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SURVEY ARTICLE

On the $\mathcal{F}$-hypercentre of a finite group

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Abstract. Our main goal here is to give a short survey of some recent results of the theory of the $\mathcal{F}$-hypercentre of finite groups.

1. Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. Moreover p is always supposed to be a prime, $\mathbb{P}$ denotes the set of all primes. $\pi(G)$ denotes the set of all primes dividing $|G|$, $\pi(\mathcal{F})$ is the union $\cup_{G\in\mathcal{F}}\pi(G)$. We use $\mathcal{N}$ and $\mathcal{U}$ to denote the classes of all nilpotent and of all supersoluble groups, respectively.

Composition formations. Let $\mathcal{F}$ be a class of groups, that is, $B \in \mathcal{F}$ whenever $B \simeq A \in \mathcal{F}$. The class $\mathcal{F}$ is said to be *hereditary (normally hereditary)* (A.I. Mal'cev [1]) if $H \in \mathcal{F}$ whenever $G \in \mathcal{F}$ and H is a subgroup (a normal subgroup, respectively) of G. If $1 \in \mathcal{F}$, then we write $G^{\mathcal{F}}$ to denote the intersection of all normal subgroups N of G with $G/N \in \mathcal{F}$.

The class $\mathcal{F}$ is said to be a *formation* if either $\mathcal{F} = \varnothing$ or $1 \in \mathcal{F}$ and every homomorphic image of $G/G^{\mathcal{F}}$ belongs to $\mathcal{F}$ for any group G. The formation $\mathcal{F}$ is said to be: (i) *solubly saturated, Baer-local* [2] or *composition* (L.A. Shemetkov [3]) if $G \in \mathcal{F}$ whenever $G/\Phi(N) \in \mathcal{F}$ for

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some *soluble* normal subgroup N of G; (ii) *saturated* or *local* if $G \in \mathcal{F}$ whenever $G/\Phi(G) \in \mathcal{F}$.

Throughout all this paper, $\mathcal{F}$ denotes a non-empty formation.

The $\mathcal{F}$-hypercentre. If H/K is a chief factor of G, then an element $x \in G$ induces the the automorphism α_a on H/K, where $\alpha_a : Kh \to KH^a$. The kernel $\mathrm{Ker}(\alpha)$ of the homomorphism $\alpha : G \to \mathrm{Aut}(H/K)$ is called the centralizer of H/K in G and denoted by $C_G(H/K)$. The quotient $G/C_G(H/K)$ is called the *group of automorphisms induced by* G *on* H/K and denoted by $\mathrm{Aut}_G(H/K)$.

At the analysis of action of G on H/K sometimes instead of the group $\mathrm{Aut}_G(H/K)$, use of the semidirect product $(H/K) \rtimes \mathrm{Aut}_G(H/K)$ appears more convenient.

Definition 1.1. A chief factor H/K of G is called $\mathcal{F}$*-central* in G provided $(H/K) \rtimes (G/C_G(H/K)) \in \mathcal{F}$, otherwise it is called $\mathcal{F}$*-excentric*.

Theorem 1.2 (D.W. Barnes and O.H. Kegel [4]) *If $G \in \mathcal{F}$, then every chief factor of G is $\mathcal{F}$-central in G.*

In general, let E be the largest normal subgroup of G such that each chief factor of G below E is $\mathcal{F}$-central in G. Such subgroup is called the $\mathcal{F}$*-hypercentre of* G and denoted by $Z_{\mathcal{F}}(G)$. A normal subgroup A of G is said to be $\mathcal{F}$*-hypercentral* in G provided $A \leq Z_{\mathcal{F}}(G)$.

It is clear that the $\mathcal{N}$-hypercentre of G coincides with the hypercentre $Z_\infty(G)$ of G, the $\mathcal{U}$-hypercentre of G is the largest normal subgroup of G such that each chief factor of G below $Z_{\mathcal{U}}(G)$ is cyclic.

