

## Symmetric modules over their endomorphism rings

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**ABSTRACT.** Let  $R$  be an arbitrary ring with identity and  $M$  a right  $R$ -module with  $S = \text{End}_R(M)$ . In this paper, we study right  $R$ -modules  $M$  having the property for  $f, g \in \text{End}_R(M)$  and for  $m \in M$ , the condition  $fgm = 0$  implies  $gfm = 0$ . We prove that some results of symmetric rings can be extended to symmetric modules for this general setting.

### 1. Introduction

Throughout this paper  $R$  denotes an associative ring with identity, and modules are unitary right  $R$ -modules. All right-sided concepts and results have left-sided counterparts. For a module  $M$ ,  $S = \text{End}_R(M)$  denotes the ring of right  $R$ -module endomorphisms of  $M$ . Then  $M$  is a left  $S$ -module, right  $R$ -module and  $(S, R)$ -bimodule. In this work, for the  $(S, R)$ -bimodule  $M$ ,  $r_R(\cdot)$  and  $l_M(\cdot)$  denote the right annihilator of a subset of  $M$  in  $R$  and the left annihilator of a subset of  $R$  in  $M$ , respectively. Similarly,  $l_S(\cdot)$  and  $r_M(\cdot)$  are the left annihilator of a subset of  $M$  in  $S$  and the right annihilator of a subset of  $S$  in  $M$ , respectively.

A ring is *reduced* if it has no nonzero nilpotent elements. In [13], Krempa introduced the notion of the rigid endomorphism of a ring. An

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endomorphism  $\alpha$  of a ring  $R$  is said to be *rigid* if  $a\alpha(a) = 0$  implies  $a = 0$  for  $a \in R$ . According to Hong-Kim-Kwak [11],  $R$  is said to be an  $\alpha$ -*rigid ring* if there exists a rigid endomorphism  $\alpha$  of  $R$ . In [15], a ring  $R$  is *symmetric* if for any  $a, b, c \in R$ ,  $abc = 0$  implies  $bac = 0$ . This is equivalent to  $abc = 0$  implies  $acb = 0$ . A ring  $R$  is called *semicommutative* if for any  $a, b \in R$ ,  $ab = 0$  implies  $aRb = 0$ . A ring  $R$  is called *abelian* if every idempotent is central, that is,  $ae = ea$  for any  $e^2 = e$ ,  $a \in R$ .

The reduced ring concept was extended to modules by Lee and Zhou in [16], that is, a right  $R$ -module  $M$  is called *reduced* if for any  $m \in M$  and any  $a \in R$ ,  $ma = 0$  implies  $mR \cap Ma = 0$ . Similarly, in [2] and [3], Harmanci et al. extended the rigid ring notion to modules. A right  $R$ -module  $M$  is called *rigid* if for any  $m \in M$  and any  $a \in R$ ,  $ma^2 = 0$  implies  $ma = 0$ . Reduced modules are certainly rigid, but the converse is not true in general. A right  $R$ -module  $M$  is said to be *semicommutative* if for any  $m \in M$  and any  $a \in R$ ,  $ma = 0$  implies  $mRa = 0$ . Abelian modules are introduced in the context by Roos in [21] and studied by Goodearl and Boyle in [9]. A module  $M$  is called *abelian* if for any  $f \in S$ ,  $e^2 = e \in S$ ,  $m \in M$ , we have  $fem = efm$ . Note that  $M$  is an abelian module if and only if  $S$  is an abelian ring. The concept of (quasi-)Baer rings was extended by Rizvi and Roman [19] to the general module theoretic setting, by considering a right  $R$ -module  $M$  as an  $(S, R)$ -bimodule. A module  $M$  is called *Baer* if for all submodules  $N$  of  $M$ ,  $l_S(N) = Se$  with  $e^2 = e \in S$ . A submodule  $N$  of  $M$  is said to be *fully invariant* if it is also a left  $S$ -submodule of  $M$ . Then the module  $M$  is said to be *quasi-Baer* if for all fully invariant submodules  $N$  of  $M$ ,  $l_S(N) = Se$  with  $e^2 = e \in S$ . Motivated by Rizvi and Roman's work on (quasi-)Baer modules, the notion of principally quasi-Baer modules initially appeared in [22]. The module  $M$  is called *principally quasi-Baer* if for any  $m \in M$ ,  $l_S(Sm) = Sf$  for some  $f^2 = f \in S$ . Finally, the concept of right Rickart rings (or right principally projective rings) was extended to modules in [20], that is, the module  $M$  is called *Rickart* if for any  $f \in S$ ,  $r_M(f) = eM$  for some  $e^2 = e \in S$ , equivalently,  $\text{Ker}f$  is a direct summand of  $M$ .

