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Expansions of numbers in positive Lüroth series and their applications to metric, probabilistic and fractal theories of numbers

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Abstract. We describe the geometry of representation of numbers belonging to $(0, 1]$ by the positive Lüroth series, i.e., special series whose terms are reciprocal of positive integers. We establish the geometrical meaning of digits, give properties of cylinders, semicylinders and tail sets, metric relations; prove topological, metric and fractal properties of sets of numbers with restrictions on use of "digits"; show that for determination of Hausdorff-Besicovitch dimension of Borel set it is enough to use connected unions of cylindrical sets of the same rank. Some applications of L-representation to probabilistic theory of numbers are also considered.

Introduction

There exist many models of real number based on positive integers. One of them is a model of number in the form of (finite and infinite) regular continued fraction. Today they study and use different models of number in the form of convergent series (number is a series, number is a sum of series, number is expanded in series). Mostly of these series are positive or alternating. Engel [12], Sylvester [16], Lüroth [8, 3, 4, 5, 7, 13], Ostrogradsky [9, 1, 2], Sierpiński [15], Pierce series et al. are among them.

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Some of them have relatively simple self-similar geometry [13, 17, 18, 14], but other have rather complicated and non-self-similar geometry [1, 2, 9, 12, 10]. Such expansions of numbers can be represented in different forms using positive integers. It is an encoding of number with symbols of infinite alphabet.

Lüroth [8] introduced in 1883 expansion of $x \in (0,1]$ in special positive series such that its terms are reciprocal to positive integers. Geometry of this expansion of numbers is self-similar and convenient for modelling of mathematical objects with non-trivial topological, metric and fractal local properties based on relatively simple metric relations generated by cylindrical sets. In papers [17, 18] we particularly studied properties of cylindrical sets and used them for study of one class of infinite Bernoulli convolutions.

In this paper we continue to study geometry of this expansion. In particular, we study properties of semicylinders, supercylinders and tail sets, solve some problems of metric and fractal theories of numbers, provide some applications of results.

1. L-representation of real numbers

Theorem 1. *Any number $x \in (0,1]$ can be uniquely expanded in Lüroth series, i.e., for x exists a unique sequence of positive integers $d_n = d_n(x)$ such that*

$$x = \frac{1}{d_1+1} + \sum_{n=2}^{\infty} \frac{1}{D_{n-1}(d_n+1)} \equiv \Delta^L_{d_1 d_2 \ldots d_n \ldots}, \tag{1}$$

where $D_n \equiv d_1(d_1+1)d_2(d_2+1)\ldots d_n(d_n+1)$.

Proof. Existence. Since $(0;1] = \bigcup\limits_{n=1}^{\infty} (\frac{1}{n+1}, \frac{1}{n}]$, it is evident that there exists d_1 such that $\frac{1}{d_1+1} < x \leqslant \frac{1}{d_1}$. Then

$$0 < x - \frac{1}{d_1+1} \equiv x_1 \leq \frac{1}{d_1} - \frac{1}{d_1+1} = \frac{1}{d_1(d_1+1)} = \frac{1}{D_1}.$$

Since $(0; \frac{1}{D_1}] = \bigcup\limits_{n=1}^{\infty} (\frac{1}{D_1(n+1)}, \frac{1}{D_1 n}]$, it is evident that for $x_1 \in (0; \frac{1}{d_1(d_1+1)}]$ there exists $d_2 \in N$ such that $\frac{1}{d_1(d_1+1)(d_2+1)} < x_1 \leq \frac{1}{d_1(d_1+1)d_2}$. Whence it follows that

$$0 < x_1 - \frac{1}{d_1(d_1+1)(d_2+1)} \equiv x_2 \leq \frac{1}{d_1(d_1+1)d_2(d_2+1)} = \frac{1}{D_2}$$

and

$$x = \frac{1}{d_1+1} + x_1 = \frac{1}{d_1+1} + \frac{1}{d_1(d_1+1)(d_2+1)} + x_2.$$

Let us perform analogous arguments for x_2 and so on to infinity and obtain (1). Series (1) is convergent because of

$$x_m = \frac{1}{d_1(d_1+1)d_2(d_2+1)\dots d_m(d_m+1)} = \frac{1}{D_m} < \frac{1}{2^m} \to 0 \ (m \to \infty).$$

Uniqueness. Suppose that x has at least two different expansions in the form (1): $x = \Delta^L_{d_1\dots d_{m-1}d_m d_{m+1}\dots} = \Delta^L_{d_1\dots d_{m-1}d'_m d'_{m+1}\dots}$, $d_m \neq d'_m$.