The hypercentre and the $\mathcal{U}$-hypercentre essentially influence on the structure of G and they are useful for descriptions of some important classes of groups. For example, if all subgroups of G of prime order and order 4 are contained in the hypercentre G, then G is nilpotent (Ito). If all these subgroups are contained in the $\mathcal{U}$-hypercentre of G, then G is supersoluble (Huppert, Doerk). In particular, if G is of odd order and every minimal subgroup of G is normal in G, then G is supersoluble (Buckley). If all minimal subgroups of G are normal in G, then G is soluble (Gaschütz). A group G is quasinilpotent if and only if $G/Z_\infty(G)$ is semisimple [5, X, Theorem 13.6]. A group G is quasisupersoluble (see Section 2) if and only if $G/Z_{\mathcal{U}}(G)$ is semisimple.

The study of $\mathcal{N}$-hypercentral subgroups and $\mathcal{U}$-hypercentral subgroups begins with the papers of Baer [6] and they have close relation to permutable subgroups. For instance, it was proved (see Maier and Schmid [7]) that if $A_G = 1$ and A is a quasinormal subgroup of G (i.e. $AH = HA$ for all subgroups H of G), then A is $\mathcal{N}$-hypercentral in G; if $A_G = 1$ and A is a modular element (in sense Kurosh [8, p. 43]) of the subgroup lattice of G, then A is $\mathcal{U}$-hypercentral in G [8, Theorem 5.2.5]). Some other results, related with the $\mathcal{U}$-hypercentral subgroups are discussed in the book [9] (see also [10, 11, 12, 13, 14, 15]).

2. Quasi-$\mathcal{F}$-groups

A group G is said to be *quasinilpotent* if for every its chief factor H/K and every $x \in G$, x induces an inner automorphism on H/K [5, p.124].

Note that since for every central chief factor H/K every element of G induces trivial automorphism on H/K, one can say that a group G is quasinilpotent if for every its *non-central* chief factor H/K and every $x \in G$, x induces an inner automorphism on H/K.

This obvious observation allows us to consider the following generalization of quasinilpotent groups.

Definition 2.1 ([16, 17]). We say that G *is a quasi-$\mathcal{F}$-group* if for every $\mathcal{F}$-eccentric chief factor H/K of G, every automorphism of H/K induced by an element of G is inner.

In particular, we say that G is a *quasisupersoluble group* if for every *non-cyclic* chief factor H/K of G, every automorphism of H/K induced by an element of G is inner.

A group G is called a *semisimple* if G is either the unit group or the direct product of non-abelian simple group. In particular any non-abelian simple group is semisimple.

The theory of quasinilpotent groups is well represented in the book [5]. A key result of this theory is the following structure theorem.

Theorem 2.2 ([5, Chapter X, Theorem 13.6]). *A a group G is a quasinilpotent if and only if $G/Z_\infty(G)$ is semisimple.*

The first question that arises when we consider the quasisupersoluble groups or the quasi-$\mathcal{F}$-groups, in general, is the following: *What can we say about the structure of the quasi-$\mathcal{F}$-groups?*

The following theorem gives a complete answer to this question in the case of quasisupersoluble groups.

Theorem 2.3 ([10]). *A group G is a quasisupersoluble if and only if $G/Z_{\mathcal{U}}(G)$ is semisimple.*

In general, we have

Theorem 2.4 ([16, 17]). *Let $\mathcal{F}$ be a saturated normally hereditary formation. Then a group G is a quasi-$\mathcal{F}$-group if and only if $G/Z_{\mathcal{F}}(G)$ is semisimple.*

Surprising similarities in the structure of the quasinilpotent groups and the quasi-$\mathcal{F}$-groups makes a real suggestion that the quasi-$\mathcal{F}$-groups inherit some other interesting properties of quasinilpotent groups. This assumption was confirmed in the above-mentioned papers [10, 16, 17].

Our immediate goal is to discuss some of the results of these papers.

The books [2, 3, 18, 19, 20] contains numerous applications of Baer-local formations. Nevertheless, it has long remained an open question how wide is the class of Baer-local formations.

It is well known that the class $\mathcal{F}$ of all nilpotent groups is a saturated formation. L.A. Shemetkov showed in [21] that the class $\mathcal{N}^*$ (we here use the notation in [5]) of all quasinilpotent groups is a Baer-local formation. Perhaps, the class $\mathcal{N}^*$ is the only classic example of the Baer-local formation which is not saturated.