In this paper, we investigate some properties of symmetric modules over their endomorphism rings. We prove that if  $M$  is a symmetric module, then  $S$  is a symmetric ring. The converse is true for Rickart or 1-epiretractable (in particular, free or regular) or principally projective modules. Among others it is shown that  $M$  is a symmetric module in one of the cases: (1)  $S$  is a strongly regular ring, (2)  $E(M)$  is a symmetric module where  $E(M)$  is the injective hull of  $M$ . Also, we give a characterization

of symmetric rings in terms of symmetric modules, that is, a ring is symmetric if and only if every cyclic projective module is symmetric.

In what follows, by  $\mathbb{Z}$ ,  $\mathbb{Q}$ ,  $\mathbb{Z}_n$  and  $\mathbb{Z}/n\mathbb{Z}$  we denote, respectively, integers, rational numbers, the ring of integers modulo  $n$  and the  $\mathbb{Z}$ -module of integers modulo  $n$ .

## 2. Symmetric modules

Let  $M$  be a simple module. By Schur's Lemma,  $S = \text{End}_R(M)$  is a division ring and clearly for any  $m \in M$  and  $f, g \in S$ ,  $fgm = 0$  implies  $gfm = 0$ . Also every module with a commutative endomorphism ring satisfies this property. A right  $R$ -module  $M$  is called *R-symmetric* ([15] and [18]) if whenever  $a, b \in R$ ,  $m \in M$  satisfy  $mab = 0$ , we have  $mba = 0$ . *R*-symmetric modules are also studied by the last two authors of this paper in [2]. Motivated by this we investigate properties of the class of modules which are symmetric over their endomorphism rings.

**Definition 2.1.** Let  $M$  be an  $R$ -module with  $S = \text{End}_R(M)$ . The module  $M$  is called *S-symmetric* whenever  $fgm = 0$  implies  $gfm = 0$  for any  $m \in M$  and  $f, g \in S$ .

From now on *S-symmetric* modules will be called *symmetric* for the sake of shortness. Note that a submodule of a symmetric module need not be symmetric. Therefore we can give the following definition.

**Definition 2.2.** Let  $M$  be an  $R$ -module with  $S = \text{End}_R(M)$  and  $N$  an  $R$ -submodule of  $M$ . The module  $N$  is called a *symmetric submodule* of  $M$  whenever  $fgn = 0$  implies  $gfn = 0$  for any  $n \in N$  and  $f, g \in S$ .

We mention some examples of modules that are symmetric over their endomorphism rings.

**Examples 2.3.** (1) Let  $M$  be a cyclic torsion  $\mathbb{Z}$ -module. Then  $M$  is isomorphic to the  $\mathbb{Z}$ -module  $(\mathbb{Z}/\mathbb{Z}p_1^{n_1}) \oplus (\mathbb{Z}/\mathbb{Z}p_2^{n_2}) \oplus \dots \oplus (\mathbb{Z}/\mathbb{Z}p_t^{n_t})$  where  $p_i$  ( $i = 1, \dots, t$ ) are distinct prime integers and  $n_i$  ( $i = 1, \dots, t$ ) are positive integers.  $\text{End}_{\mathbb{Z}}(M)$  is isomorphic to the commutative ring  $(\mathbb{Z}_{p_1^{n_1}}) \oplus (\mathbb{Z}_{p_2^{n_2}}) \oplus \dots \oplus (\mathbb{Z}_{p_t^{n_t}})$ . So  $M$  is a symmetric module.

