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Invariants of finite solvable groups

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Abstract. The article contains the results about invariants of solvable groups with given structure of Sylow subgroups and information about the nilpotent π-length of π-solvable groups. Open questions are formulated.

Introduction

We consider only finite groups. The derived and nilpotent lengths, the p-length, the π-length and the nilpotent π-length, the rank and the p-rank are carried to invariants of solvable groups.

The review of the results connected with invariants of finite solvable groups and researches of authors are provided in this article. Section 1 contains the list of used designations and definitions. In section 2 data on invariants of solvable groups with given structure of Sylow subgroups are collected. Section 3 contains information on the nilpotent π-length of π-solvable groups. Open questions are formulated.

Article has survey character. Proofs of statements aren't provided.

1. Preliminaries

All notation and definitions agree with [1]-[4].

Let $\mathbb{P}$ be the set of all primes, and let π be the set of primes. Denote the complement to π in $\mathbb{P}$ by π'. Let $\pi(a)$ be the set of primes dividing a

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positive integer a. Let G be a group and H be a subgroup of G. Assume that $\pi(G) = \pi(|G|)$ and $\pi(G : H) = \pi(|G : H|)$. Let's fix up some set of primes π. If $\pi(m) \subseteq \pi$ then a positive integer m is called a π-*number*. If $\pi(G) \subseteq \pi$ then G is called a π-*group* and if $\pi(G) \subseteq \pi'$ then G is called a π'-*group*. The Frattini and Fitting subgroups of G are denoted by $\Phi(G)$ and $F(G)$, respectively, and $O_\pi(G)$ denotes the greatest normal π-subgroup of G. The notation $G = [A]B$ is used for a semidirect product with a normal subgroup A.

A normal series of G is a finite sequence of normal subgroups G_i such that

$$1 = G_0 \subseteq G_1 \subseteq \ldots \subseteq G_m = G. \tag{1}$$

We call the groups G_{i+1}/G_i the factors of the normal series (1).

Let G be a group of order $p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k}$, where $p_1 > p_2 > \ldots > p_k$. We say that G has an ordered Sylow tower of supersolvable type if there exists a series

$$1 = G_0 \subseteq G_1 \subseteq G_2 \subseteq \ldots \subseteq G_{k-1} \subseteq G_k = G$$

of normal subgroups of G such that G_i/G_{i-1} is isomorphic to a Sylow p_i-subgroup of G for each $i = 1, 2, \ldots, k$.

For group G it is possible to construct a chain of derived subgroups

$$G \supseteq G' \supseteq (G')' \supseteq \ldots \supseteq G^{(i)} \supseteq G^{(i+1)} \supseteq \ldots.$$

Here G' is the derived subgroup of G and $G^{(i+1)} = \left(G^{(i)}\right)'$. If there is n such that $G^{(n)} = 1$ then G is called *solvable*. The smallest positive integer n such that $G^{(n)} = 1$ is called the *derived length* of G and is denoted by $d(G)$.

Let G be a group and $F_0(G) = 1$,
$F_1(G) = F(G)$ is the Fitting subgroup of G,
$F_2(G)/F_1(G) = F(G/F_1(G)), \ldots,$
$F_i(G)/F_{i-1}(G) = F(G/F_{i-1}(G)), \ldots$

It is clear that $1 = F_0(G) \subseteq F_1(G) \subseteq F_2(G) \subseteq \ldots$.

It is well known that the Fitting subgroup $F(G)$ of a solvable non-identity group G is non-identity. Then there is positive integer n such that $F_n(G) = G$. The smallest positive integer n such that $F_n(G) = G$ is called the *nilpotent length* of G and is denoted by $n(G)$. Thus $n(G) = 1$ if and only if G is nilpotent. If G contains a nilpotent normal subgroup such that the corresponding quotient group is also nilpotent then G is called *metanilpotent*. It is clear that the nilpotent length of metanilpotent group is at most 2.

If chief factors (i.e. the factors of the chief series) of group G are either elementary Abelian p-groups for $p \in \pi$ or π'-groups then G is called π-solvable [1]. As well known that the indices of the maximal subgroups of any π-solvable group are either primary π-numbers or π'-numbers ([1], Theorem 1.8.1).

Let G be a p-solvable group. Then G has a normal series

$$1 = G_0 \subseteq G_1 \subseteq \ldots \subseteq G_n = G,$$

such that every quotient factor G_{i+1}/G_i is either a p-group or a p'-group. Then we can define the (p', p)-series for G:

$$1 = P_0 \subseteq N_0 \subseteq P_1 \subseteq N_1 \subseteq P_2 \subseteq \ldots \subseteq P_l \subseteq N_l = G,$$

where $N_i/P_i = O_{p'}(G/P_i)$ is the greatest normal p'-subgroup of G/P_i and $P_{i+1}/N_i = O_p(G/N_i)$ is the greatest normal p-subgroup of G/N_i. The smallest positive integer l such that $N_l = G$ is called the *p-length* of G and is denotes by $l_p(G)$.

Let G be a p-solvable group. If p^n is the greatest orders of the p-chief factors of G then n is called *p-rank* of G and is denoted by $r_p(G)$ ([4], p. 685). A solvable group G is p-solvable for every $p \in \pi(G)$. *The rank* of non-identity solvable group G is $\underset{p\in\pi(G)}{max}\, r_p(G)$ and is denoted by $r(G)$. For the identity group 1 we put $r(1) = 0 = r_p(1)$. By the Jordan-Hölder theorem all chief series of a group are isomorphic; therefore, the values of rank and p-rank are well-defined.

If chief factors of G are either of prime orders or p'-groups then G is called p-supersolvable.

It is clear that a non-identity p-supersolvable group is p-solvable and its p-rank is equal to 1. If G is p-supersolvable for any $p \in \pi(G)$ then G is called *supersolvable.* The rank of non-identity supersolvable group is equal to 1.

2. Invariants of group depending on Sylow subgroups

P.Hall and G. Higman [5] established dependence of p-length of a p-solvable group from some invariants of its Sylow p-subgroups. Elementary theory of p-length is stated in the monograph of Huppert ([4], VI.6; [6], IX). The estimations of p-length of a p-solvable group depending on invariants of intersection of Sylow p-subgroups are found in work of A.G. Anishchenko and V.S. Monakhov [7]. The review of results about p-length of solvable groups as of 1980 contains in article of V.D. Mazurov [8].

L.A. Shemetkov extended the concept of p-length for any groups and proved that p-length of any group doesn't exceed the minimal number of generators of its Sylow p-subgroup [9]. For p-solvable groups this fact is given in Huppert's monograph ([4], Theorem VI.6.6). Naturally there was the following question formulated in review [10], page 15.

Question 1. *How much essential the condition of p-solvability in known theorems about p-length*?

In work [11] L.A. Shemetkov received the positive solution of this question for groups in which the non-solvable composition factors have cyclic Sylow p-subgroup. In Kourov's Writing-Book he wrote down the following question ([12], Question 3.60).

