Algebra and Discrete Mathematics RESEARCH ARTICLE
Volume 14 **(2012)**. Number 1. pp. 37 – 48

On locally soluble AFN-groups

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Communicated by L. A. Kurdachenko

Abstract. Let A be an $\mathbf{R}G$-module, where $\mathbf{R}$ is a commutative ring, G is a locally soluble group, $C_G(A) = 1$, and each proper subgroup H of G for which $A/C_A(H)$ is not a noetherian $\mathbf{R}$-module, is finitely generated. We describe the structure of a locally soluble group G with these conditions and the structure of G under consideration if G is a finitely generated soluble group and the quotient module $A/C_A(G)$ is not a noetherian $\mathbf{R}$-module.

Introduction

Let A be a vector space over a field F, $GL(F, A)$ be the group of all automorphisms of A. Subgroups of $GL(F, A)$ are called linear groups. If A has a finite dimension over F, $GL(F, A)$ can be considered as a group of non-singular $(n \times n)$-matrixes over F, where $n = dim_F A$. Finite dimensional linear groups have been studied by many authors. In the case when A has infinite dimension over F, the situation is rather different. Infinite dimensional linear groups were investigated a little. Study of this class of groups requires some finiteness conditions. The one from these finiteness conditions is a finitarity of infinite dimensional linear group. We recall that a linear group is called finitary if for each element $g \in G$ the subspace $C_A(g)$ has finite codimension in A (see [1], [2], for example). Many results have been obtained conserning finitary linear groups [2].

In [3] antifinitary linear groups are investigated. Let $G \leq GL(F, A)$, $A(wFG)$ be the augmentation ideal of the group ring FG, $augdim_F(G) =$

2010 MSC: 20F19.
Key words and phrases: locally soluble group, noetherian module, group ring.

$dim_F(A(wFG))$. A linear group G is called antifinitary if each proper subgroup H of infinite dimension $augdim_F(H)$ is finitely generated [3].

If $G \leq GL(F, A)$ then A can be considered as an FG-module. The natural generalization of this case is a consideration of an $\mathbf{R}G$-module A where $\mathbf{R}$ is a ring. B.A.F. Wehrfritz have considered artinian-finitary groups of automorphisms of a module M over a ring $\mathbf{R}$ and noetherian-finitary groups of automorphisms of a module M over a ring $\mathbf{R}$ which are the analogues of finitary linear groups [4, 5, 6]. A group of automorphisms $F_1 Aut_{\mathbf{R}} M$ of a module M over a ring $\mathbf{R}$ is called artinian-finitary if $A(g-1)$ is an artinian $\mathbf{R}$-module for each $g \in F_1 Aut_{\mathbf{R}} M$. A group of automorphisms $FAut_{\mathbf{R}} M$ of a module M over a ring $\mathbf{R}$ is called noetherian-finitary if $A(g-1)$ is a noetherian $\mathbf{R}$-module for each $g \in FAut_{\mathbf{R}} M$. B.A.F. Wehrfritz have investigated the relation between $F_1 Aut_{\mathbf{R}} M$ and $FAut_{\mathbf{R}} M$ [6].

In [7] the notion of the cocentralizer of a subgroup H in the module A have been introduced. Let A be an $\mathbf{R}G$-module where $\mathbf{R}$ is a ring, G is a group. If $H \leq G$ then $A/C_A(H)$ considered as an $\mathbf{R}$-module is called the cocentralizer of a subgroup H in A.

In this paper we consider the analogue of antifinitary linear groups in theory of modules over group rings. Let A be an $\mathbf{R}G$-module where $\mathbf{R}$ is a ring, G is a group. We say that a group G is an AFN-group if each proper subgroup H of G for which $A/C_A(H)$ is not a noetherian $\mathbf{R}$-module, is finitely generated.

In the paper locally soluble AFN-groups are investigated. Later on it is considered $\mathbf{R}G$-module A such that $\mathbf{R}$ is a commutative ring, $C_G(A) = 1$. The main results ate theorems 1, 2. In theorem 1 the structure of a locally soluble AFN-group is described. In theorem 2 the structure of a finitely generated soluble AFN-group G is described in the case where the cocentralizer of G in A is not a noetherian $\mathbf{R}$-module.