(2) Let  $p$  be any prime integer and  $M = (\mathbb{Z}/p\mathbb{Z}) \oplus \mathbb{Q}$  a  $\mathbb{Z}$ -module. Then  $S$  is isomorphic to the matrix ring  $\left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \mid a \in \mathbb{Z}_p, b \in \mathbb{Q} \right\}$  and so  $M$  is a symmetric module.

There are symmetric modules of which their endomorphism rings are symmetric, namely simple modules and vector spaces. Our next endeavor is to find conditions, under which the property of  $M$  being symmetric is equivalent to  $S$  being symmetric. A module  $M$  is called  $n$ -epiretractable [8] if every  $n$ -generated submodule of  $M$  is a homomorphic image of  $M$ . We show that Rickart modules and 1-epiretractable modules play an important role in this direction.

**Theorem 2.4.** *If  $M$  is a symmetric module, then  $S$  is a symmetric ring. The converse holds if  $M$  satisfies any of the following conditions.*

- (1)  $M$  is a Rickart module.
- (2)  $M$  is a 1-epiretractable module.

*Proof.* Let  $f, g, h \in S$  with  $fgh = 0$ . Since  $M$  is symmetric,  $0 = (fg)hm = (gf)hm$  for all  $m \in M$ . Then  $gfh = 0$ . Hence  $S$  is symmetric. Conversely, let  $M$  be a Rickart module with  $fgm = 0$  for  $f, g \in S$  and  $m \in M$ . Since  $M$  is a Rickart module, there exists  $e^2 = e \in S$  such that  $r_M(fg) = eM$ . Hence  $fge = 0$ . There exists  $m' \in M$  such that  $m = em'$ . By multiplying  $em'$  from the left by  $e$ , we have  $em = eem' = em' = m$ . By using symmetricity of  $S$  repeatedly, it can be easily seen that  $0 = fge = 1f(ge)$  implies  $1(ge)f = gef = 0$  and then  $gfe = 0$ . Hence  $gfm = gfem = 0$ . Thus  $M$  is symmetric. Assume now that  $M$  is 1-epiretractable. Then there exists  $h \in S$  such that  $mR = hM$ . Then we have  $fghM = 0$ , and so  $fgh = 0$ . Since  $S$  is symmetric,  $gfh = 0$ . This implies that  $gfm = 0$ . Therefore  $M$  is symmetric.  $\square$

**Corollary 2.5.** *A free  $R$ -module is symmetric if and only if its endomorphism ring is symmetric.*

*Proof.* Let  $F$  be a free  $R$ -module. Clearly, for any  $m \in F$  there exists  $f \in \text{End}_R(F)$  such that  $fF = mR$ . Thus  $F$  is a 1-epiretractable module. Therefore Theorem 2.4(2) completes the proof.  $\square$

Recall that a ring  $R$  is said to be *regular* if for any  $a \in R$  there exists  $b \in R$  with  $a = aba$ , while a ring  $R$  is called *strongly regular* if for any  $a \in R$  there exists  $b \in R$  such that  $a = a^2b$ . It is well known that a ring is strongly regular if and only if it is reduced and regular (see [14]). Also every reduced ring is symmetric by [5, Theorem I.3]. Then we have the following result.

**Corollary 2.6.** *If  $S$  is a strongly regular ring, then  $M$  is a symmetric module.*

*Proof.* Assume that  $S$  is a strongly regular ring. Then  $S$  is a symmetric and regular ring. By [4, Proposition 2.6],  $M$  is a Rickart module. The rest is clear from Theorem 2.4.  $\square$

A module  $M$  is called *regular* (in the sense of Zelmanowitz [23]) if for any  $m \in M$  there exists a right  $R$ -homomorphism  $M \xrightarrow{\phi} R$  such that  $m = m\phi(m)$ . Then we have the following result.

**Corollary 2.7.** *If  $M$  is a regular module, then the following are equivalent.*

- (1)  $M$  is a symmetric module.
- (2)  $S$  is a symmetric ring.