Question 2. *To investigate dependence between p-length of group and invariants c_p, d_p, e_p of its Sylow p-subgroups.*

Here c_p, d_p, e_p are the nilpotent class, the derived length and the exponent respectively.

By the Zassenhaus Theorem ([4], Theorem IV.2.11) *the derived subgroup of a group with cyclic Sylow subgroups is a cyclic Hall subgroup such that the corresponding quotient group is also cyclic. Hence the derived length of such group is at most 2.*

It follows from the Hall-Higman theorem ([4], Theorem IV.14.16) that *the derived length of a solvable group with abelian Sylow subgroups does not exceed the number of different prime divisors of the order of such group, $d(G) \leq |\pi(G)|$.*

Recall that a group is *metacyclic*, if it contains cyclic normal subgroup such that the corresponding quotient group is also cyclic. For the groups with metacyclic Sylow subgroup the following statements are known.

Theorem 1 ([4], Theorems IV.2.8, IV.5.10, IV.8.6)**.**

1. *Let p be a smallest prime divisor of $|G|$. If Sylow p-subgroup P of G is cyclic then G is p-nilpotent (there is a normal subgroup H such that quotient subgroup G/H is isomorphic to P). In particular, if Sylow 2-subgroup of G is cyclic then G is 2-nilpotent.*

2. *If Sylow p-subgroup of G is metacyclic and the order of G is mutually simple with p^2-1 then G is p-nilpotent. In particular, if Sylow 2-subgroup of G is metacyclic and the order of G is not divisible by 3 then G is 2-nilpotent.*

3. *If $p>2$ and Sylow p-subgroup of G is metacyclic and non-abelian then there exists a normal subgroup of index p.*

The groups with metacyclic Sylow 2-subgroup are studied in work of V.D. Mazurov [13]. Theorem 1 of this article fully describe all non-solvable group with cyclic Sylow p-subgroups for odd prime p and metacyclic Sylow 2-subgroups. Information about solvable groups with metacyclic Sylow 2-subgroups is presented in Lemmas 3, 4 of [13].

Signs of solvability of a group with metacyclic Sylow 2-subgroup are received in work of A.R. Camina and T.M. Gagen [14]. The result of this work are combined in the following theorem.

Theorem 2 ([13], [14]). *Let G be a group with metacyclic Sylow 2-subgroup G_2. Then the following statements hold.*

1. *If G_2 has a cyclic normal subgroup N such that G_2/N is cyclic and $|G_2/N| \geq 4$ then G is solvable.*

2. *If G is solvable then $|G/O_{2',2}(G)| \leq 6$.*

3. *If G is solvable and $Z(G_2)$ is non-cyclic then G has a normal series*

$$1 \subseteq U \subseteq UG_2 \subseteq G,$$

where $|U|$ is odd and $|G : UG_2| = 1$ or 3.

4. *If G is solvable and $Z(G_2)$ is cyclic then G has a normal series*

$$1 \subseteq U_1 \subseteq T_1 \subseteq U_2 \subseteq G,$$

where $|U_1|$ is odd, T_1/U_1 is 2-group, $|U_2 : T_1|$ is equal to 1 or 3, $|G : U_2| = 1$ or 2.

The groups with metacyclic Sylow subgroups are investigated in work of D. Chillag and J. Sonn [15]. Theorem 1 of this article fully describes such non-solvable groups and contains following statement for solvable groups.

Theorem 3 ([15]). *Let G be a solvable group with metacyclic Sylow p-subgroups for any $p \in \pi(G)$. Then $\{2, 3\}'$-Hall subgroup of G is normal and possesses an ordered Sylow tower of supersolvable type.*

Recall that a group is bicyclic if it is the product of two cyclic subgroups.

It is clear that metacyclic group is bicyclic. The general properties of bicyclic groups are received in works [16]–[20] and have entered into the monography [4]. In particular, bicyclic primary group of odd order is metacyclic ([4], Theorem III.11.5). Bicyclic 2-groups and non-primary bicyclic groups of odd order can be not metacyclic.

Example 1. In Huppert's article [16] the 2-group

$$G =< a, b, c \mid a^2 = b^8 = c^2 = 1,\ [a, b] = c,\ [b, c] = b^4,\ [a, c] = 1 >,$$

which contains a normal elementary abelian subgroup

$$N =< a > \times < b^4 > \times < c >$$

of order 8 such that G/N is cyclic and $|G/N| = 4$ is constructed. Besides,

$$Z(G) =< b^4 >,\ G' =< c > \times < b^4 >,\ \Phi(G) = \mho_1(G) =< b^2, c >,$$

$$(ab)^2 = cb^2 \notin < b >,\ (ab)^4 = b^4,\ G =< ab >< b > .$$

Then G is bicyclic of order 2^5. Since it contains a normal elementary abelian subgroup N of order 8, it is non-metacyclic.

Example 2. The calculations in the computer system GAP [21] show that the group G of order $189 = 3^3 7$ having number 7 in the library SmallGroups,

$$G =< a, b, c, d \mid b^3 = c^3 = d^7 = 1,\ a^3 = c,\ [a, b] = c^{-1},$$

$$[a, d] = d^{-1},\ [a, c] = [b, c] = [b, d] = [c, d] = 1 >,$$

is the product of two cyclic subgroups $A =< bd >$ of order 21 and $B =< ab >$ of order 9. Hence G is bicyclic non-primary group of odd order. There are only three non-identity cyclic normal subgroups in G: $N_1 =< c >$ of order 3, $N_2 =< d >$ of order 7, $N_3 =< cd >$ of order 21. Since G/N_i is non-cyclic, it follows that G is non-metacyclic.

Bicyclic group $G = AB$ is supersolvable, the derived subgroup of G is abelian and A contains non-identity normal subgroup in G for $|A| \geq |B|$ ([4], Theorems VI.4.4 and VI.10.1). From definition of bicyclic group follows that the Sylow subgroups and its any quotient groups are bicyclic. Let G be a bicyclic primary group. Then the Frattini subgroup of G, the center of G and $\mho_1(G) =< x^p \mid x \in G >$ are bicylic [16]. Other properties of bicyclic primary groups are listed in the following theorem.

Theorem 4 ([16], [22]). *Let G be a bicyclic p-group. Then the following statements hold.*

1. *Let $p > 2$. Then:*

1.1) *G is metacyclic;*

1.2) *any two subgroups of G is permutable;*

1.3) *if N is a complemented normal subgroup of G then either $N = G$ or N is cyclic.*

2. *Let $p = 2$. Then:*

2.1) *any normal subgroup of G is generated no more than three elements;*

2.2) *if N is a complemented normal subgroup of G then $|N/\Phi(N)| \leq 4$.*

V.D. Mazurov proved that *if $\alpha \neq 1$ is automorphism of odd order of metacyclic 2-group T then the order of α is equal to 3, T is either a quaternion group of order 8 or a direct product of two isomorphic cyclic subgroup, but in the latter case α acts on T without fixed points* ([13], Lemma 1). It is simple to check up that the statement remains for bicyclic 2-groups.