Following Robinson [22], a group G is said to be an *SC-group* if every chief factor of G is a simple group. SC-Groups have many interesting properties. In particular, the class of all such groups is a new example of the Baer-local formation. By above Theorem 2.3 we see that every quasisupersoluble group is an SC-group. These observations are a motivation for attempts to find new series of Baer-local formations among classes of quasi-$\mathcal{F}$-groups. We use $\mathcal{F}^*$ to denote the class of all quasi-$\mathcal{F}$-groups.

Theorem 2.5 ([10]). *The class $\mathcal{U}^*$ of all quasisupersoluble groups is a normally hereditary Baer-local formation.*

In general, we have

Theorem 2.7 ([16, 17]). *Suppose that $\mathcal{F}$ is a saturated formation containing all nilpotent groups. Then:*

(1) *$\mathcal{F}^*$ is a Baer-local formation.*

(2) *$\mathcal{F}$ is normally hereditary, then $\mathcal{F}^*$ is normally hereditary.*

(3) *If $\mathcal{F}$ is closed under taking products of normal subgroups (i.e. $\mathcal{F}$ contains each group $G = AB$ where A and B are normal in G and $A, B \in \mathcal{F}$), then $\mathcal{F}^*$ is also closed under taking products of normal subgroups.*

On the base of Theorems 2.3, one can easily obtain examples of quasisupersoluble groups. For example, let $A = C_7 \rtimes \langle\alpha\rangle$, where $|C_7| = 7$ and α is an automorphism of C_7 with $|\alpha| = 3$. Let $B = A \times A_7$. Then by

Theorem 2.3, B is quasisupersoluble and not quasinilpotent. The group $C = B \rtimes \langle\beta\rangle$, where β is an inner automorphism of A_7 with $|\beta| = 2$ and α acts trivially on A, is an SC-group but not a quasisupersoluble group.

3. On the intersections of $\mathcal{F}$-maximal subgroups

Throughout this section, $\mathcal{F}$ denotes a hereditary saturated formation. A group G is called *$\mathcal{F}$-critical* if G is not in $\mathcal{F}$ but all proper subgroups of G are in $\mathcal{F}$.

Recall that a subgroup U of G is called *$\mathcal{F}$-maximal* in G provided that (a) $U \in \mathcal{F}$, and (b) if $U \leq V \leq G$ and $V \in \mathcal{F}$, then $U = V$ [2, p. 288].

We use $\mathrm{Int}_{\mathcal{F}}(G)$ to denote the intersection of all $\mathcal{F}$-maximal subgroups of G. It is not difficult to show that for any group G we have $Z_{\mathcal{F}}(G) \leq \mathrm{Int}_{\mathcal{F}}(G)$. Moreover, for the case when $\mathcal{F} = \mathcal{N}$ is the class of all nilpotent groups,

$$Z_\infty(G) = \mathrm{Int}_{\mathcal{N}}(G),$$

so the hypercentre of G may be characterized as the intersection of all maximal nilpotent (i.e. $\mathcal{N}$-maximal) subgroups of G (Baer [23]).

Some other classes $\mathcal{F}$ for which the equality

$$\mathrm{Int}_{\mathcal{F}}(G) = Z_{\mathcal{F}}(G) \qquad (*)$$

holds for each soluble group G were found by A.V. Sidorov in the paper [24]. Nevertheless, in general, $Z_{\mathcal{F}}(G) < \mathrm{Int}_{\mathcal{F}}(G)$, even when $\mathcal{F} = \mathcal{U}$ and G is soluble.

L.A. Shemetkov asked in 1995 at the Gomel Algebraic seminar the following question (the formulation of this question was also given in [24, p. 41]): *What are the non-empty hereditary saturated formations $\mathcal{F}$ with the property that for each group G, the equality*

$$\mathrm{Int}_{\mathcal{F}}(G) = Z_{\mathcal{F}}(G) \qquad (*)$$

holds?