*Proof.* Every cyclic submodule of a regular module is a direct summand, and so it is 1-epiretractable. It follows from Theorem 2.4.  $\square$

In [7], Evans introduced principally projective modules as follows: An  $R$ -module  $M$  is called *principally projective* if for any  $m \in M$ ,  $r_R(m) = eR$ , where  $e^2 = e \in R$ . The ring  $R$  is called *right principally projective* [10] if the right  $R$ -module  $R$  is principally projective. The concept of left principally projective rings is defined similarly.

In this note, we call the module  $M$  *principally projective* if  $M$  is principally projective as a left  $S$ -module, that is, for any  $m \in M$ ,  $l_S(m) = Se$  for some  $e^2 = e \in S$ .

It is straightforward that all Baer modules are principally projective. However quasi-Baer modules need not be principally projective. Namely, matrix rings over a commutative domain  $R$  are quasi-Baer rings; but if the commutative domain  $R$  is not Prüfer, matrix rings over  $R$  will not be principally projective rings. And every quasi-Baer module is principally quasi-Baer. There are principally projective modules which may not be quasi-Baer or Baer (see [6, Example 8.2]).

**Example 2.8.** Let  $R$  be a Prüfer domain (a commutative ring with an identity, no zero divisors, and all finitely generated ideals are projective) and  $M$  denote the right  $R$ -module  $R \oplus R$ . By ([12], page 17),  $S$  is a  $2 \times 2$  matrix ring over  $R$  and it is a Baer ring. Hence  $M$  is Baer and so a principally projective module.

Note that the endomorphism ring of a principally projective module may not be a right principally projective ring in general. For if  $M$  is a principally projective module and  $g \in S$ , then we distinguish the two

cases:  $\text{Kerg} = 0$  and  $\text{Kerg} \neq 0$ . If  $\text{Kerg} = 0$ , then for any  $f \in r_S(g)$ ,  $gf = 0$  implies  $f = 0$ . Hence  $r_S(g) = 0$ . Assume that  $\text{Kerg} \neq 0$ . There exists a nonzero  $m \in M$  such that  $gm = 0$ . By hypothesis,  $g \in l_S(m) = Se$  for some  $e^2 = e \in S$ . In this case  $g = ge$  and so  $r_S(g) \leq (1 - e)S$ . The following example shows that this inclusion is strict.

**Example 2.9.** Let  $Q$  be the ring and  $N$  the  $Q$ -module constructed by Osofsky in [17]. Since  $Q$  is commutative, we can just as well think of  $N$  as a right  $Q$ -module. If  $S = \text{End}_Q(N)$ , then  $N$  is a principally projective module. Identify  $S$  with the ring  $\begin{bmatrix} Q & 0 \\ Q/I & Q/I \end{bmatrix}$  in the obvious way, and consider  $\varphi = \begin{bmatrix} 0 & 0 \\ 1 + I & 0 \end{bmatrix} \in S$ . Then  $r_S(\varphi) = \begin{bmatrix} I & 0 \\ Q/I & Q/I \end{bmatrix}$ . This is not a direct summand of  $S$  because  $I$  is not a direct summand of  $Q$ . Therefore  $S$  is not a right principally projective ring.

**Theorem 2.10.** *If  $M$  is a principally projective module, then the following are equivalent.*

- (1)  $M$  is a symmetric module.
- (2)  $S$  is a symmetric ring.

*Proof.* (2)  $\Rightarrow$  (1) Let  $S$  be a symmetric ring and assume that  $fgm = 0$  for some  $f, g \in S$  and  $m \in M$ . Since  $M$  is principally projective, there exists  $e^2 = e \in S$  such that  $l_S(gm) = Se$ . Due to  $f \in l_S(gm)$ , we have  $f = fe$  and  $egm = 0$ . Similarly, there exists an idempotent  $e_1 \in S$  such that  $l_S(m) = Se_1$ . Since  $eg \in l_S(m)$ ,  $eg = ege_1$  and  $e_1m = 0$ . By hypothesis,  $Se_1m = 0$  implies  $e_1Sm = 0$  and so  $ege_1Sm = egSm = 0$ . Note that symmetric rings are abelian (indeed, since  $ae(1 - e) = 0 = a(1 - e)e$  for any  $e = e^2, a \in S$ , we have  $ea(1 - e) = 0 = (1 - e)ae$ . This implies that  $ea = ae$ ). Hence  $0 = egfm = gfem = gfm$ . Therefore  $M$  is symmetric.