It is known also that *quotient group G/G' of bicyclic 2-group G with non-cyclic derived subgroup G' is abelian of type $(n, 1)$, where $n > 1$ is integer* [18]. In the same work bicyclic 2-groups G with cyclic derived subgroup G' are considered. It has appeared that *if there is no cyclic subgroup N, which is not containing G', then quotient group G/G' is abelian of type $(2^r, 2)$*. Blackburn [19] proved that *non-cyclic derived subgroup of bicyclic 2-group G with quotient group G/G' of type $(n, 1)$, $n > 2$ is abelian of type $(m, 1)$ for some m.*

The invariants of the groups with bicyclic Sylow subgroups were found in work of V.S. Monakhov and E.E. Gribovskaya [22]. Let's give the full formulation of these results.

Theorem 5 ([22], Theorems 1–2, Lemma 2)**. 1**. *Let G be a solvable group with bicyclic Sylow subgroups. Then the following statements hold:*

1.1) *$n(G) \leq 4$ and $d(G) \leq 6$;*

1.2) *$\{2, 3\}'$-Hall subgroup is normal and possesses an ordered Sylow tower of supersolvable type;*

1.3) *$2'$-Hall subgroup $G_{2'}$ possesses an ordered Sylow tower of supersolvable type, its derived subgroup is nilpotent and $d(G_{2'}) \leq 3$;*

1.4) *$3'$-Hall subgroup $G_{3'}$ possesses an ordered Sylow tower of supersolvable type, its supersolvable residual is nilpotent and $d(G_{3'}) \leq 4$;*

1.5) *$n(G_{\{2,3\}}) \leq 3$ and $d(G_{\{2,3\}}) \leq 4$;*

1.6) *$\{2, p\}$-Hall subgroup $G_{\{2,p\}}$ possesses an ordered Sylow tower of supersolvable type and $d(G_{\{2,p\}}) \leq 4$ for every $p > 3$.*

2. *If G is a p-solvable group with bicyclic Sylow p-subgroup then $l_p(G) \leq 1$ for $p > 2$ and $l_2(G) \leq 2$ for $p = 2$.*

Corollary 1 ([22], Corollary)**.** *Any group G of odd order with metacyclic Sylow subgroups is possesses an ordered Sylow tower of supersolvable type* *and its derived subgroup is nilpotent. In particular, G is metanilpotent and $d(G) \le 3$.*

In 2009 authors of the review showed that in formulation of Theorem 5 it is possible to add condition "Sylow p-subgroups have an order p^3". The estimation of derived length will remain.

Theorem 6 (V.S. Monakhov, A.A. Trofimuk [23]). *Let G be a solvable group in which the Sylow p-subgroups are either bicyclic or of order p^3 for any $p \in \pi(G)$. Then the derived length of G is at most 6 and $l_2(G) \le 2$, $l_3(G) \le 2$ and $l_p(G) \le 1$ for every prime $p > 3$. In particular, if G is an A_4-free group, then the derived length of G is at most 5.*

A group G is A_4-free if there is no section isomorphic to the alternating group A_4 of degree 4.

Corollary 2. *([23], Corollary) Let G be a group of odd order in which Sylow p-subgroups are either bicyclic or of order p^3 for any $p \in \pi(G)$. Then the derived length of G is at most 3.*

Example 3. The group $GL(2,3)$ is generated matrices

$$a = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}, \quad c = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

then

$$a^8 = b^2 = c^3 = 1, \; a^b = a^3, \; (a^2)^c = ab, \; (ab)^c = aba^2, \; c^b = c^{-1}$$

([24], Lemma XII.5.3). It is clear that $GL(2,3) =< a > ([< c >] < b >)$ ($GL(2,3)$ is a product of subgroup $< a >$ of order 8 and $[< c >] < b >$ of order 6). The derived length of $GL(2,3)$ is equal to 4. Hence the estimation of the derived length, which is obtained in Theorem 5(1.5), is exact.

Example 4. Let S be a extraspecial group of order 7^3 and

$$H = \langle a, \; b, \; c \mid a^2 = b^3 = c^4 = abc \rangle$$

be a group of order $48 = 2^4 \cdot 3$ in which Sylow 2-subgroup is quaternion group Q_{16} of order 16. With the computer algebra system GAP we can construct the group $G = [S]H$ of order $16464 = 2^4 \cdot 3 \cdot 7^3$. The derived length of G is equal to 6 and the nilpotent length of G is equal 4. Hence the estimations of the derived and the nilpotent lengths in Theorem 6 are exact.

Example 5. Let E_{7^3} be an elementary abelian group of order 7^3, S be a extraspecial group of order 27, Q_8 be a quaternion group of order 8. With the computer algebra system GAP we can construct the group $G = [E_{7^3}]([S]Q_8)$ of order $74088 = 2^3 \cdot 3^3 \cdot 7^3$ such that it is A_4-free. The derived length of G is equal to 5 and the p-length is equal to 1 for any $p \in \pi(G)$. Hence the estimation of the derived length, which is obtained in Theorem 6, is exact.

Let's notice that Theorems 5 and 6 cover all solvable groups $G = AB$ with cyclic Sylow subgroups in factors A and B. But such groups possess new properties, which not inherited to groups with bicyclic Sylow subgroups.

Let p is prime. A *z_p-group* is a group with cyclic Sylow p-subgroup and a *z-group* is a group, in which all Sylow subgroups are cyclic.

Theorem 7 (J.G. Berkovich [25], R. Maier [26])**.** *Let $G = AB$, where A and B are the z_p-subgroups. Then the following statements hold.*

1. *If $|G|$ is odd then G is p-supersolvable.*

2. *If G is p-solvable, A and B are p-nilpotent then G is p-supersolvable.*

3. *If $|G|$ is odd, A and B are the z-subgroups then G is supersolvable.*

If the order of $G = AB$ is even, A and B are the z-subgroup then G can be non-solvable. As example there is the symmetric group S_5 of degree 5, which is product of z-subgroup of order 20 and cyclic subgroup of order 6.

V.D. Mazurov [27] proved that *$G = AB$ is non-simple if A and B are the z-subgroups.* Besides, V.D. Mazurov ([28], p. 75) for any prime $p > 2$ has specified examples not p-supersolvable solvable group of even order, which is product of two z_p-groups.

New information about solvable groups that are product of two z-subgroups is received by V.S. Monakhov.

Theorem 8 ([29], Theorem, Corollaries 2–4)**.** **1**. *If $G = AB$ is a 3-solvable group, where A and B are the z_p-subgroups for $p = 2$ and $p = 3$, then G is 3-supersolvable.*

2. *If $G = AB$ is a $\{2,3\}$-group, where A and B are the z-subgroups and $l_2(G) \leq 1$, then G supersolvable.*

3. *If $G = AB$ is a S_4-free $\{2,3\}$-group, where A and B are the z-subgroups, then G supersolvable.*

4. *A solvable group $G = AB$, where A and B are the z-subgroups, has a normal series of length ≤ 3 with supersolvable factors.*

Let's note that for receiving 3-supersolvability of G in a condition of Theorem 8 can't be replaced a factorization with the requirement that Sylow 2- and 3-subgroups in group are bicyclic. As example, the group $[E_9]Z_8$. It is a extension of elementary abelian group E_9 of order 3^2 by cyclic group Z_8 of order 2^3, which irreducible acts on E_9.