The answer to this question was obtained on the base of the theory of the intersections of $\mathcal{F}$-maximal subgroups which was developed in [25, 26].

First of all, in the paper [25] the general studying methods of the subgroup $\mathrm{Int}_{\mathcal{F}}(G)$ were developed. It has appeared that such subgroups possess practically all such general properties which the $\mathcal{F}$-hypercentre has.

Proposition 3.1 ([25]) *Let H, E be subgroups of G, N a normal subgroup of G and$I = \mathrm{Int}_{\mathcal{F}}(G)$.*

(a) $\mathrm{Int}_{\mathcal{F}}(H)N/N \leq \mathrm{Int}_{\mathcal{F}}(HN/N)$.

(b) $\mathrm{Int}_{\mathcal{F}}(H) \cap E \leq \mathrm{Int}_{\mathcal{F}}(H \cap E)$.

(c) *If $H/H \cap I \in \mathcal{F}$, then $H \in \mathcal{F}$.*

(d) *If $H \in \mathcal{F}$, then $IH \in \mathcal{F}$.*

(e) *If $N \leq I$, then $I/N = \mathrm{Int}_{\mathcal{F}}(G/N)$.*

(f) $\mathrm{Int}_{\mathcal{F}}(G/I) = 1$.

(g) *If every $\mathcal{F}$-critical subgroup of G is soluble and $\psi_0(N) \leq I$, then $N \leq I$.*

(h) $Z_{\mathcal{F}}(G) \leq I$.

It this proposition $\psi_0(N)$ denotes the subgroup of N generated by all its cyclic subgroups of prime order and order 4 (if the Sylow 2-subgroups of N are non-abelian).

Then for any $p \in \pi(\mathcal{F})$ we write $\mathcal{F}(p)$ to denote the intersection of all formations containing the set $\{G/O_{p',p}(G) \mid G \in \mathcal{F}\}$, and let $F(p)$ denote the class of all groups G such that $G^{\mathcal{F}(p)}$ is a p-group.

Definition 3.2. We say that $\mathcal{F}$ satisfies:

(1) The *boundary condition* if $G \in \mathcal{F}$ whenever G is an $F(p)$-critical group, for some $p \in \pi(\mathcal{F})$.

(2) The *boundary condition in the class of all soluble groups* if $G \in \mathcal{F}$ whenever G is a soluble $F(p)$-critical group, for any $p \in \pi(\mathcal{F})$.

If $\mathcal{F}$ is the class of all identity groups, then for any group G we have $Z_{\mathcal{F}}(G) = 1 = \mathrm{Int}_{\mathcal{F}}(G)$. In the other limited case, when $\mathcal{F} = \mathcal{G}$ is the class of all groups, we have $Z_{\mathcal{F}}(G) = G = \mathrm{Int}_{\mathcal{F}}(G)$.

For the general case, we have the following.

Theorem 3.3 ([26]). *Let $\mathcal{F}$ be a hereditary saturated formation with $(1) \neq \mathcal{F} \neq \mathcal{G}$. Equality $(*)$ holds for each group G if and only if $\mathcal{F}$ satisfies the boundary condition.*

Theorem 3.4 ([26]). *Let $\mathcal{F}$ be a hereditary saturated formation with $(1) \neq \mathcal{F} \neq \mathcal{G}$. Equality $(*)$ holds for each soluble group G if and only if $\mathcal{F}$ satisfies the boundary condition in the class of all soluble groups.*

Since for any concrete formation $\mathcal{F}$ and for any prime p the both classes $\mathcal{F}(p)$ and $F(p)$ either are well-known or can be easily found, general Theorems 3.3 and 3.4 allow to answer to above Shemetkov's question respectively $\mathcal{F}$.

Now we demonstrate this on some examples.

Example 3.5. Let $\mathcal{F} = \mathcal{N}$. Then $F(p)$ is the class of all p-groups. Hence every $F(p)$-critical group has prime order, so is nilpotent. Thus the above of Baer's result follows from Theorem 3.3.

A group G is called p-decomposable if there exists a subgroup H of G such that $G = P \times H$ for some (and hence the unique) Sylow p-subgroup P of G.