1. Prelimlnary results

We begin by assembling some elementary facts about AFN-groups.

Lemma 1. *Let A be an $\mathbf{R}G$-module.*

(1) If $L \leq H \leq G$ and the cocentralizer of a subgroup H in A is a noetherian $\mathbf{R}$-module, then the cocentralizer of a subgroup L in A is a noetherian $\mathbf{R}$-module.

(2) If $L, H \leq G$ and the cocentralizers of subgroups L, H in A are noetherian $\mathbf{R}$-modules, then the cocentralizer of $\langle L, H\rangle$ in A is a noetherian $\mathbf{R}$-module.

Corollary 1. *Let A be an $\mathbf{R}G$-module, $ND(G)$ be a set of all elements $x \in G$ such that the cocentralizer of $\langle x \rangle$ in A is a noetherian $\mathbf{R}$-module. Then $ND(G)$ is a normal subgroup of G.*

Proof. By lemma 1 $ND(G)$ is a subgroup of G. Since $C_A(x^g) = C_A(x)g$ for all $x, g \in G$ then $ND(G)$ is a normal subgroup of G. □

Corollary 2. *Let A be an $\mathbf{R}G$-module, G be an* AFN*-group. If G has proper non-finitely generated subgroups K and L then the cocentralizer of $\langle K, L \rangle$ in A is a noetherian $\mathbf{R}$-module.*

Lemma 2. *Let A be an $\mathbf{R}G$-module, G be an* AFN*-group. Suppose that H is a subgroup of G and K is a normal subgroup of H such that $H/K = Dr_{\lambda\in\Lambda}(H_\lambda/K)$ where $H_\lambda \neq K$ for every $\lambda \in \Lambda$ and the index set Λ is infinite. Then the cocentralizer of H in A is a noetherian $\mathbf{R}$-module.*

Proof. The quotient group H/K is decomposed in the direct product $H/K = H_1/K \times H_2/K$ such that H_1/K and H_2/K are non-finitely generated quotient groups. Since G is an AFN-group then by Lemma 1 the cocentralizer of H in A is a noetherian $\mathbf{R}$-module. □

Corollary 3. *Let A be an $\mathbf{R}G$-module, G be an* AFN*-group. Suppose that H is a subgroup of G and K is a normal subgroup of H such that $H/K = Dr_{\lambda\in\Lambda}(H_\lambda/K)$, $H_\lambda \neq K$ for every $\lambda \in \Lambda$ and the index set Λ is infinite. If g is an element of G such that H_λ is $\langle g \rangle$-invariant for every $\lambda \in \Lambda$, then $g \in ND(G)$.*

Proof. The subgroup K is $\langle g \rangle$-invariant. Since the index set Λ is infinite,

$$Dr_{\lambda\in\Lambda}(H_\lambda/K)\langle gK \rangle = (H_1/K)((H_2/K)\langle gK \rangle),$$

where H_1 and $H_2\langle g \rangle$ are proper non-finitely generated subgroups of G. It follows that the cocentralizer of $\langle H, g \rangle$ in A is a noetherian $\mathbf{R}$-module. By lemma 1 the cocentralizer of $\langle g \rangle$ in A is a noetherian $\mathbf{R}$-module. □

Corollary 4. *Let A be an $\mathbf{R}G$-module, G be an* AFN*-group. Suppose that H is a subgroup of G and K is a normal subgroup of H such that $H/K = Dr_{\lambda\in\Lambda}(H_\lambda/K)$, $H_\lambda \neq K$ for every $\lambda \in \Lambda$ and the index set Λ is infinite. If H_λ is G-invariant for every $\lambda \in \Lambda$, then $G = ND(G)$.*

Corollary 5. *Let A be an $\mathbf{R}G$-module, G be an* AFN*-group. Suppose that H is a subgroup of G and K is a normal subgroup of H such that H/K is an infinite elementary abelian p-group for some prime p. If g is an element of G such that H and K are $\langle g \rangle$-invariant and $g^k \in C_G(H/K)$ for some $k \in \mathbb{N}$ then $g \in ND(G)$.*