Without loss of generality we assume that $d'_m < d_m$. Then

$$\delta \equiv \Delta^L_{d_1\dots d_{m-1}d'_m d'_{m+1}\dots} - \Delta^L_{d_1\dots d_{m-1}d_m d_{m+1}\dots} = \frac{1}{D_m}\cdot\delta_1,$$

$$\delta_1 \equiv \frac{1}{d_m+1} - \frac{1}{d'_m+1} + \sum_{n=1}^{\infty}\frac{1}{D'_{m+n-1}(d'_{m+n}+1)} - \sum_{n=1}^{\infty}\frac{1}{D_{m+n-1}(d_{m+n}+1)}.$$

However,

$$\delta_1 > \left(\frac{d_m - d'_m}{(d'_m+1)(d_m+1)} - \sum_{n=1}^{\infty}\frac{1}{D_{m+n-1}(d_{m+n}+1)}\right) \geq$$
$$\geq \frac{1}{(d'_m+1)(d_m+1)} - \frac{1}{d_m(d_m+1)}\left(\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\dots\right) = 0.$$

Thus, $\delta_1 > 0$. This contradicts the assumption that there are two different expansions of the same number. □

Definition 1. Brief notation $x = \Delta^L_{d_1d_2\dots d_n\dots}$ of the expansion (1) is called *L-representation of* x, and $d_n = d_n(x)$ is its nth L-symbol.

Theorem 2 ([17])**.** *A number $x \in (0,1]$ is rational if its L-representation is periodic.*

2. Geometry of L-representation: cylinders and semicylinders

Definition 2. Let $(c_1, c_2, \dots, c_m)$ be a fixed m-tuple of positive integers. *Cylinder* of rank m with the base $c_1c_2\dots c_m$ is a set

$$\Delta^L_{c_1c_2\dots c_m} := \{x : x = \Delta^L_{c_1c_2\dots c_m d_{m+1}d_{m+2}\dots}, d_{n+i} \in N\}.$$

Cylinders have the following *properties*.

1. $\Delta^L_{c_1c_2\ldots c_m} = \bigcup\limits_{i_1=1}^{\infty}\ldots\bigcup\limits_{i_k=1}^{\infty}\Delta^L_{c_1\ldots c_m i_1 i_2\ldots i_k} \forall k\in N.$

2. Cylinder $\Delta^L_{c_1c_2\ldots c_m}$ is a half-interval with endpoints

$$\inf \Delta^L_{c_1\ldots c_m} = \frac{1}{c_1+1} + \frac{1}{b_1(c_2+1)} + \ldots + \frac{1}{b_{m-1}(c_m+1)} = a_m;$$

$$\max \Delta^L_{c_1\ldots c_m} = a_m + \frac{1}{b_m}, \quad \text{where } b_m = c_1(c_1+1)\ldots c_m(c_m+1).$$

3. The length of cylinder is equal to

$$\left|\Delta^L_{c_1\ldots c_m}\right| = \frac{1}{c_1(c_1+1)\ldots c_m(c_m+1)} = \prod_{i=1}^{m}\frac{1}{c_i(c_i+1)}.$$

4. For any sequence of positive integers (c_n), the intersection

$$\bigcap_{m=1}^{\infty}\Delta^L_{c_1c_2\ldots c_m\ldots} = x \equiv \Delta^L_{c_1c_2\ldots c_m\ldots} \in (0,1].$$

5. If $d_j(a) = d_j(b)$ for $j<m$ and $d_m(a) > d_m(b)$, then $a<b$.

6. Rearrangement of L-symbols in the base does not change the length of cylinder.

7. Basic metric relation: $\left|\Delta^L_{c_1\ldots c_m}\right| = i(i+1)\left|\Delta^L_{c_1\ldots c_m i}\right|$.

8. $\sum\limits_{j=a}^{\infty}\left|\Delta^L_{c_1\ldots c_m j}\right| = a\left|\Delta^L_{c_1\ldots c_m}\right|$. 9. $\left|\Delta^L_{c_1\ldots c_m a}\right| = \sum\limits_{j=a(a+1)}^{\infty}\left|\Delta^L_{c_1\ldots c_m j}\right|$.

10. $\left|\Delta^L_{c_1\ldots c_m(i+1)}\right| = \dfrac{2i}{i+2}\left|\Delta^L_{c_1\ldots c_m i1}\right|$.

11. If $a<b$ and $d_j(a)=d_j(b)$ for $j<m$, but $d_m(a)\neq d_m(b)$, then 1) $(a,b]\subset\Delta^L_{d_1(a)\ldots d_{m-1}(a)}$, 2) $\Delta^L_{d_1(a)\ldots d_{m-1}(a)d_m(b)(d_{m+1}(b)+1)}\subset(a,b]$.

12. If $d_m(a)>d_m(b)$, but $d_j(a)=d_j(b)$ for $j<m$, then

$$\Delta^L_{d_1(a)\ldots d_{m-1}(a)d_m(b)(d_{m+1}(b)+1)}\subset(a,b).$$

Definition 3. Let (c_n) be a fixed sequence of positive integers and (k_n) be a fixed increasing sequence of positive integers. *Semicylinder* with the base $\begin{pmatrix} k_1 & k_2 & \ldots & k_n \\ c_1 & c_2 & \ldots & c_n \end{pmatrix}$ is a set

$$\Delta^{k_1k_2\ldots k_n}_{c_1c_2\ldots c_n} \equiv \{x : x = \Delta^L_{d_1d_2\ldots d_k\ldots},\ d_{k_i}(x) = c_i, i=\overline{1,n}\}.$$

Lemma 1. *Semicylinders have the following properties.*

1. $\Delta^{12\ldots n}_{c_1c_2\ldots c_n} = \Delta^L_{c_1c_2\ldots c_n}$.