Example 3.6. Let $\mathcal{F}$ be the class of all p-decomposable groups. Then evidently $F(p)$ is the class of all p-groups and $F(q)$ is the class of all p'-groups for all primes $q \neq p$. Hence for any prime r every $F(r)$-critical group has prime order, so is p-decomposable. Thus by Theorem 3.3 for any group G we have $Z_{\mathcal{F}}(G) = \mathrm{Int}_{\mathcal{F}}(G)$.

Example 3.7. Let $\mathcal{F} = \mathcal{U}$. Then $\mathcal{F}(7)$ is the class of all abelian groups of exponent dividing 6. Hence A_4 is $F(7)$-critical, but not supersoluble. Hence $\mathcal{F}$ does not satisfy the boundary condition in the class of all soluble groups, so by Theorem 3.4 for some soluble group G we have $Z_{\mathcal{F}}(G) < \mathrm{Int}_{\mathcal{F}}(G)$.

Example 3.8. Let $\mathcal{F}$ be one of the following formations:

(1) *the class of all p-soluble groups;*

(2) *the class of all p-supersoluble groups;*

(3) *the class of all p-nilpotent groups;*

(4) *the class of all soluble groups.*

Then for any prime $q \neq p$ we have $\mathcal{F} = F(q)$. Hence clearly $\mathcal{F}$ does not satisfy the boundary condition, so by Theorem 3.3 in some group G we have $Z_{\mathcal{F}}(G) < \mathrm{Int}_{\mathcal{F}}(G)$.

Some other properties of the subgroup $\mathrm{Int}_{\mathcal{F}}(G)$ were found by J. C. Beidleman and H. Heineken in the paper [27].

4. On two questions of L.A. Shemetkov concerning of $\mathcal{U}$-hypercentral subgroups

Recall that a subgroup A of a group G is said to be *S-quasinormal, S-permutable*, or *$\pi(G)$-permutable* in G (Kegel [28]) if $AP = PA$ for all

Sylow subgroups P of G; the subgroup A of G is said to be *c-normal* in G (Wang [29]) if G has a normal subgroup T such that $AT = G$ and $A \cap T \leq A_G$. A is said to be *c-supplemented* in G (Ballester-Bolinches, Wang and Guo [30]) if G has a subgroup T such that $AT = G$ and $A \cap T \leq A_G$, the largest normal subgroup of G contained in A.

If $\mathcal{F}$ is a saturated formation containing all supersoluble groups and G is a group with a normal subgroup E, then the following results are true.

(1) If $G/E \in \mathcal{F}$ and every cyclic subgroup of E of prime order and order 4 is either S-quasinormal (Ballester-Bolinches and Pedraza-Aguilera [31], Asaad and Csörgő [32]) or c-normal (Ballester-Bolinches and Wang [33]) or c-supplemented (Ballester-Bolinches, Wang and Guo [30], Wang and Li [34]) in G, then $G \in \mathcal{F}$.

(2) If $G/E \in \mathcal{F}$ and every cyclic subgroup of every Sylow subgroup of $F^*(E)$ of prime order and order 4 is either S-quasinormal (Li and Wang [35]) or c-normal (Wei, Wang and Li [36]) or c-supplemented (Wang, Wei and Li [39], Wei, Wang and Li [38]) in G, then $G \in \mathcal{F}$.

(3) If $G/E \in \mathcal{F}$ and every maximal subgroup of every Sylow subgroup of E is either S-quasinormal (Asaad [40]) or c-normal (Wei [41]) or c-supplemented (Ballester-Bolinches and Guo [42]) in G, then $G/E \in \mathcal{F}$.

(4) If $G/E \in \mathcal{F}$ and every maximal subgroup of every Sylow subgroup of $F^*(E)$ is either S-quasinormal (Li and Wang [38]) or c-normal (Wei, Wang and Li [36]) or c-supplemented (Wei, Wang and Li [37]) in G, then $G \in \mathcal{F}$.

In these results $F^*(E)$ denotes the generalized Fitting subgroup of E, that is, the product of all normal quasinilpotent subgroups of E.