A.R. Camina and T.M. Gagen proved that *if $G = AB$ is non-solvable group, where A is cyclic and B is metacyclic then $G/S(G) \simeq PGL(2,p)$, $p > 3$, p is prime,* [30]. Here $S(G)$ is a largest normal solvable subgroup of G.

V.S. Monakhov proved that *if $G = AB$, $A \cap B = 1$, A is a z_2-subgroup of even order, B is proper subgroup then G contains a subgroup of index 2* ([31], Lemma 2). V.D. Mazurov [27] weakened the requirement $A \cap B = 1$ to following: the intersection $A \cap B$ has an odd order.

A group G is said to be a *mutually m-permutable product* of the subgroups G_1 and G_2 if $G = G_1G_2$ and G_1 permutes with every maximal subgroup of G_2 and G_2 permutes with every maximal subgroup of G_1.

In 2011 V.S. Monakhov and M. Assaad [32] give a generalization of Berkovich's result [25] for an arbitrary number of factors.

Theorem 9 (V.S. Monakhov and M. Assaad [32],Theorems 1.1 and 1.3, Corollary 1.2)**.** *Let $G = G_1G_2 \cdots G_n$ be the product of the pairwise permutable subgroups $G_1, \ldots, G_n$. Then the following statements hold.*

1. *If G is a group of odd order and the Sylow p-subgroups of any G_i are cyclic then G is p-supersolvable.*

2. *If G is a group of odd order and the Sylow subgroups of any G_i are cyclic then G is supersolvable.*

3. *If G is a group of even order, $G_2G_3 \cdots G_n$ is of odd order, G_1G_i is a mutually m-permutable product $i = \overline{2, \ldots, n}$ and the Sylow subgroups of any G_i are cyclic then G is supersolvable.*

The normal rank $r_n(P)$ of a p-group P is defined thus:

$$r_n(P) = \max_{X \lhd P} log_p |X/\Phi(X)|, \tag{2.1}$$

where X runs over all normal subgroups of P including P. Here $\Phi(X)$ is the Frattini subgroup of X. The basis theorem of Burnside ([4], Theorem III.3.15) implies that the normal rank, $r_n(P)$, is the least positive integer k such that every normal subgroup of a p-group P is generated by at most k elements. It is obvious that p-group is cyclic, if its normal rank is equal to 1. The normal rank of metacyclic p-group is at most 2. The normal rank of bicyclic p-group is at most 3.

The development of Theorem 5 is the following theorem.

Theorem 10 ([33])**.** **1**. *If G is solvable with Sylow subgroups of normal rank ≤ 3 then the nilpotent length of G is at most 5 and the p-length is at most 2 for any prime p.*

2. *If G is solvable with Sylow 2-subgroup of normal rank ≤ 3 and Sylow p-subgroups of normal rank ≤ 2 for all $p > 2$ then $n(G) \leq 4$, $l_2(G) \leq 2$, $l_3(G) \leq 2$ and $l_p(G) \leq 1$ for all prime $p > 3$.*

3. *If G is an odd order with Sylow subgroups of normal rank ≤ 2 then G is metanilpotent.*

4. *If G is solvable with Sylow subgroups of normal rank ≤ 2 then $n(G) \leq 4$, $l_2(G) \leq 2$, $l_3(G) \leq 2$ and $l_p(G) \leq 1$ for all prime $p > 3$. Besides, for Hall subgroups of G the following statements hold:*

4.1) $n(G_{2'}) \leq 2$;
4.2) $n(G_{3'}) \leq 3$;
4.3) $n(G_{\{2,3\}}) \leq 3$;
4.4) $n(G_{\{2,p\}}) \leq 2$ *for all* $p > 3$.

V.S. Monakhov [34] established dependence of invariants solvable group from orders of Sylow subgroups.

Theorem 11 ([34])**.** *Let the order of a soluble group G not be divisible by $(n+1)$th degrees of primes. Then:*

1) *if* $n \in \{2, 3, 4\}$ *then* $d(G/\Phi(G)) \leq 3 + n$;
2) *if* $n \in \{5, 6, 7\}$ *then* $d(G/\Phi(G)) \leq 8$;
3) *if* $n \in \{8, 9\}$ *then* $d(G/\Phi(G)) \leq 1 + n$;
4) *if* $n \in \{10, \ldots, 17\}$ *then* $d(G/\Phi(G)) \leq 11$;
5) *if* $n \in \{18, \ldots, 25\}$ *then* $d(G/\Phi(G)) \leq 12$;
6) *if* $n \in \{26, \ldots, 33\}$ *then* $d(G/\Phi(G)) \leq 13$;
7) *if* $n \in \{34, \ldots, 65\}$ *then* $d(G/\Phi(G)) \leq 14$;
8) *if* $n \geq 66$ *then* $d(G/\Phi(G)) \leq 1 + 5log(n-2) + 53/10$.

In particular, $d(G/\Phi(G)) \leq 3 + n$ *in any case.*

Theorem 12 ([34])**.** *Assume that the order of a soluble group G is odd and is not divisible by $(n+1)$th degrees of primes. Then:*

1) *if* $n \leq 2$ *then* $d(G/\Phi(G)) \leq 2$;
2) *if* $n \in \{3, 4\}$ *then* $d(G/\Phi(G)) \leq 3$;
3) *if* $5 \cdot 7^x \leq n < 15 \cdot 7^x$ *then* $d(G/\Phi(G)) \leq 4 + 2x$;
4) *if* $15 \cdot 7^x \leq n < 5 \cdot 7^{x+1}$ *then* $d(G/\Phi(G)) \leq 5 + 2x$.

It should be noted that the estimation of the derived length received on the basis of the general technique of research of solvable groups with restrictions on orders of Sylow subgroups, is non-exact for small values of orders. For example, if orders of Sylow subgroups of solvable group G are cube-free then $d(G) \leq 3$, but $d(G/\Phi(G)) \leq 5$ by Theorem 11. Information on a structure of groups cube-free order is presented in the following theorem.

Theorem 13 ([35]). *Let G be a solvable group in which the Sylow p-subgroups are either cyclic or of order p^2 for any $p \in \pi(G)$. Then the following statements hold:*

1) *the derived length of G is at most 3;*
2) *$2'$-Hall subgroup is metabelian.*

The chief factor H/K is called the *Fitting* chief factors, if the Fitting subgroup $F(G)$ of G contains a subgroup H.

In 1978 Gashutz [36] proved the following statement: *let H/K be a chief factor of the greatest order of G. Then $H \leq F(G)$.* From this statement does not follow that each chief factor of order $p^{r(G)}$ is a Fitting chief factor, where p is prime. Any supersolvable non-nilpotent group is an example of this fact.