2. $\Delta^{k_1\ldots k_n}_{c_1\ldots c_n} = \Delta^{k_1}_{c_1} \cap \Delta^{k_2}_{c_2} \cap \ldots \cap \Delta^{k_n}_{c_n} = \Delta^{k_1\ldots k_m}_{c_1\ldots c_m} \cap \Delta^{k_{m+1}\ldots k_n}_{c_{m+1}\ldots c_n}$.

3. Semicylinder is a union of cylinders of rank k_n.

4. Semicylinders Δ^k_c *and* Δ^m_d *are metrically independent iff* $k \neq m$.

5. The Lebesgue measure of $\Delta^{k_1k_2\ldots k_n}_{c_1c_2\ldots c_n}$ *is calculated by formula*

$$\lambda(\Delta^{k_1k_2\ldots k_n}_{c_1c_2\ldots c_n}) = \prod_{i=1}^{n} \frac{1}{c_i(c_i+1)}.$$

Proof. Properties 1–3 follows immediately from the definition of semicylinder.

It is evident that for $k=1$ the set Δ^k_c is an L-cylinder of 1st rank Δ^L_c, and according to Property 3

$$\lambda(\Delta^1_c) = |\Delta^L_c| = \frac{1}{c(c+1)}.$$

If $k=2$, then by definition of the set Δ^k_c and properties of cylinders $\Delta^2_c = \bigcup_{i\in N} \Delta^L_{ic}$. So, the Lebesgue measure is equal to

$$\lambda(\Delta^2_c) = \sum_{i=1}^{\infty} |\Delta^L_{ic}| = \frac{1}{c(c+1)} \sum_{i=1}^{\infty} \frac{1}{i(i+1)} = \frac{1}{c(c+1)}.$$

For $k=3$, we have

$$\Delta^3_c = \bigcup_{i_1=1}^{\infty} \bigcup_{i_2=1}^{\infty} \Delta^L_{i_1i_2c},$$

$$\lambda(\Delta^3_c) = \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{\infty} |\Delta^L_{i_1i_2c}| = \frac{1}{c(c+1)} \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{\infty} \frac{1}{i_1(i_1+1)i_2(i_2+1)} = \frac{1}{c(c+1)}.$$

For any k the Lebesgue measure of the set Δ^{k+1}_c is defined by equality

$$\lambda(\Delta^{k+1}_c) = \sum_{i_1=1}^{\infty} \cdots \sum_{i_{k-1}=1}^{\infty} |\Delta^L_{i_1\ldots i_k c}|.$$

Using Property 7 (basic metric relation) we have

$$\lambda(\Delta^{k+1}_c) = \sum_{i_1=1}^{\infty} \cdots \sum_{i_k=1}^{\infty} |\Delta^L_{i_1\ldots i_k c}| = \frac{1}{c(c+1)} \sum_{i_1=1}^{\infty} \cdots \sum_{i_k=1}^{\infty} |\Delta^L_{i_1\ldots i_k}| = \frac{1}{c(c+1)}.$$

The last equality follows from that fact: $\sum\limits_{i_1=1}^{\infty} \dots \sum\limits_{i_k=1}^{\infty} |\Delta^L_{i_1\dots i_k}| = 1$.

For $n = 2$

$$\Delta^{k_1k_2}_{c_1c_2} = \Delta^{k_1}_{c_1} \bigcap \Delta^{k_2}_{c_2} = \bigcup_{\substack{i_j \in N \\ j\in\{1,2,\dots,k_2-1\}\setminus\{k_1\}}} \Delta^L_{i_1\dots i_{k_1-1}c_1 i_{k_1+1}\dots i_{k_2-1}c_2},$$

$$\lambda(\Delta^{k_1k_2}_{c_1c_2}) = \frac{1}{c_1(c_1+1)}\frac{1}{c_2(c_2+1)} \sum_{\substack{i_j \in N \\ j\in\{1,2,\dots,k_2-1\}\setminus\{k_1\}}} |\Delta^L_{i_1\dots i_{k_1-1} i_{k_1+1}\dots i_{k_2-1}}| =$$

$$= \frac{1}{c_1(c_1+1)}\frac{1}{c_2(c_2+1)} = \lambda(\Delta^{k_1}_{c_1})\lambda(\Delta^{k_2}_{c_2}).$$

The last equality provide *metric independence* of semicylinders $\Delta^{k_1}_{c_1}$ and $\Delta^{k_2}_{c_2}$, i.e., semicylinders Δ^k_c and Δ^m_d for $k \neq m$. If $k = m$, then $\Delta^k_c \cap \Delta^m_d$ is an empty set for $c \neq d$ and $\Delta^k_c = \Delta^m_d$ for $c = d$. Thus $\lambda(\Delta^k_c \cap \Delta^m_d) \neq \lambda(\Delta^k_c)\lambda(\Delta^m_d)$. So, semicylinders are not metrically independent.