(1)  $\Rightarrow$  (2) Clear. □

A proof of the following proposition can be given in the same way as the proof of [3, Lemma 2.12].

**Proposition 2.11.** *If  $M$  is a symmetric module and  $m \in M, f_i \in S$  for  $1 \leq i \leq n$ , then  $f_1 \dots f_n m = 0$  if and only if  $f_{\sigma(1)} \dots f_{\sigma(n)} m = 0$ , where  $n \in \mathbb{N}$  and  $\sigma \in S_n$ .*

Lemma 2.12 is a corollary to Lemma 2.18. But we give a proof in detail.

**Lemma 2.12.** *If  $M$  is a symmetric module and  $N$  a direct summand of  $M$ , then  $N$  is a symmetric module.*

*Proof.* Let  $S_1 = \text{End}_R(N)$  and  $M = N \oplus K$  for some submodule  $K$  of  $M$ . Let  $f, g \in S_1$  and  $n \in N$  with  $fgn = 0$ . Define  $f_1(n, k) = (fn, 0)$  and  $g_1(n, k) = (gn, 0)$  where  $f_1, g_1 \in S = \text{End}_R(M)$ ,  $k \in K$ . Then  $f_1g_1(n, 0) = f_1(gn, 0) = (fgn, 0) = (0, 0)$ . Since  $M$  is symmetric and  $f_1, g_1 \in S$ ,  $g_1f_1(n, 0) = (0, 0)$ . But  $(0, 0) = g_1f_1(n, 0) = g_1(fn, 0) = (gfn, 0)$ . Hence  $gfn = 0$ . Therefore  $N$  is symmetric.  $\square$

**Corollary 2.13.** *Let  $R$  be a symmetric ring and  $e \in R$  an idempotent. Then  $eR$  is a symmetric module.*

**Theorem 2.14.** *Let  $R$  be a ring. Then the following conditions are equivalent.*

- (1) *Every free  $R$ -module is symmetric.*
- (2) *Every projective  $R$ -module is symmetric.*

*Proof.* (1)  $\Rightarrow$  (2) Let  $M$  be a projective  $R$ -module. Then  $M$  is a direct summand of a free  $R$ -module  $F$ . By (1),  $F$  is symmetric and so is  $M$  from Lemma 2.12.

(2)  $\Rightarrow$  (1) Clear.  $\square$

**Theorem 2.15.** *A ring  $R$  is symmetric if and only if every cyclic projective  $R$ -module is symmetric.*

*Proof.* The sufficiency is clear. For the necessity, let  $M$  be a cyclic projective  $R$ -module. Then  $M \cong I$  for some direct summand right ideal  $I$  of  $R$ . Since  $R$  is symmetric, by Lemma 2.12,  $I$  is symmetric and so is  $M$ .  $\square$

Any direct sum of symmetric modules need not be symmetric, as the following example shows.

**Example 2.16.** Consider the  $\mathbb{Z}$ -modules  $\mathbb{Z}/2\mathbb{Z}$  and  $\mathbb{Z}/4\mathbb{Z}$ . Clearly, these modules are symmetric. Let  $M$  denote the  $\mathbb{Z}$ -module  $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ . Then the endomorphism ring  $\text{End}_{\mathbb{Z}}(M)$  of  $M$  is  $\begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ \mathbb{Z}_2 & \mathbb{Z}_4 \end{bmatrix}$ . Consider  $f = \begin{bmatrix} \bar{0} & \bar{1} \\ \bar{0} & \bar{1} \end{bmatrix}$ ,  $g = \begin{bmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{0} \end{bmatrix}$  and  $e = \begin{bmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{1} \end{bmatrix}$  of  $\text{End}_{\mathbb{Z}}(M)$ . Then  $fge = 0$  but  $gfe \neq 0$ . Hence  $\text{End}_{\mathbb{Z}}(M)$  is not a symmetric ring. By Theorem 2.4,  $M$  is not a symmetric module.