Proof. Let $1 \neq h_1K \in H/K, H_1/K = \langle h_1K\rangle^{\langle gK\rangle}$. Since the element g induced on the quotient group H/K an automorphism of finite order, H_1/K is finite. Since the quotient group H/K is elementary abelian then $H/K = H_1/K \times C_1/K$. Note that the set $\{C_1^y | y \in \langle g\rangle\}$ is finite. Let

$$\{C_1^y | y \in \langle g\rangle\} = \{U_1, \ \cdots \ , U_m\}.$$

Then the $\langle g\rangle$-invariant subgroup $D_1 = U_1 \ \cap \ \cdots \ \cap \ U_m = Core_{\langle g\rangle}(C_1)$ has finite index in H. Moreover, since the subgroup K is $\langle g\rangle$-invariant, $K \leq D_1$. Let $1 \neq h_2K \in D_1/K, H_2/K = \langle h_2K\rangle^{\langle gK\rangle}$. Then

$$\langle H_1/K, H_2/K\rangle = H_1/K \times H_2/K.$$

Again we have $H/K = (H_1/K \times H_2/K) \times C_2/K$ for some subgroup C_2. Reasoning in a similar way, we construct an infinite family $\{H_n/K | n \in \mathbb{N}\}$ of non-identity $\langle g\rangle$-invariant subgroups such that

$$\langle H_n/K | n \in \mathbb{N}\rangle = Dr_{n\in\mathbb{N}} H_n/K.$$

By corollary 3 $g \in ND(G)$. □

2. On locally soluble AFN-groups

A group G is said to have finite 0-rank $r_0(G) = r$ if G has a finite subnormal serires with exactly r infinite cyclic factors, all other factors being periodic. It is well known that the 0-rank is independent of the chosen series.

Lemma 3. *Let A be an* **R***G-module, G be an* AFN*-group. Suppose that a group G has a normal subgroup K such that G/K is an abelian quotient group of infinite* 0*-rank. Then the cocentralizer of G in A is a noetherian* **R***-module.*

Proof. Let B/K be a free abelian subgroup of G/K such that G/B is periodic. If $\pi(G/B)$ is infinite then the cocentralizer of G in A is a noetherian **R**-module by lemma 2. Suppose that $\pi(G/B)$ is finite and choose a prime q such that $q \notin \pi(G/B)$. Put $C/K = (B/K)^q$ so that B/C is a Sylow q-subgroup of G/C. Let P/C be the Sylow q'-subgroup of G/C. Then G/P is an infinite elementary abelian q-group. By lemma 2 the cocentralizer of G in A is a noetherian **R**-module. □

Corollary 6. *Let A be an* **R***G-module, G be an* AFN*-group. Suppose that G has a normal subgroup K such that G/K is an abelian-by-finite* *group of infinite* 0*-rank. Then the cocentralizer of* G *in* A *is a noetherian* $\mathbf{R}$*-module.*

Proof. Let L/K be a normal abelian subgroup of G/K such that G/L is finite. Then $r_0(L/K)$ is infinite. Pick $g \in G \setminus L$. Let B/K be a free abelian subgroup of L/K such that the quotient group L/B is periodic. The rank $r_0(B/K)$ is infinite. Choose an element $a_1 \in B \setminus K$. Put $A_1/K = (\langle a_1 \rangle K/K)^{\langle gK \rangle}$. Since G/L is finite, A_1/K is a finitely generated abelian group. It follows that $A_1/K \cap B/K$ is finitely generated. Choose the subgroup C_1/K of B/K which maximal under

$$(A_1/K \cap B/K) \cap C_1/K = \langle 1 \rangle.$$

Then L/C_1 is a group of finite 0-rank. Since G/L is finite, the family $\{(C_1/K)^{yK} | y \in \langle g \rangle\}$ is finite. Let

$$\{(C_1/K)^{yK} | y \in \langle g \rangle\} = \{D_1/K, \cdots, D_n/K\},$$

and put

$$E/K = D_1/K \ \cap \ \cdots \ \cap \ D_n/K.$$

Then $E/K \leq B/K$, E/K is $\langle g \rangle$-invariant. By Remak's theorem L/E has finite 0-rank. In particular, E/K has infinite 0-rank. Choose an element $a_2 \in E \setminus K$. Put $A_2/K = (\langle a_2 \rangle K/K)^{\langle gK \rangle}$. Then $A_2/K \leq E/K$, $(A_1/K) \cap (A_2/K) = 1$. Proceeding in the same way, we construct a family $\{A_n/K | n \in \mathbb{N}\}$ of non-identity $\langle g \rangle$-invariant subgroups such that