One can prove Property 5 by induction. □

Lemma 2. *The family of supercylindrical sets (finite or countable unions of cylinders in W_L) is an algebra, i.e., closed with respect to finite union and complement class of sets.*

Proof. It is evident that union of two supercylindrical sets A and A' is a such set. Let us show that intersection of two supercylindrical sets A and A' is a supercylindrical set. Let $A = \bigcup\limits_i A_i$, $A' = \bigcup\limits_j A'_j$, where A_i and A'_j are cylindrical sets. Then $A \bigcap A' = \bigcup\limits_i \bigcup\limits_j [A_i \bigcap A'_j]$. However, $A_i \bigcap A'_j$ is a cylindrical set. Thus $A \bigcap A'$ is a supercylindrical set by definition.

Now we prove that complement $\overline{B}$ of supercylindrical set B is a such set. Complement of $\Delta^L_{c_1\dots c_m}$ is a union of sets in the form $\Delta^L_{s_1\dots s_m}$, where m-tuple $(s_1 \dots s_m)$ takes all possible combinations of L-symbols except for $(c_1 \dots c_m)$, i.e., complement of cylinder is a countable union of cylinders of the same rank. It is evident that $\overline{B_1 \cup B_2} = \overline{B_1} \cap \overline{B_2}$. So, if we take into account that intersection of two supercylindrical sets is a such set, then we have that complement of supercylindrical set is a such set. □

3. Set of numbers with given sequence of fixed digits

Let (c_n) be a fixed sequence of positive integers, (k_n) be a fixed increasing sequence of positive integers. We consider the set

$$\Delta^{k_1k_2\dots k_n\dots}_{c_1c_2\dots c_n\dots} \equiv \{x : x = \Delta^L_{d_1d_2\dots d_k\dots},\ d_{k_i}(x) = c_i,\ i \in N\}.$$

Theorem 3. *Let $g_n := k_{n+1} - k_n$. 1. If $g_n = 1$ for all n and $k_1 = 1$, then set $\Delta^{k_1k_2\ldots k_n\ldots}_{c_1c_2\ldots c_n\ldots}$ consists from one point $\Delta^L_{c_1c_2\ldots c_n\ldots}$. If inequality $g_n > 1$ is fulfilled for finitely many n, then this set is countable. If inequality $g_n > 1$ is fulfilled for infinitely many n, then it is a continuum set.*

2. Lebesgue measure of the set $\Delta^{k_1k_2\ldots k_n\ldots}_{c_1c_2\ldots c_n\ldots}$ is equal to 0.

Proof. 1. If $g_n = 1$ starting from some n_0, then the set $\Delta^{k_1k_2\ldots k_n\ldots}_{c_1c_2\ldots c_n\ldots}$ is countable because only for finite set of the first $n_0 - 1$ positions there exists an alternative for L-symbols from at most countable set. If $g_n > 1$ for infinitely many n, then $\Delta^{k_1k_2\ldots k_n\ldots}_{c_1c_2\ldots c_n\ldots}$ is a continuum set, because one can establish one-to-one correspondence f between this set and half-interval $(0, 1]$ by formula $f(\Delta^L_{d_1d_2\ldots d_n\ldots}) = \alpha_1 2^{-1} + \alpha_2 2^{-2} + \ldots + \alpha_n 2^{-n} + \ldots$, where $\alpha_n = 0$ if $g_n = 1$, and $\alpha_n = 1$ if $g_n > 1$.

2. If set $\Delta^{k_1k_2\ldots k_n\ldots}_{c_1c_2\ldots c_n\ldots}$ is countable, then its Lebesgue measure is equal to 0 by the properties of the Lebesgue measure. So, it is enough to prove statement 2 if it is a continuum set.

Let F_k be a closure of a union of all cylinders of rank k whose interior contains point from the set $\Delta^{k_1k_2\ldots k_n\ldots}_{c_1c_2\ldots c_n\ldots}$. Since $F_{k_n} \supset F_{k_n+1}$ and

$$\Delta^{k_1k_2\ldots k_n\ldots}_{c_1c_2\ldots c_n\ldots} = \bigcap_{n=1}^{\infty} F_{k_n},$$

we have $\lambda(\Delta^{k_1k_2\ldots k_n\ldots}_{c_1c_2\ldots c_n\ldots}) = \lim\limits_{n\to\infty} \lambda(F_{k_n})$ by the continuity from above of the Lebesgue measure.

From the basic metric relation it follows that

$$|\Delta^L_{c_1c_2\ldots c_{k+1}}| = \frac{1}{c_{k+1}(c_{k+1}+1)} |\Delta^L_{c_1c_2\ldots c_k}|.$$