Hence the following Question was arose: Let G be a solvable group. Are there the Fitting chief factors of order $p^{r(G)}$ for some prime p? The following theorem gives answer on this Question.

Theorem 14 (V.S. Monakhov [37], Theorem 2). *Every finite solvable non-identity group G contains a nilpotent normal subgroup K such that $\Phi(G) \leq K$, the quotient group $K/\Phi(G)$ is a chief factor of G and $|K/\Phi(G)| = p^{r(G/\Phi(G))}$ for some prime p.*

In particular, Huppert's well-known result from here follows.

Corollary 3 ([38], Theorem 13; [4], Theorem V.9.9). *Let G be a finite solvable group. If*

$$\Phi(G) = N_0 \lhd N_1 \lhd \ldots \lhd N_{m-1} \lhd N_m = F(G),$$

is a normal series such that $N_i \lhd G$ and N_i/N_{i-1} has a prime order, $i = 1, 2, \ldots, m$ then G is supersolvable.

A.A. Trofimuk [39] noticed that, to estimate the derived length, it suffices to consider the orders of the Sylow subgroups only Fitting subgroup of a group. Besides, a substantial influence on the upper bound of the derived length of a group is due the Sylow subgroups in the Fitting subgroup, which are not bicyclic (if they exist), rather than all Sylow subgroups of this kind.

Theorem 15 (A.A. Trofimuk [39]). *Let G be a solvable non-primary group and F is its Fitting subgroup. Then the following statements hold.*

1. *If $\pi^*(F) \neq \emptyset$ then $d(G) \leq \rho(t(F)) + max\{d(F_p) \mid p \in \pi(F)\}$.*

2. *If $\pi^*(F) = \emptyset$ then $d(G) \leq 6$.*

Here $\pi^*(F)$ is the set of all primes p in $\pi(F)$ for which Sylow p-subgroup of F is not bicyclic. If $\pi^*(F) = \emptyset$ then all Sylow subgroups of F are bicyclic. The function $t_p(F)$ and $t(F)$ are defined thus:

$$t_p(F) = log_p(|F_p|), \quad t(F) = \max_{p \in \pi^*(F)} t_p(F).$$

Here $\rho(n)$ is the maximum of the derived lengths of the completely reducible solvable subgroups of $GL(n, \mathbb{F})$, where $\mathbb{F}$ is a field. By the Zassenhaus Theorem [40] this function $\rho(n) : \mathbb{N} \to \mathbb{N}$ exists and is independent on the field $\mathbb{F}$. The values of $\rho(n)$ are known for every n, [41]–[44].

Corollary 4 ([39]). *Let G be a solvable non-primary group and F is its Fitting subgroup. Then the following statements hold.*

1. *If the Sylow subgroups of F are bicyclic then the derived length is at most 6.*

2. *If $\pi^*(F) \neq \emptyset$ then $d(G) \leq \rho(t(F)) + \delta(t(F)) + 1$.*

Here $\delta(n) = max\{d \in \mathbb{N} \mid n \geq 2^d + 2d - 2\}$.

The dependence of the rank and the derived length of solvable group from index of Fitting subgroups in their normal closure is established in [45].

Theorem 16 (A.A. Trofimuk [45], Theorem 1). *Let G be a solvable group. Then the following statements hold.*

1. $r(G/\Phi(G)) \leq 1 + t^F(G)$.
2. $d(G/\Phi(G)) \leq 1 + \rho(1 + t^F(G)) \leq 4 + t^F(G)$.

Theorem 17 (A.A. Trofimuk [45], Theorem 2). *Let G be a solvable group and $t^F(G) \leq 2$. Then $d(G/\Phi(G)) \leq 6$, $n(G) \leq 4$ and $l_p(G) \leq 2$ for any prime p. In particular, if G is A_4-free then $d(G/\Phi(G)) \leq 5$.*

Here the function $t_p^F(G)$ and $t^F(G)$ are defined thus:

$$t_p^F(G) = \max\{n \mid p^n \mid\mid |H^G : H|, \ H \leq F(G)\}, \ p \in \pi(G);$$

$$t^F(G) = \max_{p \in \pi(G)} t_p^F(G).$$

We denote by $p^k \mid\mid n$ that p^k divides n, but p^{k+1} does not divide n; H^G is the smallest normal subgroup of G that contains H.

The examples that show accuracy of received estimations in Theorem 17 are constructed.

Example 6. Let E_{7^3} be an elementary abelian group of order 7^3 and K be a extraspecial group of order 27. With the computer algebra system GAP we can construct the group $G = [E_{7^3}]([K]SL(2,3))$ of order 222 264. The Fitting subgroup of G is coincide with E_{7^3} and $t^F(G) = 2$. The Frattini subgroup $\Phi(G)$ is identity, the derived length of G is equal to 6 and nilpotent length of G is equal 4.

Example 7. The A_4-free group $G = [E_{7^3}]([S]Q_8)$ of order $74088 = 2^3 \cdot 3^3 \cdot 7^3$ of example 5 has $F(G) = E_{7^3}$ and $t^F(G) = 2$. Besides, the derived length of G is equal to 5. Hence the estimation of the derived length, which is obtained in Theorem 17, is exact.

Finding of invariants of solvable groups with the set properties of Sylow subgroups has found development in researches of a structure of groups on properties of Sylow subgroups in factors of their normal series.

If G has a normal series with cyclic Sylow subgroups in factors then G is supersolvable. Therefore, G possesses an ordered Sylow tower of supersolvable type, the derived subgroup of G is nilpotent and the nilpotent length of G is at most 2. As any p-group has a normal series with factors of prime orders then the derived length of such groups to limit from above it is impossible. However, the derived length of $G/\Phi(G)$ is at most 2.

The research of the solvable groups having a normal series whose factors have bicyclic Sylow subgroups, is spent to 2009 by authors of the present review.

Theorem 18 (V.S. Monakhov, A.A. Trofimuk [46])**.** *Let G be a solvable group having a normal series such that every Sylow subgroup of its factors is bicyclic. Then the following statements hold.*

1. *The nilpotent length of G is at most 4 and the derived length of $G/\Phi(G)$ is at most 5;*

2. *$l_2(G) \leq 2$, $l_3(G) \leq 2$ and $l_p(G) \leq 1$ for every prime $p > 3$;*

3. *If G is an A_4-free group then the following statements hold:*

3.1) *$l_p(G) \leq 1$ for every prime p;*

3.2) *the derived length of $G/\Phi(G)$ is at most 3.*

4. *If G is a group of odd order then the derived subgroup of G is nilpotent. In particular, $G/\Phi(G)$ is metabelian.*

Clearly, that the Theorem 18 covers all groups having a normal series with bicyclic factors.

Examples that show accuracy of the estimations in Theorem 18 are constructed.