Bearing in mind the above results L.A. Shemetkov asked in 2004 at Gomel Algebraic Seminar the following two questions:

(I) *Is it true that all the abovementioned results can be strengthened by proving that every G-chief factor below E is cyclic?*

(II) *Is it true that the conclusion about the cyclic character of the G-chief factors below E still holds if we omit the condition "$G/E \in \mathcal{F}$ "?*

A partial solution of these problems has been obtained in [43, Theorem 1.4]. A complete answer to the above questions was obtained in [11].

Our main ingredient is the S-quasinormal embedding introduced in [44]: a subgroup H of a group G is said to be *S-supplemented* in G if

G has a subgroup T such that $G = HT$ and $T \cap H \leq H_{sG}$, where H_{sG} is the subgroup generated by all subgroups of H which are S-quasinormal in G. We prove:

Theorem 4.1 ([11]). *Let E be a normal subgroup of a group G. Suppose that for every non-cyclic Sylow subgroup P of E, every maximal subgroup of P or every cyclic subgroup of P of prime order and order 4 is S-supplemented in G. Then $E \leq Z_{\mathcal{U}}(G)$.*

Theorem 4.2 ([11]). *Let $\mathcal{F}$ be any formation and G a group. If E is a normal subgroup of G and $F^*(E) \leq Z_{\mathcal{F}}(G)$, then $E \leq Z_{\mathcal{F}}(G)$.*

Corollary 4.3 ([11]). *Let E be a normal subgroup of a group G. If $F^*(E) \leq Z_{\mathcal{U}}(G)$, then $E \leq Z_{\mathcal{U}}(G)$.*

It is rather clear that if $\mathcal{F}$ is a saturated formation containing all supersoluble groups and G is a group with a cyclic normal subgroup E such that $G/E \in \mathcal{F}$, then $G \in \mathcal{F}$. Hence Theorem 4.1 and Corollary 4.3 allow us to give affirmative answers to both Questions I and II. Finally, note that in view of Theorem 4.1 and Corollary 4.3 not only generalize all the results in [31]- [42] mentioned above but also gives new methods for proofs of them.

5. On factorizations of groups with $\mathcal{F}$-hypercentral intersections of the factors

One of the highlights of the proof of the above Theorem 4.1 is the following result is allowing to carry out inductive reasonings

Theorem 5.1 ([15, Corollary 3.2]) *Let A, B and E be normal subgroups of a group G. Suppose that $G = AB$ and $E \leq Z_{\mathcal{U}}(A) \cap Z_{\mathcal{U}}(B)$. If either $(|G : A|, |G : B|) = 1$ or $G' \leq F(G)$, then $E \leq Z_{\mathcal{U}}(G)$.*

But in fact this theorem is a generalization of the following well-known results of the theory of supersolvable groups.

Corollary 5.2 (Baer [45]). *Let $G = AB$ where A, B are normal supersoluble subgroups of G. If $G' \leq F(G)$, then G is supersoluble.*

Corollary 5.3 (Friesen [46]). *Let $G = AB$ where A, B are normal supersoluble subgroups of G. If $(|G : A|, |G : B|) = 1$, then G is supersoluble.*

These important observations have led to the following general problem:

Problem. *Let $G = AB$ be the product of two subgroups A and B of G. What we can say about the structure of G if $A \cap B \leq Z_{\mathcal{F}}(A) \cap Z_{\mathcal{F}}(B)$ for some class of groups $\mathcal{F}$?*

The paper [15] is devoted to the analysis of this topic. In particular the following facts were proved.

Theorem 5.4 ([15, Theorem 3.5]). *Suppose that G has three subgroups A_1, A_2 and A_3 whose indices $|G : A_1|$, $|G : A_2|$, $|G : A_3|$ are pairwise coprime. If $A_i \cap A_j \leq Z_{\mathcal{S}}(A_i) \cap Z_{\mathcal{S}}(A_j)$ for all $i \neq j$, then G is soluble.*

In this theorem $\mathcal{S}$ denotes the class of all soluble groups.