Thus

$$\lambda(F_{k_{n+1}}) = \sum_{i_1\in N,\ldots,i_{k_{n+1}-1}\in N} |\Delta^L_{i_1i_2\ldots c_1\ldots c_2\ldots c_n\ldots i_{k_{n+1}-1}c_{k_{n+1}}}| =$$

$$= \frac{1}{c_{k_{n+1}}(c_{k_{n+1}}+1)} \sum_{i_1\in N,\ldots,i_{k_{n+1}-1}\in N} |\Delta^L_{i_1i_2\ldots c_1\ldots c_2\ldots c_n\ldots i_{k_{n+1}-1}}| =$$

$$= \frac{1}{c_{k_{n+1}}(c_{k_{n+1}}+1)} \lambda(F_{k_{n+1}-1}).$$

From the definition of $\Delta^{k_1k_2\ldots k_n\ldots}_{c_1c_2\ldots c_n\ldots}$ it follows $F_{k_n} = F_{k_n+1} = \ldots = F_{k_{n+1}-1}$. Thus,

$$\lambda(F_{k_{n+1}}) = \frac{\lambda(F_{k_n})}{c_{k_{n+1}}(c_{k_{n+1}}+1)} = \lambda(F_{k_1}) \prod_{i=2}^{n+1} \frac{1}{c_{k_i}(c_{k_i}+1)} \xrightarrow{n\to\infty} 0. \quad \square$$

4. Shift operator for L-representation

In the set $\mathcal{Z}^L_{(0,1]}$ of all L-representations of numbers belonging to $(0, 1]$ we introduce a binary relation of equivalence "to have the same tail" (we denote it by $\sim$).

Definition 4. Two L-representations $\Delta^L_{\alpha_1\alpha_2\ldots\alpha_n\ldots}$ and $\Delta^L_{\beta_1\beta_2\ldots\beta_n\ldots}$ *have the same tail* or *they are in relation* $\sim$ if there exist positive integers m and k such that $\alpha_{m+j} = \beta_{k+j}$ for any $j \in N$. It is evident that $\sim$ is an equivalence relation (i.e., it is reflexive, symmetric, and transitive) and partitions set where it is defined on the equivalence classes. Any equivalence class is a *tail set.* Any tail set is determinated uniquely by arbitrary its element (representative). Two numbers x and y have the same tail (or they are in relation $\sim$), if their L-representations are in relation $\sim$. We denote it by $x \sim y$.

Lemma 3. *Any tail set is a countable dense in* $(0, 1]$ *set.*

Proof. Let H be any equivalence class, and $x_0 = \Delta^L_{c_1\ldots c_k\ldots}$ be its representative. Then for any positive integer m there exists set H_m of numbers x such that $\alpha_{m+j}(x) = \alpha_{k+j}(x_0)$ for any $j \in N$, $k = 1, 2, \ldots$. Set $H = \bigcup\limits_{m\in N} H_m$ is a countable union of countable set. So, it is countable.

Since number x belongs to set H independently of any finite number of the first L-symbols, we have that there exits point from H in any cylinder of any rank m. Thus, H is an everywhere dense in $(0, 1]$ set. $\square$

Corollary. *Factor set* $G \equiv (0, 1]/\sim$ *is a continuum set.*

In the set $\mathcal{Z}^L_{(0,1]}$ we consider shift operator φ for L-symbols defined by equality $\varphi(\Delta^L_{\alpha_1\alpha_2\ldots\alpha_n\ldots}) = \Delta^L_{\alpha_2\alpha_3\ldots\alpha_n\ldots}$. This operator is a function $\varphi\colon (0, 1] \to (0, 1]$.

It is clear that function φ has a countable set of invariant points $\{\Delta^L_{(c)}$, where $c \in N\}$. It is surjective but not injective, because preimages of $\Delta^L_{c_1c_2\ldots c_k\ldots}$ are points $\Delta^L_{cc_1c_2\ldots c_k\ldots}$, where $c \in N$ (countable set).

Lemma 4. *Function* φ *is: 1) decreasing on any cylinder of* $1st$ *rank; 2) continuous at any point of cylinder of* $1st$ *rank and left-continuous at right endpoint of this interval.*

Proof. 1. Let us consider two points $x_1 = \Delta^L_{\alpha_1\alpha_2(x_1)\ldots\alpha_n(x_1)\ldots}$ and $x_2 = \Delta^L_{\alpha_1\alpha_2(x_2)\ldots\alpha_n(x_2)\ldots}$ belonging to interval $\Delta^L_{\alpha_1}$ such that $x_1 < x_2$. Since

$\alpha_n(\varphi(x)) = \alpha_{n+1}(x)$ and their L-symbols satisfy conditions (1), we have $\varphi(x_1) > \varphi(x_2)$, and this proves first statement.

2. Since function φ is monotonic and bounded on any cylinder of 1st rank, it has finite right and left limits at any point of this interval. Moreover, it has finite left limit at the right endpoint and finite right limit at the left endpoint.

Let $x = \Delta^L_{\alpha_1\alpha_2(x)\ldots\alpha_n(x)\ldots}$ be any irrational point of $\operatorname{int}\Delta^L_{\alpha_1}$, and (x_k) be any sequence of points x_k such that $\lim\limits_{k\to\infty} x_k = x$.

It is easy to prove that $\lim\limits_{k\to\infty} x_k = x$ is equivalent to $\lim\limits_{k\to\infty} m_k = \infty$, where m_k is minimal positive integer such that $\alpha_{m_k}(x_k) \neq \alpha_{m_k}(x)$. In fact, $\lim\limits_{k\to\infty} x_k = x$ is equivalent to the following fact: for any $M > 0$ there exists $m_k > M$ and cylinder $\Delta^L_{\alpha_1\alpha_2(x)\ldots\alpha_{m_k}(x)}$ of rank m_k containing all x_k starting from some k_0.

