with order $|H| = |D|$ and with order $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) are S-quasinormal in G. Then G is supersolvable.

3. Main results

As an improvement of Lemma 2.1, we have

Lemma 3.1. Suppose that G is a quaternion-free group. Let P be a Sylow 2-subgroup of G and $D \leqslant P$ with $2 \leqslant |D| < |P|$. If every subgroup of P of order $|D|$ is normal in G, then G is 2-nilpotent.

Proof. Suppose that the lemma is false and let G be a counterexample of minimal order. If $|D| = 2$, then every subgroup of P of order 2 is normal in G and hence G is 2-nilpotent by Lemma 2.1, a contradiction. Thus we may assume that $2 < |D| < |P|$. Let H be a subgroup of P such that $|H| = |D|$. Then H is normal in G by the hypothesis of the lemma. Let K be a subgroup of P such that $H \leqslant K$ and $|K| = 2|H|$. It is clear that HR is a subgroup of G, where R is any Sylow subgroup of G of odd order. If H is cyclic, then R is normal in HR by [5]. If H is not cyclic, then K is not cyclic. Hence there exists a maximal subgroup L of K such that $L \neq H$. Clearly, $K = HL$. By the hypothesis of the lemma H and L are normal in G. Then K is normal in G, so KR is a subgroup of G. Clearly, all maximal subgroups of K are normal in KR. Then R is normal in KR by Lemma 2.3 and so R is normal in HR. Thus $HR = H \times R$, where R is any Sylow subgroup of G of odd order. Then by [11, p. 221], $1 \neq H \leqslant Z_\infty(G)$, so $Z(G) \neq 1$. Let $A \leqslant Z(G)$ such that $|A| = 2$. Then A is normal in G. Now consider G/A. Clearly, every subgroup of P/A of order $\frac{|D|}{|A|}$ is normal in G/A (recall that $|D| > 2$). Then G/A is 2-nilpotent by our minimal choice of G and since $A \leqslant Z(G)$, it follows that G is 2-nilpotent, a contradiction. □

As an improvement of Lemma 2.2, we have

Lemma 3.2. Suppose that G is a quaternion-free group. Let P be a Sylow 2-subgroup of G and $D \leqslant P$ with $2 \leqslant |D| < |P|$. If every subgroup of P of order $|D|$ is S-quasinormal in G, then G is 2-nilpotent.

Proof. Suppose that the lemma is false and let G be a counterexample of minimal order. Then there exists a subgroup H of P of order $|D|$ such that H is not normal in G by Lemma 3.1. By the hypothesis, H is S-quasinormal in G. Then by Lemma 2.5 $O^2(G) \leqslant N_G(H) < G$. Let M be a maximal subgroup of G such that $N_G(H) \leqslant M < G$. Then $|G/M| = 2$. Let M_2 be a Sylow 2-subgroup of M. If $|D| = |M_2|$, then every maximal subgroup of P is S-quasinormal in G and so G is 2-nilpotent by Lemma 2.3, a contradiction. Thus we may assume that M_2 has a subgroup D such that $2 \leqslant |D| < |M_2|$. By Lemma 2.4 (1), every subgroup of M_2 of order $|D|$ is S-quasinormal in M. Then M is 2-nilpotent by the minimal choice of G and so G is 2-nilpotent, a contradiction. □

As an immediate consequence of Lemma 3.2, we have

Lemma 3.3. Suppose that G is a quaternion-free group. Let P be a Sylow 2-subgroup of G. If P is cyclic or P has a subgroup D with $2 \leqslant |D| < |P|$ such that every subgroup of P of order $|D|$ is S-quasinormal in G, then G is 2-nilpotent.

As a corollary of the proofs of Lemmas 3.1 and 3.2, we have

Lemma 3.4. Let G be a group of odd order, p be the smallest prime dividing $|G|$ and P a Sylow p-subgroup of G. If P is cyclic or P has a subgroup D with $p \leqslant |D| < |P|$ such that every subgroup of P of order $|D|$ is S-quasinormal in G, then G is p-nilpotent.

Lemma 3.5. Let G be a supersolvable group of composite order. Then $G = UW$, where U is a normal nilpotent Hall subgroup of G of odd order, W is a supersolvable Hall subgroup of G with $(|U|, |W|) = 1$.

Proof. Since G is supersolvable of composite order, it follows that G possesses a Sylow tower of supersolvable type. Hence P is normal in G, where P is a Sylow p-subgroup of G and p $(p > 2)$ is the largest prime dividing $|G|$. Let U be a normal nilpotent Hall subgroup of G of odd order such that $P \leqslant U$. By the Schur–Zassenhaus Theorem, G has a Hall subgroup W such that $G = UW$ with $(|U|, |W|) = 1$. Clearly W is supersolvable. □

Proof of Theorem 1.2. By the hypothesis of the theorem N is a nontrivial subgroup of G. Then $F^*(N) \neq 1$ (see [6, X. 13]).