Example 8. It is well known that S_4 has the normal series

$$1 \leq E_4 \leq A_4 \leq S_4$$

with bicyclic factors and $l_2(S_4) = 2$. The group $G = [E_{3^2}]SL(2,3)$ has the normal series

$$1 \leq E_{3^2} \leq [E_{3^2}]Z_2 \leq [E_{3^2}]Q_8 \leq [E_{3^2}]SL(2,3)$$

with bicyclic factors and $l_3(G) = 2$.

Example 9. Let E_{7^2} be an elementary Abelian group of order 7^2. The automorphism group of E_{7^2} is the general linear group $GL(2,7)$ with cyclic center $Z = Z(GL(2,7))$ of order 6. We choose a subgroup C of order 2 in Z. Evidently, C is normal in $GL(2,7)$. The calculations in the computer system GAP show that $GL(2,7)$ has a subgroup S of order 48 such that S/C is isomorphic to the symmetric group S_4 of degree 4. The semidirect product $G = [E_{7^2}]S$ is a group of order $2352 = 2^4 7^2 3$. In particular, $\Phi(G) = 1$. The nilpotent length of G is equal to 4, the derived length of G is equal to 5. The group G has the chief series

$$1 \subset E_{7^2} \subset [E_{7^2}]Z_2 \subset [E_{7^2}]Q_8 \subset [[E_{7^2}]Q_8]Z_3 \subset [E_{7^2}]S = G$$

with bicyclic factors:

$$E_{7^2},\ ([E_{7^2}]Z_2)/(E_{7^2}) \simeq Z_2,\ ([E_{7^2}]Q_8)/([E_{7^2}]Z_2) \simeq E_4,$$

$$([[E_{7^2}]Q_8]Z_3)/([E_{7^2}]Q_8) \simeq Z_3,\ (G/[[E_{7^2}]Q_8]Z_3) \simeq Z_2.$$

Hence the estimations of the nilpotent length and the derived length, which are obtained in Theorem 18, are exact.

Example 10. Let E_{5^2} be an elementary Abelian group of order 5^2. The automorphism group of E_{5^2} is the general linear group $GL(2,5)$. The group $GL(2,5)$ has a subgroup, which is isomorphic to the symmetric group S_3 of degree 3. The semidirect product $G = [E_{5^2}]S_3$ is an A_4-free group with identity Frattini subgroup. The derived length of G is equal to 3. The group G has the chief series

$$1 \subset E_{5^2} \subset [E_{5^2}]Z_3 \subset [E_{5^2}]S_3 = G$$

with bicyclic factors:

$$E_{5^2},\ ([E_{5^2}]Z_3)/(E_{5^2}) \simeq Z_3,\ ([E_{5^2}]S_3)/([E_{5^2}]Z_3) \simeq Z_2.$$

Consequently, the estimation of the derived length, which is obtained in Theorem 18, is exact.

It is well known that a p-solvable group of p-rank 1 is called p-supersolvable ([4], p. 713). If G is solvable of rank 1 then it has the following properties: *G possesses an ordered Sylow tower of supersolvable type; the nilpotent length of G and the derived length of $G/\Phi(G)$ are at most 2; the p-length $l_p(G)$ equals 1 for all $p \in \pi(G)$* ([4], VI.9).

Huppert [38] and Rose [47] studied solvable groups of rank 2 and proved the two theorems.

Theorem 19 ([38], Theorem 14; [4], Theorem VI.9.1(d))**.** *Consider a solvable group G of rank ≤ 2 and the greatest prime divisor p of $|G|$. If $p > 3$ then the Sylow p-subgroup is normal in G. In particular, if the order of the group is not divisible by 2 or 3 then the group possesses an ordered Sylow tower of supersolvable type.*

Theorem 20 ([47], Corollary 1)**.** *Consider a solvable group G and the greatest prime divisor p of $|G|$. If $r_t(G) \leq 2$ for all $t \in \pi(G) \setminus \{p\}$ and G includes no sections isomorphic to the alternating group A_4 then G possesses an ordered Sylow tower of supersolvable type.*

Theorem 19 implies that each solvable group of rank ≤ 2 has a normal $\{2,3\}'$-Hall subgroup that possesses an ordered Sylow tower of supersolvable type.

In [48] new properties of solvable groups of rank 2 are established and study solvable groups of rank 3. In particular, we prove the following theorems.

Theorem 21 (V.S. Monakhov and A.A. Trofimuk [48], Theorem 1)**.** *Let G be a solvable group with $r(G) \leq 2$. Then the following statements hold.*

1. *The nilpotent length of G is at most 4.*
2. *The derived length of $G/\Phi(G)$ is at most 5.*
3. *$l_p(G) \leq 1$ for every prime $p > 3$, while $l_2(G) \leq 2$ and $l_3(G) \leq 2$.*

Theorem 22 (V.S. Monakhov and A.A. Trofimuk [48], Theorem 2)**.** *Let G be a solvable group with $r(G) \leq 3$. Then the following statements hold.*

1. *The nilpotent length of G is at most* 4.
2. *The derived length of $G/\Phi(G)$ is at most* 6.
3. $l_p(G) \leq 1$ *for every prime* $p > 3$, *while* $l_2(G) \leq 2$ *and* $l_3(G) \leq 2$.

Solvable groups of rank ≤ 2 and rank ≤ 3 admit identical upper bounds on nilpotent length and p-length, while the upper bounds on derived length differ. But if we bound the p-rank of a solvable group G for a small values of p by 3, and for the remaining p by 2 then we can retain the upper bound on the derived length of $G/\Phi(G)$ as in Theorem 21.

Theorem 23 (V.S. Monakhov and A.A. Trofimuk [48], Theorem 3)**.** *Let G be a solvable group with $r_p(G) \leq 2$ for every prime $p > 5$ and $r_p(G) \leq 3$ for every $p \in \{2, 3, 5\}$. Then the following statements hold.*

1. *The derived length of $G/\Phi(G)$ is at most* 5.
2. *The group G includes a normal $\{2, 3, 5, 7, 13, 31\}'$-Hall subgroup that possesses an ordered Sylow tower of supersolvable type.*

The next theorem establishes the dependence of the derived length of $G/\Phi(G)$ and the p-length on the rank of a solvable group G in the general case.

Theorem 24 (V.S. Monakhov and A.A. Trofimuk [48], Theorem 4)**.**
1. *If G is a solvable group then the derived length of $G/\Phi(G)$ is at most $1 + \rho(r(G))$. In particular, it is at most $3 + r(G)$.*
2. *If G is a p-solvable group then $l_p(G) < 2 + \log_2 r_p(G)$.*

For p-solvable group G it is known that $l_p(G) \leq r_p(G)$ ([4], Theorem VI.6.6). We refine this as

Corollary 5 (V.S. Monakhov and A.A. Trofimuk [48], Corollary 1)**.** *Let G be a p-solvable group. If $l_p(G) = r_p(G)$ then either $r_p(G) = 1$ or $r_p(G) = 2$ and $p \in \{2, 3\}$. In particular, if $r_p(G) \geq 3$ then $l_p(G) \leq r_p(G) - 1$.*

Example 11. Let S be a extraspecial group of order 27. The semidirect product $G = [S]GL(2, 3)$ is a solvable group of rank 2 with the Frattini subgroup $\Phi(G)$ of order 3. The nilpotent length of G equals 4, the derived length of $G/\Phi(G)$ equals 5, 2- and 3-lengths of this group equal 2. Hence, the estimations of Theorem 21 are exact.