So, from equalities $\lim\limits_{k\to\infty} x_k = x$ and $\alpha_n(\varphi(x)) = \alpha_{n+1}(x)$ it follows that $\lim\limits_{k\to\infty} \varphi(x_k) = \varphi(x)$, and this proves continuity of the function φ at the point x.

Now let $x = \Delta^L_{\alpha_1\alpha_2(x)\ldots\alpha_n(x)}$ be any rational point of $\operatorname{int}\Delta^L_{\alpha_1}$. Let us consider sequence $x'_k = \Delta^L_{\alpha_1\alpha_2(x)\ldots\alpha_n(x)k}$ converging to x and $x'_k < x$. It is evident that $\lim\limits_{k\to\infty} \varphi(x'_k) = \varphi(x)$, i.e., function φ is left continuous at point x.

Now let us consider sequence $x''_k = \Delta^L_{\alpha_1\alpha_2(x)\ldots(\alpha_n(x)-1)1k}$ converging to x and $x''_k > x$. It is evident that $\lim\limits_{k\to\infty} \varphi(x''_k) = \varphi(x)$, i.e., function φ is right continuous at point x. □

Remark. All points x, $\varphi^n(x)$, $n \in N$, belong to the same tail set, and $x \sim y$ iff there exists positive integers k and m such that $\varphi^k(x) = \varphi^m(y)$.

5. Sets with restrictions on use of L-symbols

Definition 5. A number x is called *L-rational* if its L-representation has a period (1), i.e., $x = \Delta^L_{c_1c_2\ldots c_m(1)}$. A number is called *$L$-irrational* if it is not L-rational.

Any L-rational number is a right endpoint of cylinder, moreover number $\Delta^L_{c_1c_2\ldots c_m(1)}$ is a right endpoint of $\Delta^L_{c_1c_2\ldots c_m}$. Vice versa, right endpoint of any cylinder is L-rational number. It is easy to prove that any L-rational number is rational, but not all rational numbers are L-rational. For example, number $\Delta_{(12)}$ is rational, but is not L-rational.

Theorem 4. *The set* $C \equiv C[L,V] = \{x : x = \Delta^L_{d_1 d_2 \ldots d_n \ldots},\ d_n(x) \in V \subset N\}$ *is*

1. a half-interval $(0,1]$ *if* $V = N$*;*

2. a nowhere dense non-closed set of zero Lebesgue measure coinciding with its closure with respect to countable set if $V \neq N$*;*

3. self-similar if V *is a finite set and* N*-self-similar if* V *is an infinite set; moreover, its self-similar* (N*-self-similar*) *dimension* α_s *is a solution of equation*

$$\sum_{v \in V} \left(\frac{1}{v(v+1)}\right)^x = 1 \qquad \text{if} \quad |V| < \infty; \tag{2}$$

and is a number

$$\alpha_s = \sup_n \left\{ x : \sum_{v : V \ni v \le n} \left(\frac{1}{v(v+1)}\right)^x = 1 \right\} \qquad \text{if} \quad |V| = \infty. \tag{3}$$

Proof. Statement 1 is evident. 2. Let $V \neq N$. It is easy to see that

$$C \subset \bigcup_{k \in V} \Delta^L_k, \ C \subset \bigcup_{\substack{k_i \in V \\ i \in N}} \Delta^L_{k_1 k_2 \ldots k_n} \equiv F_n \subset F_{n-1}, \ C = \bigcap_{k=1}^{\infty} F_k = \lim_{k \to \infty} F_k.$$

Let (a,b) be any subinterval of $(0,1]$. It is evident that cylinder $\Delta^L_{d_1(b) \ldots d_m(b) d_{m+1}(b)+1} \subset (a,b)$, where $d_m(b) \neq d_m(a)$. Let α and β be the endpoints of the cylinder $\Delta^L_{d_1(b) \ldots d_m(b)(d_{m+1}(b)+1)v}$, where $v \in N \backslash V$. Then the interval (α, β) does not contain points of the set C. So, the set C is a nowhere dense set by definition.

For Lebesgue measure λ of the set C the following relation holds:

$$\lambda(C) \le \sum_{k_1 \in V} \ldots \sum_{k_n \in V} |\Delta^L_{k_1 \ldots k_n}| = \sum_{k_1 \in V} \ldots \sum_{k_n \in V} \prod_{i=1}^{n} \frac{1}{k_i(k_i+1)} = b^n \xrightarrow{n \to \infty} 0,$$

where $0 < b^n = \sum\limits_{k \in V \neq N} \dfrac{1}{k(k+1)} < 1$. So, $\lambda(C) = 0$.

3. Since $C = \bigcup\limits_{v \in V} [\Delta^L_v \cap C]$ and

$$1)\ C \overset{k_v}{\sim} \Delta^L_v \cap C, \text{ where } k = \frac{1}{v(v+1)}, \quad 2)\ (\Delta^L_{v_i} \cap C) \cap (\Delta^L_{v_j} \cap C) = \varnothing,$$

the set C is self-similar if V is finite, and N-self-similar if V is infinite.