$$\langle A_n/K | n \in \mathbb{N} \rangle = Dr_{n \in \mathbb{N}}(A_n/K).$$

By corollary 3 $g \in ND(G)$. We can choose a finitely generated subgroup F of G such that $G/K = (FK/K)(L/K)$ and for each element g of F $g \in ND(G)$. Since F is a finitely generated subgroup then $F \leq ND(G)$. By lemma 3 the cocentralizer of L in A is a noetherian $\mathbf{R}$-module. Since $G = FL$ then by lemma 1 the cocentralizer of G in A is a noetherian $\mathbf{R}$-module. □

Lemma 4. *Let* A *be an* $\mathbf{R}G$*-module,* G *be an* AFN*-group. Suppose that* G *has subgroups* $L \leq K \leq H$ *such that* L *and* K *are normal subgroups of* H*,* K/L *is a divisible Chernikov group and* H/K *is a polycyclic-by-finite group. If the cocentralizer of* H *in* A *is not a noetherian* $\mathbf{R}$*-module, then* $H = G$*. Moreover, either* $G = K$ *(so that* G/L *is a Prüfer* p*-group for some prime* p*) or* G/K *is a cyclic* q*-group for some prime* q*.*

Proof. Suppose that H/L is finitely generated. By P. Hall theorem (theorem 5.34 [8]) H/L satisfies the maximal condition for normal subgroups. In particular, K/L satisfies the condition $max - H$. Since K/L is a divisible Chernikov group, this is impossible. Therefore H/L can not be finitely generated and thus H is non finitely generated subgroup. Since the cocentralizer of H in A is not a noetherian $\mathbf{R}$-module, then $H = G$.

Suppose that $G \neq K$. Then $G = \langle K, M \rangle$ for some finite set M. Since M is finite, we may choose a subset S of M such that $G = \langle K, S \rangle$ but $G \neq \langle K, X \rangle$ for any proper subset X of S. Let

$$S = \{x_1, \cdots, x_m\}.$$

If $m > 1$, then $\langle K, x_1, \cdots, x_{m-1} \rangle$ and $\langle K, x_m \rangle$ are proper non finitely generated subgroups of G. Since G is an AFN-group then the cocentralizers of subgroups $\langle K, x_1, \cdots, x_{m-1} \rangle$ and $\langle K, x_m \rangle$ in A are noetherian $\mathbf{R}$-modules. Since $G = \langle \langle K, x_1, \cdots, x_{m-1} \rangle, \langle K, x_m \rangle \rangle$, by lemma 1 the cocentralizer of G in A is a noetherian $\mathbf{R}$-module. This is a contradiction that shows that $m = 1$. Therefore $G/K = \langle xK \rangle$ is cyclic. If G/K is infinite, then G must be a product of two proper non finitely generated subgroups, what again gives a contradiction. If G/K is finite but $|\pi(G/K)| > 1$, we again have a contradiction. Hence G/K is a cyclic q-group for some prime q. □

Lemma 5. *Let A be an $\mathbf{R}G$-module, G be an* AFN*-group. Suppose that H is a normal subgroup of G such that G/H is an infinite abelian-by-finite periodic group. If the cocentralizer of G in A is not a noetherian $\mathbf{R}$-module, then either G/H is a Prüfer p-group for some prime p or G has a normal subgroup K such that G/K is a cyclic q-group for some prime q, $H \leq K$ and K/H is a Chernikov divisible p-group for some prime p.*