Example 12. The group $G = [E_{7^3}]([K]SL(2, 3))$ of order 222 264 of an example 6 has $r_2(G) = r_3(G) = 2$, $r_7(G) = 3$. Besides, the Frattini subgroup of this group is identity, the nilpotent length equals 4 and the derived length equals 6. Hence the estimations of the nilpotent and the derived lengths in Theorem 22 are exact.

3. The nilpotent π-length of finite π-solvable group

For a group G consider a series

$$1 = P_0 \subseteq N_0 \subseteq P_1 \subseteq N_1 \subseteq P_2 \subseteq N_2 \subseteq \dots,$$

$$N_i/P_i = O_{\pi'}(G/P_i),\ P_{i+1}/N_i = O_\pi(G/N_i),\ i = 0, 1, 2, \dots.$$

Here $O_{\pi'}(X)$ and $O_\pi(X)$ denote the largest normal π'- and π-subgroup of the group X respectively. If G is π-solvable then $N_k = G$ for some positive integer k. The least positive integer k possessing this property is called *the π-length of a π-solvable group* G and is denoted by $l_\pi(G)$. For $\pi = \{p\}$ the definition of the π-length of a π-solvable group turns to the definition of the p-length.

The concept of the π-length of a π-solvable group is related to the following Shemetkov's problem ([12], Question 11.119).

Question 5. *Let π be a non-empty set of prime numbers. Is it true that the π-length of a π-solvable group is bounded from above by the derived length $d(G_\pi)$ of its π-Hall subgroup?*

N.S.Chernikov and A.P. Petravchuk [49] proved that $l_\pi(G) \leq 2d(G_\pi)$ and L.S. Kazrin [50] obtained a positive answer for the case $2 \notin \pi$. In general case this question is opened.

R. Carter, B. Fischer, T. Hawkes [51] suggested the concept of the nilpotent π-length of a solvable group as generality of the nilpotent length and the p-length simultaneously. The nilpotent π-length of a π-solvable group G is defined as follows. Let

$$P_0^n = 1,\ N_i^n/P_i^n = O_{\pi'}(G/P_i^n),\ P_{i+1}^n/N_i^n = F(G/N_i^n),\ i = 0, 1, 2, \dots.$$

For π-solvable group G there exists a number k such that $N_k^n = G$. The least positive integer k such that $N_k^n = G$ is called the *nilpotent π-length* of G and is denoted by $l_\pi^n(G)$. Since P_{i+1}^n/N_i^n is a nilpotent π-group and N_i^n/P_i^n is a π'-group then $l_\pi(G) \leq l_\pi^n(G)$. If $\pi = \{p\}$ then $l_\pi^n(G) = l_\pi(G) = l_p(G)$. It is also clear that the equality $l_\pi(G) = l_\pi^n(G)$ remains valid for a π-solvable group with a nilpotent π-Hall subgroup.

If $\pi(G) \subseteq \pi$ then a π-solvable group G becomes solvable and the value of the nilpotent π-length of G coincides with the value of the nilpotent length. If G is solvable and $\pi \subseteq \pi(G)$, $\pi \neq \pi(G)$ then the concept of the nilpotent π-length and the π-length are of individual interest.

Work of Numata [52] was one of the first works on the nilpotent π-length of a π-solvable group. In it following three facts are established.

Theorem 25. 1. *If G is π-solvable then $l_\pi^n(G) \le 1 + c_\pi^m(G)$, where $c_\pi^m(G)$ is the number of classes of the conjugacy non-normal maximal subgroups of G, whose indexes belong to π.*

2. *If G is solvable then $n(G) \le 1 + c^m(G)$, where $c^m(G)$ is the number of classes of the conjugacy non-normal maximal subgroups of G.*

3. *For any n there is a solvable group G such that the nilpotent length of G is equal to n and the number of classes of the conjugacy non-normal maximal subgroups is equal to $n-1$.*

In view of item 3 as composed in item 1 and 2 Theorem 25 it is impossible to get rid of unit. But if to consider all maximal subgroups, then it is possible to make. This supervision leads to the following theorem.

Theorem 26 ([53])**. 1**. *If G is π-solvable then $l_\pi^n(G)$ is at most the number of classes of the conjugacy maximal subgroups of its π-Hall subgroups.*

2. *Let G be a π-solvable group. Let's fix up the maximal subgroup H, which index is π-number. Let K be a intersection of all non-conjugacy with H maximal subgroups of G, which index is π-number. Then $l_\pi^n(K) \le 2$.*

3. *The nilpotent length of solvable group is at most the number of classes of the conjugacy maximal subgroups.*

In connection with item 3 of Theorems 25 and 26 there is the following question.

Question 6. *How the derived length of solvable group is connected with number of classes of conjugacy (non-normal) maximal subgroups?*

The work of N.S. Chernikov and A.P. Petravchuk [49] devoted to estimations of the nilpotent π-length of a π-solvable group.

Theorem 27 ([49], Theorem 1)**.** *Let G be a π-solvable group. If one of the following conditions is hold: G_2 is abelian; $2 \in \pi$ and G_π is 2-separable; $2 \notin \pi$ and G_π is 3-separable; $2 \notin \pi$ and G_3 is abelian, then $l_\pi^n(G) \le d(G_\pi)$.*

V.S. Monakhov and O.A. Shpyrko proved the following theorem.

Theorem 28 ([54])**.** *Let G be a π-solvable group. Then the following statements hold:*

1) *if $2 \notin \pi$ then $l_\pi^n(G) \le d(G_\pi)$;*
2) *$l_\pi^n(G) \le 2d(G_\pi)$.*

Since $l_\pi(G) \le l_\pi^n(G)$ for every π-solvable group G, this theorem generalizes the results given above L.S. Kazarin [50], N.S. Tchernikov and A.P. Petravchuk [49] in which similar estimates are received for $l_\pi(G)$. In connection with Theorem 28 there is the following question.

Question 7. *Let π be a non-empty set of prime numbers. Whether it is true that the nilpotent π-length of a π-solvable group G is limited from above by the derived length $d(G_\pi)$ of its π-Hall subgroup?*

It is clear that the positive solution of this question will lead to the decision formulated above L.A. Shemetkov's task.

The following hypothesis offered by V.S. Monakhov ([12], Question 15.61) connected with concept of the nilpotent π-length.

Question 8. *Whether it is true that $l_\pi^n(G) \leq n(G_\pi) - 1 + \max_{p\in\pi} l_p(G)$ for any π-solvable group?*

Here $n(G_\pi)$ is the nilpotent length of π-Hall subgroup G_π of G.