According to the definition, a self-similar (N-self-similar) dimension is a solution of (2) (or determined by (3) respectively). $\square$

6. Random variable with independent L-symbols

Theorem 5. *Random variable $\xi = \Delta^L_{\tau_1\tau_2\ldots\tau_k\ldots}$ with the following distributions of L-symbols τ_k: $\mathrm{P}\{\tau_k = i\} = p_{ik}$, $i \in N$, has a pure Lebesgue type, moreover, 1. discrete iff*

$$M = \prod_{k=1}^{\infty} \max_i \{p_{ik}\} > 0;$$

2. absolutely continuous iff

$$S = \prod_{k=1}^{\infty} \left(\sum_{i=1}^{\infty} \sqrt{\frac{p_{ik}}{i(i+1)}} \right) > 0; \tag{4}$$

3. singular in other cases, i.e., if $M = 0 = S$.

Proof. Let $\{(\Omega_k, B_k, \mu_k)\}$ and $\{(\Omega_k, B_k, \nu_k)\}$ be two sequences of probability spaces such that $\Omega_k = N$, B_k is a σ-algebra of all subsets of Ω_k,

$$\mu_k(i) = p_{ik}, \ \nu_k(i) = \frac{1}{i(i+1)}, \ k \in N,$$

where p_{ik} is an element of the matrix $\|p_{ik}\|$ determining the distribution of the random variable ξ. It is evident that measure μ_k is absolutely continuous with respect to measure ν_k ($\mu_k \ll \nu_k$) for all $k \in N$. Let us consider the infinite products of probability spaces

$$(\Omega, B, \mu) = \prod_{k=1}^{\infty} (\Omega_k, B_k, \mu_k), \qquad (\Omega, B, \nu) = \prod_{k=1}^{\infty} (\Omega_k, B_k, \nu_k).$$

From Kakutani's theorem [6] it follows that $\mu \ll \nu$ iff

$$\prod_{k=1}^{\infty} \int_{\Omega_k} \sqrt{\frac{d\mu_k}{d\nu_k}} d\nu_k > 0, \quad \text{where integral is the Hellinger integral.}$$

In this case the last inequality is equivalent to condition (4). Therefore, from the condition (4) it follows that the measure μ is absolutely continuous with respect to the measure ν. Let us consider the mapping $\Omega \xrightarrow{f} [0;1]$ defined by equality

$$\forall \omega = (\omega_1, \ldots, \omega_k, \ldots) \in \Omega : \ f(\omega) = \Delta^L_{\omega_1\ldots\omega_k\ldots}.$$

For any Borel set E, we define the measures μ^* and ν^* as the image measures of μ and ν under mapping f: $\mu^*(E) = \mu(f^{-1}(E))$, $\nu^*(E) = \nu(f^{-1}(E))$. The measure μ^* coincides with the probabilistic measure P_ξ and the measure ν^* coincides with the probabilistic measure P_ψ, which equivalent to Lebesgue measure λ. From the absolutely continuity of the measure μ with respect to the measure ν it follows that the measure μ^* is absolutely continuous with respect to the measure ν^*. Since $\nu^* \sim \lambda$, from condition (4) it follows that the random variable ξ is of absolutely continuous distribution. $\square$

7. L-representation and fractal analysis of subsets of $[0, 1]$

Definition 6. *Hausdorff-Besicovitch dimension* of bounded set $E \subset R^1$ is a number $\alpha_0(E) = \sup\{\alpha : H^\alpha(E) \neq 0\} = \inf\{\alpha : H^\alpha(E) = 0\}$, where $H^\alpha(E)$ is a *α-dimensional Hausdorff measure* of E defined by equality

$$H^\alpha(E) = \lim_{\varepsilon \to 0} \inf_{d(E_i)<\varepsilon} \left\{ \sum_i d^\alpha(E_i) : \ E \subset \bigcup_i E_i \right\},$$

$d(E_i)$ is a diameter of the set E_i.

Let W be a class of sets such that they are unions of L-cylinders of the following form:

$$(1) \quad \bigcup_{i=k}^{n} \Delta^L_{c_1 \ldots c_m i}, \qquad\qquad (2) \quad \bigcup_{i=k}^{\infty} \Delta^L_{c_1 \ldots c_m i},$$

where k, n are arbitrary positive integers. It is clear that any cylinder belongs to class W, because for $k = 1$ set (2) is a cylinder as well as set (1) is a cylinder for $k = n$.

Lemma 5. *For any $u \equiv (a, b) \subset (0, 1]$ there exists at most 4 sets belonging to class W covering u and having length not exceeding $|u|$.*

Proof. The following cases are possible: 1. Numbers a and b belong to different L-cylinders of rank 1; 2. a and b belong to the same L-cylinder of rank 1.

Consider every case separately.