Proof. Let L/H be an abelian normal subgroup of G/H such that G/L is finite. If $\pi(L/H)$ is infinite, then the cocentralizer of L in A is a noetherian $\mathbf{R}$-module by lemma 2. By corollary 4 $G = ND(G)$. Since G/L is finite, it follows that the cocentralizer of G in A is a noetherian $\mathbf{R}$-module by lemma 1. This contradiction proves that $\pi(L/H)$ is finite. Then there exists a prime p such that the Sylow p-subgroup P/H of L/H is infinite. Let F/H be the Sylow p'-subgroup of L/H. There is a finite subgroup S/H such that $G/H = (L/H)(S/H)$. If F/H is infinite then both subgroups $(P/H)(S/H)$ and $(F/H)(S/H)$ are not finitely generated. Therefore the cocentralizers of subgroups PS and FS in A are noetherian $\mathbf{R}$-modules. By lemma 1 the cocentralizer of G in A is a noetherian $\mathbf{R}$-module. This contradiction shows that F/H is finite. Put $B/H = (P/H)^p$. If P/B is infinite then P/B is not finitely generated. Therefore the cocentralizer of P in A is a noetherian $\mathbf{R}$-module. By corollary 5 $G = ND(G)$. Since G/P is finite, it follows that the cocentralizer of G in A is a noetherian $\mathbf{R}$-module by lemma 1. This contradiction proves that $(P/H)/(B/H)$ is finite. By lemma 3 [9] $P/H = (V/H) \times (D/H)$ where D/H is divisible and V/H is finite. D is a G-invariant subgroup. Put $K = D$. Since G/D is finite, it is suffices to apply lemma 4. □

Lemma 6. *Let A be an $\mathbf{R}G$-module, G be an* AFN*-group. Suppose that G has normal subgroups $K \leq H$ such that G/H is finite and H/K is torsion-free abelian. If the cocentralizers of G in A is not a noetherian $\mathbf{R}$-module, then H/K is finitely generated.*

Proof. By corollary 6 H/K has finite 0-rank. Let B/K be a free abelian subgroup of H/K such that H/B is periodic. Since $r_0(H/K)$ is finite then B/K is finitely generated. Suppose that H/K is not finitely generated. Since G/H is finite, $C/K = (B/K)^{G/K}$ is finitely generated. By lemma 5 $|\pi(G/C)| \leq 2$. Choose the distinct primes r, s such that $r, s \notin \pi(G/C)$. Put $D/K = (C/K)^{rs}$. Then G/D is abelian-by-finite, periodic and not finitely generated. Moreover $|\pi(G/D)| \geq 3$. This contradicts lemma 5. Therefore H/K is finitely generated. □

Lemma 7. *Let A be an $\mathbf{R}G$-module, G be an* AFN*-group. Suppose that G has two normal subgroups $K \leq H$ such that G/H is finite and H/K is abelian and not finitely generated. If the cocentralizer of G in A is not a noetherian $\mathbf{R}$-module, then H/K is Chernikov.*

Proof. By corollary 6 H/K has finite 0-rank. Let T/K be the periodic part of H/K. By lemma 6 H/T is finitely generated. Then H/K has a finitely generated subgroup B/K such that H/B is periodic. Since G/H is finite, $C/K = (B/K)^{G/K}$ is finitely generated. By lemma 5 G/C is a Chernikov group. It follows that T/K is Chernikov too. Let D/K be the divisible part of T/K. Then G/D is finitely generated and abelian-by-finite. It is suffices to apply lemma 4. □

Lemma 8. *Let A be an $\mathbf{R}G$-module, G be a soluble* AFN*-group. If G is not a Prüfer p-group for some prime p then $G/ND(G)$ is a polycyclic quotient group.*

Proof. Put $D = ND(G)$. If the cocentralizer of G in A is a noetherian $\mathbf{R}$-module, then $G = ND(G)$. Therefore we suppose that $G \neq ND(G)$.