If π-Hall subgroup is nilpotent then the hypothesis is fair ([49], Lemma 4). If π-Hall subgroup is either supersolvable, or a minimal non- supersolvable group then the hypothesis is fair. It follows from following theorem.

Theorem 29 ([54], [55])**. 1**. *Let G be a π-solvable group such that the derived subgroup of its π-Hall subgroup is nilpotent. Then $l_\pi^n(G) \leq 1 + \max_{p\in\pi} l_p(G)$.*

2. *Let G be a π-solvable and let all proper subgroups of its π-Hall subgroups are supersolvable. Then*

$$l_\pi^n(G) \leq n(G_\pi) - 1 + \max_{p\in\pi} l_p(G) \leq 2 + \max_{p\in\pi} l_p(G).$$

A number of the corollaries giving new estimations of nilpotent π-length of a π-solvable group depending on either the structure of its π-Hall subgroup, or from types of Sylow p-subgroups for $p \in \pi$, follows from this theorem.

Corollary 6. 1. *If Sylow p-subgroups of π-solvable group G are cyclic for all $p \in \pi$ then $l_\pi^n(G) \leq 2$.*

2. *Let G be a π-solvable group with metabelian π-Hall subgroup. Then $l_\pi^n(G) \leq 3$, but if $2 \notin \pi$ then $l_\pi^n(G) \leq 2$.*

3. *If G is a π-solvable group with totally factorizable π-Hall subgroup then $l_\pi^n(G) \leq 1 + \max_{p\in\pi} l_p(G) \leq 2$.*

4. *If G is π-solvable and its π-Hall subgroup is Schmidt subgroup then $l_\pi^n(G) \leq 2$.*

Let's remind that a group G is called totally factorizable, if there is the complements to any subgroup. A Schmidt group is a non-nilpotent group all of whose proper subgroups are nilpotent.

Let $\mathfrak{N}$ denote the class of all nilpotent groups and k be a positive integer. Then $\mathfrak{N}^k$ is the class of all solvable group of the nilpotent length $\leq k$. It is known that $\mathfrak{N}^k$ is a saturated formation and is the Fitting class ([3], Theorem 5.39). Therefore, in each solvable group there are $\mathfrak{N}^k$-projectors and $\mathfrak{N}^k$-injector ([3], Theorems 5.15 and 5.45).

K. Doerk [56] proved the following statements.

Theorem 30 ([56], [3], Theorems 4.30 and 5.54)**. 1**. *If M is a maximal subgroup of solvable group G then $n(M) = n(G) - i$, $i \in \{0, 1, 2\}$.*

2. *Let G be a solvable group. If G has a subgroup H such that H is a $\mathfrak{N}^k$-projector and a $\mathfrak{N}^k$-injector, $k \geq 2$ then $G \in \mathfrak{N}^k$.*

Example 13. All three values of $i \in \{0, 1, 2\}$ in item 1 of Theorem 30 are possible. In a non-identity nilpotent group it is true that $i = 0$ for any maximal subgroup. In the symmetric group S_3 it is true that $i = 1$ for any maximal subgroup. In the symmetric group S_4 it is true that $i = 2$ for the maximal subgroup coinciding with Sylow 2-subgroup.

The second statement is broken for $k = 1$. As an example serves the symmetric group S_4 of degree 4 in which Sylow 2-subgroup are a $\mathfrak{N}$-projector and a $\mathfrak{N}$-injector.

A. Ballester-Bolinches and M. Perez-Ramos [57] have transferred the first statement for the $\mathfrak{F}$-length. If $\mathfrak{F}$ is a saturated formation then the $\mathfrak{F}$-length of solvable group G is the nilpotent length of its $\mathfrak{F}$-residual ([57], [58], V.5.2). In other words the $\mathfrak{F}$-length of solvable group G is the least positive integer $n = n_{\mathfrak{F}}(G)$ such that $G \in \mathfrak{N}^n\mathfrak{F}$. Obviously, it is a definition of the nilpotent length for $\mathfrak{F} = (1)$.

Theorem 31 ([57], Theorem 1)**.** *If $\mathfrak{F}$ is a subgroup closed saturated formation and M is a maximal subgroup of G then $n_{\mathfrak{F}}(M) = n_{\mathfrak{F}}(G) - i$, $i \in \{0, 1, 2\}$.*

V.S. Monakhov and O.A. Shpyrko have transferred the Doerk's theorem on π-solvable groups. In the beginning we will define for each positive integer k following classes:

$\mathfrak{L}_\pi(k)$ is a class of all solvable group of π-length $\leq k$,

$\mathfrak{L}^n_\pi(k)$ is a class of all solvable group of nilpotent π-length $\leq k$.

Both classes are saturated formations and Fitting classes ([3], Theorem 5.39). Then every solvable group has a $\mathfrak{L}_\pi(k)$- and a $\mathfrak{L}^n_\pi(k)$-projector, a $\mathfrak{L}_\pi(k)$- and a $\mathfrak{L}^n_\pi(k)$-injector ([3], Theorems 5.15 and 5.45).

It is clear that if π is a set of prime then $\mathfrak{L}^n_\pi(k) = \mathfrak{N}^k$.

Theorem 32 (V.S. Monakhov, O.A. Shpyrko [59]). **1**. *If G is π-solvable and M is its maximal subgroup then $l_\pi^n(M) = l_\pi^n(G) - i$, $i \in \{0, 1, 2\}$.*

2. *If G is solvable and some $\mathfrak{L}_\pi^n(k)$-projector is $\mathfrak{L}_\pi^n(k)$-injector, $k \geq 2$ then $G \in \mathfrak{L}_\pi^n(k)$.*

Theorem 33 (V.S. Monakhov, O.A. Shpyrko [59]). **1**. *If G is π-solvable and M is its maximal subgroup then $l_\pi(M) = l_\pi(G) - i$, $i \in \{0, 1\}$.*

2. *If G is solvable and some $\mathfrak{L}_\pi(k)$-projector is $\mathfrak{L}_\pi(k)$-injector then $G \in \mathfrak{L}_\pi(k)$.*

For $\pi = \{p\}$ we will receive

Corollary 7. **1**. *Let p be a prime and G is p-solvable. If M is a maximal subgroups then $l_p(M) = l_p(G) - i$, $i \in \{0, 1\}$.*

2. *If G is solvable and some $\mathfrak{L}_p(k)$-projector is $\mathfrak{L}_p(k)$-injector then $G \in \mathfrak{L}_p(k)$.*

The results of Doerk are special cases of Theorem 32 in the case $\pi = \pi(G)$. If $\pi \subseteq \pi(G)$, $\pi \neq \pi(G)$ then all statements of Theorems 32 and 33, Corollary 7 are new to any finite solvable group G.

Other results connected with π-length and nilpotent π-length contained in works [60]–[64].

In connection with Theorems 30–33 there is the following task.

Question 9. *To find the smallest positive integer k, if it exists, such that $d(G) - d(M) \leq k$ for any finite solvable group G and any its maximal subgroup M.*

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