1.1. Let a and b belong to neighbouring L-cylinders of 1st rank $\Delta^L_{d_1(b)+1}$ and $\Delta^L_{d_1(b)}$ respectively, and $c = \sup \Delta^L_{d_1(b)+1}$.

a) If $a = c$ (it is equivalent to $d_j(a) = 1$ for $j > 1$), then for covering u it is enough two sets from W:

$$\bigcup_{j=d_2(b)+1}^{\infty} \Delta^L_{d_1(b)j}, \qquad \Delta^L_{d_1(b)d_2(b)}, \tag{5}$$

having the length not exceeding $b - a$ (first one belongs to $(a, b]$, second satisfies Property 6 of cylinders).

b) If $a \neq c$, then there exists $d_k(a) \neq 1$. Let us consider the least such k. Then $\Delta^L_{d_1(a)\dots d_{k-1}(a)1} \subset (a, c]$ and sets

$$\bigcup_{j=1}^{d_k(a)-1} \Delta^L_{d_1(a)\dots d_k(a)j} \qquad \text{and} \qquad \Delta^L_{d_1(a)\dots d_k(a)} \tag{6}$$

cover $(a, c]$ and have length not exceeding $c-a$, and therefore, not exceeding $b - a$. Half-interval $(c, b]$ is covered by two sets (6).

So, for covering $(a, b]$ it is enough 4 sets belonging to W.

1.2. If there exists cylinder $\Delta^L_m \subset (a, b]$, then $(a, b]$ is covered by the sets

$$\bigcup_{j=m}^{\infty} \Delta^L_j, \qquad \bigcup_{j=d_2(b)+1}^{\infty} \Delta^L_{d_1(b)j}, \qquad \Delta^L_{d_1(b)d_2(b)},$$

belonging to W and having length lesser than $b - a$.

2. Let a and b belong to the same cylinder of 1st rank $\Delta^L_{d_1(b)}$. Then there exists positive integer m such that a and b belong to the same cylinder of rank m, but to different cylinders of rank $m + 1$:

$$\Delta^L_{d_1(b)\dots d_m(b)d_{m+1}(a)} \qquad \text{and} \qquad \Delta^L_{d_1(b)\dots d_m(b)d_{m+1}(b)}.$$

Repeating the same arguments as in the case 1, we obtain the same result: for covering $(a, b]$ it is enough at most four sets belonging to W and having length not exceeding $b - a$. □

Theorem 6. *For determination of Hausdorff-Besicovitch dimension of any Borel subset of* $(0, 1]$ *it is enough to use covering by sets belonging to class* W.

Proof. In fact, if u is an arbitrary half-interval belonging to covering E, then there exists at most 4 sets ω_1, ω_2, ω_3, ω_4 belonging to W such that $|\omega_i|^\alpha \leq |u^\alpha|$ for any $\alpha > 0$. If

$$l^\alpha_\varepsilon(E) = \inf_{|v_k| \leq \varepsilon} \sum_k |v_k|^\alpha,$$

where $E \subset \bigcup\limits_k v_k$ and $v_k \in W$, then $m_\varepsilon^\alpha(E) \le l_\varepsilon^\alpha(E) \le 4m_\varepsilon^\alpha(E)$ for any $\varepsilon > 0$. Therefore $H^\alpha(E) \le H_L^\alpha(E) \equiv \lim\limits_{\varepsilon\to\infty} l_\varepsilon^\alpha(E) \le 4H^\alpha(E)$, that is $H_L^\alpha(E)$ and $H^\alpha(E)$ simultaneously (with respect to α) take the values 0 and ∞. Consequently, $\alpha_0(E) = \inf\{\alpha :\ H_L^\alpha(E)\}$. □

Theorem 7. *Continuous strictly increasing probability distribution function F of the random variable with independent identically distributed L-symbols preserve the Hausdorff-Besicovitch dimension iff*

$$p_i = \frac{1}{i(i+1)},\ \forall\ i \in N. \tag{7}$$

Proof. If Equality (7) holds, then distribution is uniform on $[0,1]$, and it is evident that probability distribution function preserve the Hausdorff-Besicovitch dimension.

Suppose that there exists $p_m \neq \frac{1}{m(m+1)}$. Let $p_m < \frac{1}{m(m+1)}$. Then there exists $p_c > \frac{1}{c(c+1)}$, i.e., there exist p_m and p_c such that

$$\left(p_m - \frac{1}{m(m+1)}\right)\left(p_c - \frac{1}{c(c+1)}\right) < 0.$$

Then for any $a \in N$, $m \neq a \neq c$, there exists $g \in \{m, c\}$ such that

$$\left(p_a - \frac{1}{a(a+1)}\right)\left(p_g - \frac{1}{g(g+1)}\right) \ge 0. \tag{8}$$

Let us consider set $C \equiv C[L, \{a, g\}]$ and its image $C' = F(C)$ under transformation F. These sets are self-similar and their self-similar dimensions coincides with Hausdorff-Besicovitch dimensions and are solutions of the following equations

$$a^{-x}(a+1)^{-x} + g^{-x}(g+1)^{-x} = 1 \ \text{ and } \ p_a^x + p_g^x = 1$$

respectively. However, from (8) and $p_g \neq g^{-1}(g+1)^{-1}$ it follows that their solutions does not coincide. Thus, $\alpha_0(C) \neq \alpha_0(C')$. This contradiction proves the theorem. □

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