Let $D = D_0 \leq D_1 \leq \cdots \leq D_n = G$ be a series of subnormal subgroups of G whose factors are abelian. Consider the factor D_j/D_{j-1}, $j < n$. If this factor is not finitely generated, then the subgroup D_j cannot be finitely generated and the cocentralizer of D_j in A is a noetherian $\mathbf{R}$-module. In particular, $D_j \leq ND(G)$. It follows that D_j/D_{j-1} is finitely generated for every $j = 1, \cdots, n-1$. Put $K = D_{n-1}$. If G/K is finitely generated, then G/D is polycyclic, and all is done. Suppose that G/K is not finitely generated. By lemma 7 G/K is a Chernikov group. Let P/K be the divisible part of G/K. If $P/K \neq G/K$, then P is not finitely generated proper subgroup of G. Thus the cocentralizer of P in A is a noetherian $\mathbf{R}$-module. Therefore $P \leq ND(G)$. But in this case $G/ND(G)$ is finite. Contradiction. Hence $G/K = P/K$. Clearly in this case G/K is a Prüfer p-group for some prime p. Let $g \in G\backslash K$. Since $g \notin ND(G)$, $\langle g, K\rangle$ is finitely generated. The finiteness of $\langle g\rangle K/K$ implies that K is finitely generated (theorem 1.41 [8]). Since G is not a Prüfer p-group for some prime p, then $K \neq 1$. It follows that K has a proper G-invariant subgroup L of finite index such that G/L is Chernikov and not divisible. As above, in this case $G/ND(G)$ is finite. □

Lemma 9. *Let A be an $\mathbf{R}G$-module, G be a locally soluble* AFN*-group. If the cocentralizer of G in A is a noetherian $\mathbf{R}$-module, then G contains a normal hyperabelian subgroup N such that G/N is soluble.*

Proof. Since the cocentralizer of G in A is a noetherian $\mathbf{R}$-module, then $A/C_A(G)$ is a finitely generated $\mathbf{R}$-module. Put $C = C_A(G)$. A has the finite series of $\mathbf{R}G$-submodules

$$\langle 0\rangle = C_0 \leq C_1 = C \leq C_2 = A,$$

such that C_2/C_1 is a finitely generated $\mathbf{R}$-module.

By theorem 13.5 [10] the quotient group $\overline{G} = G/C_G(C_2/C_1)$ contains a normal hyperabelian, locally nilpotent subgroup $\overline{N} = N/C_G(C_2/C_1)$ such that $\overline{G}/\overline{N}$ is imbedded in the Cartesian product $\overline{\mathbf{\Pi}}_{\alpha\in\mathcal{A}}G_\alpha$ of finite dimensional linear groups G_α of degree $f \leq n$ where n depends on the number of generating elements of $\mathbf{R}$-module C_2/C_1 only. Since G is a locally soluble group then $\overline{G}$ is locally soluble too. It follows that the projection H_α of $\overline{G}/\overline{N}$ on each subgroup G_α is a locally soluble finite dimensional linear group of degree at most n. By corollary 3.8 [10] H_α is a soluble group for each $\alpha \in \mathcal{A}$. By theorem 3.6 [10] each group H_α contains a normal subgroup K_α such that $|H_\alpha : K_\alpha| \leq \mu(n)$, K_α is a triangularizable group, K_α has a nilpotent subgroup M_α of step at most

$n-1$, M_α is a normal subgroup of H_α and K_α/M_α is abelian. Therefore $H = \overline{\overline{\mathbf{\Pi}}}_{\alpha\in\mathcal{A}} H_\alpha$ contains a normal nilpotent subgroup $M = \overline{\overline{\mathbf{\Pi}}}_{\alpha\in\mathcal{A}} M_\alpha$ of step at most $n-1$, H/M has a normal abelian subgroup K/M where $K = \overline{\overline{\mathbf{\Pi}}}_{\alpha\in\mathcal{A}} K_\alpha$ and $(H/M)/(K/M)$ is a locally finite group of the finite period at most $\mu(n)!$. It follows that H is a soluble group of the derived length at most $n-1+1+\mu(n)! = n+\mu(n)!$. Therefore $\overline{G}/\overline{N}$ is a soluble group of the derived length at most $n+\mu(n)!$. It follows that G has the series of normal subgroups $C_G(C_2/C_1) \leq N \leq G$. As $G/N \simeq \overline{G}/\overline{N}$ then G/N is a soluble group of the derived length at most $n+\mu(n)!$. Since $C_G(A/C_A(G))$ is abelian and $N/C_G(C_2/C_1)$ is hyperabelian then N is hyperabelian too. □

Theorem 1. *Let A be an $\mathbf{R}G$-module, G be a locally soluble* AFN-*group. Then G has an ascending series of normal subgroups*

$$\langle 1\rangle = L_0 \leq L_1 \leq L_2 \leq \cdots \leq L_\gamma \leq \cdots \leq L_\delta = G$$

such that each factor $L_{\gamma+1}/L_\gamma, \gamma < \delta$, is hyperabelian.