Corollary 5.5 (H. Wielandt [47]). *If G has three soluble subgroups A_1, A_2 and A_3 whose indices $|G : A_1|$, $|G : A_2|$, $|G : A_3|$ are pairwise coprime, then G is itself soluble.*

In the following theorem, $c(G)$ denotes the nilpotent class of a nilpotent group G.

Theorem 5.6 [15, Theorem 3.7]. *Suppose that G has three subgroups A_1, A_2 and A_3 whose indices $|G : A_1|$, $|G : A_2|$, $|G : A_3|$ are pairwise coprime. Let p be a prime. Then:*

(1) *If $A_i \cap A_j \leq Z_{\mathcal{F}}(A_i) \cap Z_{\mathcal{F}}(A_j)$ for all $i \neq j$, where $\mathcal{F}$ is the class of all p-closed groups, then G is p-closed.*

(2) *If $A_i \cap A_j \leq Z_{\mathcal{F}}(A_i) \cap Z_{\mathcal{F}}(A_j)$ for all $i \neq j$, where $\mathcal{F}$ is the class of all p-decomposable groups, then G is p-decomposable.*

(3) *If $A_i \cap A_j \leq Z_n(Z_\infty(A_i)) \cap Z_n(Z_\infty(A_j))$ for all $i \neq j$, then G is nilpotent and $c(G) \leq n$.*

The following corollaries are well known.

Corollary 5.7 (O. Kegel). *If G has three nilpotent subgroups A_1, A_2 and A_3 whose indices $|G : A_1|$, $|G : A_2|$, $|G : A_3|$ are pairwise coprime, then G is itself nilpotent.*

Corollary 5.8 (K. Doerk). *If G has three abelian subgroups A_1, A_2 and A_3 whose indices $|G : A_1|$, $|G : A_2|$, $|G : A_3|$ are pairwise coprime, then G is itself abelian.*

Theorem 5.9 ([15, Theorem 3.11]). *Suppose that G has four subgroups A_1, A_2, A_3 and A_4 whose indices $|G : A_1|$, $|G : A_2|$, $|G : A_3|$, $|G : A_4|$ are pairwise coprime. If $A_i \cap A_j \leq Z_{\mathcal{U}}(A_i) \cap Z_{\mathcal{U}}(A_j)$ for all $i \neq j$, then G is supersoluble.*

Corollary 5.10 (K. Doerk). *If G has four supersoluble subgroups A_1, A_2, A_3 and A_4 whose indices $|G : A_1|$, $|G : A_2|$, $|G : A_3|$, $|G : A_4|$ are pairwise coprime, then G is supersoluble.*

Recall that a subgroup H of G is said to be abnormal if $x \in \langle H, H^x \rangle$. It is clear that if H is a abnormal in G, then $N_G(H) = H$.

Theorem 5.11 ([15, Theorem 3.13]). *Suppose that G has three abnormal subgroups A_1, A_2 and A_3 whose indices $|G : A_1|$, $|G : A_2|$, $|G : A_3|$ are pairwise coprime.*

(1) *If $A_i \cap A_j \leq Z_{\mathcal{F}}(A_i) \cap Z_{\mathcal{F}}(A_j)$ for all $i \neq j$, where $\mathcal{F}$ is the class of all metanilpotent groups, then G is metanilpotent.*

(2) *If $A_i \cap A_j \leq Z_{\mathcal{U}}(A_i) \cap Z_{\mathcal{U}}(A_j)$ for all $i \neq j$, then G is supersoluble.*

Corollary 5.12 (A.F. Vasilyev and T.I. Vasilyeva [48]). *If G has three abnormal supersoluble subgroups A_1, A_2 and A_3 whose indices $|G : A_1|$, $|G : A_2|$, $|G : A_3|$ are pairwise coprime, then G is itself supersoluble.*

Finally, we mention the following result.

Theorem 5.13. *A group G is supersoluble if and only if every maximal subgroup V of every Sylow subgroup of G either is normal or has a supplement T in G such that $V \cap T \leq Z_{\mathcal{U}}(T)$.*

Corollary 5.14 (W. Guo, K. P. Shum and A. N. Skiba [49]). *A group G is supersoluble if and only if every maximal subgroup of every Sylow subgroup of G either is normal or has a supersoluble supplement in G.*

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