(1) $\Longrightarrow$ (2) If every noncyclic Sylow subgroup P of $F^*(N)$ has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with $|H| = |D|$ are S-quasinormal in G, then all subgroups H of P with $|H| = |D|$ are S-quasinormal in $F^*(N)$ by Lemma 2.4 (1). By Lemmas 3.3 and 3.4, $F^*(N)$ possesses an ordered Sylow tower of supersolvable type. Then $F^*(N)$ is solvable and so $F^*(N) = F(N)$ (see [6, Ch. X. 13]).

Let p be the smallest prime dividing $|F(N)|$. If $p = 2$, then $O_2(N) \neq 1$ and $O_2(N)R$ is a subgroup of G for any Sylow subgroup R of G of odd order. Then by Lemmas 2.4 (1) and 3.3, $O_2(N)R$ is 2-nilpotent. Hence $O_2(N)R = O_2(N) \times R$ for any Sylow subgroup R of G of odd order. Now it follows easily that every subgroup of $O_2(N)$ of order $2|D|$ is S-quasinormal in G. Hence G is supersolvable by Lemma 2.6 and consequently $G = UW$, where U is a normal nilpotent Hall subgroup of G of odd order, W is a supersolvable Hall subgroup of G with $(|U|, |W|) = 1$ by Lemma 3.5.

Since $O_2(N)R = O_2(N) \times R$ for any Sylow subgroup R of G of odd order, it follows that $O_2(N) \leqslant Z_\infty(G)$ by [11, Theorem 6.3, p. 221]. Thus (2) holds.

(2) $\Longrightarrow$ (3) Since $G/U \simeq W$ is supersolvable and U is nilpotent, it follows that G is solvable and so N is solvable. Let R be any Sylow subgroup of G. If $R \leqslant U$, then R is normal in G. If $(|R|, |U|) = 1$, then by (2), every non-cyclic Sylow subgroup P of $F(N)$ of odd order has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with $|H| = |D|$ permute with R. Thus either R is normal in G or $(|R|, |U|) = 1$, we have that every non-cyclic Sylow subgroup P of $F(N)$ of odd order has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with $|H| = |D|$ are S-quasinormal in G. On the other hand, $O_2(N) \leqslant Z_\infty(G)$. Then by [11, Theorem 6.2, p. 221], every subgroup of $O_2(N)$ is S-quasinormal in G. Thus (3) holds.

(3) $\Longrightarrow$ (1) It is clear. □

Corollary 3.6. Let G be a group of composite order such that G is quaternion-free. Then the following statements are equivalent:

(1) Every non-cyclic Sylow subgroup P of $F^*(G)$ has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with $|H| = |P|$ are S-quasinormal in G.

(2) $G = UW$, where U is a normal nilpotent Hall subgroup of G of odd order, W is a supersolvable Hall subgroup of G with $(|U|, |W|) = 1$, every non-cyclic Sylow subgroup P of $F(G)$ of odd order has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with $|H| = |D|$ permute with R, where R is any Sylow subgroup of G with $(|R|, |U|) = 1$ and $O_2(G) \leqslant Z_\infty(G)$.

(3) G is solvable and every non-cyclic Sylow subgroup P of $F(G)$ has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with $|H| = |D|$ are S-quasinormal in G.

Proof. This is an immediate consequence of Theorem 1.2 if $N = G$. □

As an immediate corollary of Theorem 1.2 we get Theorem 1.1.

References

[1] M. Asaad, P. Csörgő, The influence of minimal subgroups on the structure of finite groups, *Archiv der Mathematik* **72** (1999), 401–404.

[2] M. Asaad, P. Csörgő, Characterization of Finite Groups with Some S-quasinormal Subgroups, *Monatsh. Math.* **146** (2005), 263–266.

[3] M. Asaad, A. A. Heliel, On S-quasinormally embedded subgroups of finite groups, *J. Pure and Applied Algebra* **165** (2001), 129–135.

[4] Junhong Yao, Changqun Wang, Yangming Li, A note on characterization of finite groups with some S-quasinormal subgroups, *Monatsh. Math.*, **161** (2010), 233–236.

[5] B. Huppert, *Endliche Gruppen I*, Springer-Verlag, Berlin, 1979.

[6] B. Huppert, N. Blackburn, *Finite Groups III*, Springer-Verlag, Berlin, 1982.

[7] O. H. Kegel, Sylow Gruppen und Subnormalteiler endlicher Gruppen, *Math. Z.* **78** (1962), 205–221.

[8] Y. Li, Y. Wang, The influence of minimal subgroups on the structure of finite groups, *Proc. Amer. Math. Soc.* **131** (2003), 337–341.

[9] P. Schmidt, Subgroups permutable with all Sylow subgroups, *J. Algebra* **267** (1998), 285–293.

[10] A. N. Skiba, On weakly s-permutable subgroups of finite groups, *J. Algebra* **315** (2007), 192–209.

[11] M. Weinstein (Ed.), Between Nilpotent and Solvable, Polygonal Publishing House, Passaic, 1982.

Contact information

M. Asaad
Cairo University, Faculty of Science, Department of Mathematics,
Giza 12613, Egypt
E-Mail: moasmo45@hotmail.com

Piroska Csörgő
Eötvös University, Department of Algebra and Number Theory,
Pázmány Péter sétány 1/c,
H–1117 Budapest, Hungary
E-Mail: ska@cs.elte.hu

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