Proof. If the cocentralizer of G in A is a noetherian $\mathbf{R}$-module then we apply lemma 9. Later on we consider the case where the cocentralizer of G in A is not a noetherian $\mathbf{R}$-module. If G is a soluble group then the theorem is valid. Let G be non soluble. By corollary 5.27 [8] G cannot be simple. Therefore G has a proper normal subgroup H_1. If H_1 is finitely genetated, then it is soluble. It follows that H_1 has the series of G-admissible subgroups

$$\langle 1\rangle = B_0 \leq B_1 \leq B_2 \leq \cdots \leq B_k = H_1$$

such that the factors B_t/B_{t-1}, $t = 1, \cdots, k$, are abelian. If H_1 is not finitely generated, then the cocentralizer of H_1 in A is a noetherian $\mathbf{R}$-module. By lemma 9 H_1 contains a normal hyperabelian subgroup N_1 such that H_1/N_1 is soluble. Then H_1 has the series of G-admissible subgroups

$$\langle 1\rangle = R_0 \leq R_1 \leq R_2 \leq \cdots \leq R_m = H_1$$

such that the factors R_t/R_{t-1}, $t = 2, \cdots, m$, are abelian, R_1 is a hyperabelian subgroup. If G/H_1 is a soluble group, then G has the series of normal subgroups

$$H_1 = G_0 \leq G_1 \leq G_2 \leq \cdots \leq G_r = G$$

such that the factors G_t/G_{t-1}, $t = 1, \cdots, r$, are abelian. Therefore G has an ascending series of normal subgroups

$$\langle 1\rangle = L_0 \ \leq\ L_1 \ \leq\ L_2 \ \leq\ \cdots\ \leq\ L_n = G,$$

such that each factor L_t/L_{t-1}, $t = 1, \cdots, n$, is hyperabelian. If G/H_1 is not a soluble group, then G/H_1 has a proper normal subgroup H_2/H_1. As above H_2/H_1 has the series of G-admissible subgroups

$$H_1 = D_0 \ \leq\ D_1 \ \leq\ D_2 \ \leq\ \cdots\ \leq\ D_j = H_2$$

such that each factor D_t/D_{t-1}, $t = 1, \cdots, j$, is hyperabelian.

We proceed in this way. At step with the ordinal α we have that G/H_α is a soluble quotient group. It follows that G has an ascending series of normal subgroups

$$\langle 1\rangle = L_0 \ \leq\ L_1 \ \leq\ L_2 \ \leq\ \cdots\ \leq\ L_\gamma \ \leq\ \cdots\ \leq\ L_\delta = G$$

such that each factor $L_{\gamma+1}/L_\gamma$, $\gamma < \delta$, is hyperabelian. □

Lemma 10. *Let A be an* $\mathbf{R}G$*-module, G be a finitely generated soluble* AFN*-group. Then the cocentralizer of $ND(G)$ in A is a noetherian* $\mathbf{R}$*-module.*

Proof. Put $D = ND(G)$ and let

$$\langle 1\rangle = D_0 \ \leq\ D_1 \ \leq\ \cdots\ \leq\ D_n = D$$

be the derived series of D. If each factor D_{j+1}/D_j, $j = 0, 1, \cdots, n-1$, is finitely generated, then D is polycyclic, and, in particular, D is finitely generated. By lemma 1 the cocentralizer of D in A is a noetherian $\mathbf{R}$-module. Therefore, we suppose that some of the factors D_{j+1}/D_j, $j = 0, 1, \cdots, n-1$, is not finitely generated. Let t be a number such that D_t/D_{t-1} is not finitely generated but D_{j+1}/D_j is finitely generated for every $j \geq t$. It follows that D/D_t is polycyclic. Since G is a finitely generated group then D_t is a proper non finitely generated subgroup of G. Therefore the cocentralizer of D_t in A is a noetherian $\mathbf{R}$-module. Since D/D_t is polycyclic, $D = KD_t$ for some finitely generated subgroup K. As $K \leq ND(G)$, we have that the cocentralizer of K in A is a noetherian $\mathbf{R}$-module. By lemma 1 the cocentralizer of $ND(G)$ in A is a noetherian $\mathbf{R}$-module. □

Theorem 2. *Let A be an $\mathbf{R}G$-module, G be a finitely generated soluble* AFN*-group. If the cocentralizer of G in A is not a noetherian $\mathbf{R}$-module, then the following conditions holds:*

(1) the cocentralizer of $ND(G)$ in A is a noetherian $\mathbf{R}$-module;

(2) G has the series of normal subgroups $B \leq R \leq W \leq G$ such that B is abelian, R/B is locally nilpotent, W/R is nilpotent and G/W is a polycyclic group.

Proof. By lemma 10 the cocentralizer of $ND(G)$ in A is noetherian $\mathbf{R}$-module. Let $C = C_A(ND(G))$. Since A/C is a noetherian $\mathbf{R}$-module, then A has the finite series of $\mathbf{R}G$-submodules $\langle 0 \rangle = C_0 \leq C_1 = C \leq C_2 = A$, such that A/C is a finite generated $\mathbf{R}$-module.

By theorem 13.5 [10] the quotient group $S = G/C_G(C_2/C_1)$ contains the normal locally nilpotent subgroup $D = N/C_G(C_2/C_1)$ such that the quotient group S/D is embedded in the Cartesian product $\overline{\mathbf{\Pi}}_{\alpha\in\mathcal{A}} G_\alpha$ of finite dimensional linear groups G_α of degree $f \leq n$ where n depends on the number of generating elements of an $\mathbf{R}$-module C_2/C_1 only. Since the group G is soluble then the quotient group S is soluble too. Therefore the projection H_α of S on each subgroup G_α is a soluble finite dimensional linear group of degree at most n. By theorem 3.6 [10] each group H_α contains the normal subgroup K_α such that $|H_\alpha : K_\alpha| \leq \mu(n)$, the subgroup K_α is triangularizable, K_α contains the nilpotent subgroup M_α of step at most $n-1$ such that M_α is a normal subgroup of G_α and the quotient group K_α/M_α is abelian. Therefore $H = \overline{\mathbf{\Pi}}_{\alpha\in\mathcal{A}} H_\alpha$ contains the normal nilpotent subgroup $M = \overline{\mathbf{\Pi}}_{\alpha\in\mathcal{A}} M_\alpha$ of step at most $n-1$, the quotient group H/M has the normal abelian subgroup K/M where $K = \overline{\mathbf{\Pi}}_{\alpha\in\mathcal{A}} K_\alpha$ and the quotient group $(H/M)/(K/M)$ is a locally finite group of the period at most $\mu(n)!$. Since S/D is embedded in the Cartesian product $H = \overline{\mathbf{\Pi}}_{\alpha\in\mathcal{A}} H_\alpha$ then S has the series of normal subgroups $D \leq L \leq F \leq S$ such that D is locally nilpotent, L/D is nilpotent, F/L is abelian and S/F is a locally finite group of the finite period. Since G is a finitely generated group then S is finitely generated too. Therefore the quotient group S/F is finite. It follows that S/L is an almost abelian group. Since S/L is finitely generated then S/L is a polycyclic group. Therefore S has the series of normal subgroups $D \leq L \leq S$ such that D is locally nilpotent, L/D is nilpotent, S/L is a polycyclic group.

Let $B = C_G(C_1) \cap C_G(C_2/C_1)$. Each element of B acts trivially in each factor $C_{j+1}/C_j, j = 0, 1$. It follows that B is abelian. By Remak's theorem

$$G/B \leq G/C_G(C_1) \times G/C_G(C_2/C_1).$$

As $ND(G) \leq C_G(C_1)$ then the quotient group $G/C_G(C_1)$ is polycyclic by lemma 8. Since $S = G/C_G(C_2/C_1)$ has the series of normal subgroups $D \leq L \leq S$ such that D is locally nilpotent, L/D is nilpotent, S/L is a polycyclic group then G has the series of normal subgroups

$$B \leq R \leq W \leq G$$

such that B is abelian, R/B is locally nilpotent, W/R is nilpotent and G/W is a polycyclic group. □

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Received by the editors: 21.04.2012
and in final form 02.10.2012.