**Proposition 2.17.** *Let  $M_1$  and  $M_2$  be modules over a ring  $R$ . If  $M_1$  and  $M_2$  are symmetric and  $\text{Hom}_R(M_i, M_j) = 0$  for  $i \neq j$ , then  $M_1 \oplus M_2$  is a symmetric module.*

*Proof.* Let  $M = M_1 \oplus M_2$  and  $S_i = \text{End}_R(M_i)$  for  $i = 1, 2$ . We may describe  $S$  as  $\begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}$ . Let  $f = \begin{bmatrix} f_1 & 0 \\ 0 & f_2 \end{bmatrix}, g = \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix} \in S$  with  $f_1, g_1 \in S_1$  and  $f_2, g_2 \in S_2$  and  $m = (m_1, m_2) \in M$  with  $m_1 \in M_1, m_2 \in M_2$  such that  $fgm = 0$ . Then we have  $f_1g_1m_1 = 0$  and  $f_2g_2m_2 = 0$ . Since  $M_1$  and  $M_2$  are symmetric,  $g_1f_1m_1 = 0$  and  $g_2f_2m_2 = 0$ . This implies that  $gfm = 0$ . Therefore  $M$  is symmetric.  $\square$

**Lemma 2.18.** *Let  $M$  be an  $R$ -module and  $N$  a submodule of  $M$ . If  $M$  is symmetric and every endomorphism of  $N$  can be extended to an endomorphism of  $M$ , then  $N$  is also symmetric.*

*Proof.* Let  $S = \text{End}_R(M)$  and  $f, g \in \text{End}_R(N), n \in N$  with  $fgn = 0$ . By hypothesis, there exist  $\alpha, \beta \in S$  such that  $\alpha|_N = f$  and  $\beta|_N = g$ . Then  $\alpha|_N\beta|_Nn = 0$ , and so  $\alpha\beta n = 0$ . Since  $M$  is symmetric, we have  $\beta\alpha n = 0$ . This and  $\alpha n \in N$  imply that  $0 = \beta|_N\alpha|_Nn = gfn$ . Therefore  $N$  is a symmetric module.  $\square$

It is well known that every endomorphism of any module  $M$  can be extended to an endomorphism of the injective hull  $E(M)$  of  $M$ . By considering this fact, we can say the next result.

**Theorem 2.19.** *Let  $M$  be a module. If  $E(M)$  is symmetric, then so is  $M$ .*

*Proof.* Clear from Lemma 2.18.  $\square$

Recall that a module  $M$  is *quasi-injective* if it is  $M$ -injective. Then we have the following.

**Theorem 2.20.** *Let  $M$  be a quasi-injective module. If  $M$  is symmetric, then so is every submodule of  $M$ .*

*Proof.* Let  $N$  be a submodule of  $M$  and  $f \in \text{End}_R(N)$ . By quasi-injectivity of  $M$ ,  $f$  extends to an endomorphism of  $M$ . Lemma 2.18 completes the proof.  $\square$

Let  $M$  be an  $R$ -module with  $S = \text{End}_R(M)$ . Consider

$$T({}_S M) = \{m \in M \mid fm = 0 \text{ for some nonzero } f \in S\}.$$

The subset  $T({}_S M)$  of  $M$  need not be a submodule of the modules  ${}_S M$  and  $M_R$  in general, as the following example shows.

**Example 2.21.** Let  $e_{ij}$  denote  $3 \times 3$  matrix units and consider the ring  $R = \{(e_{11} + e_{22} + e_{33})a + e_{12}b + e_{13}c + e_{23}d : a, b, c, d \in \mathbb{Z}_2\}$  and the  $R$ -module  $M = \{e_{12}a + e_{13}b + e_{23}c : a, b, c \in \mathbb{Z}_2\}$ . Let  $f, g \in S$  defined by  $f(e_{12}a + e_{13}b + e_{23}c) = e_{12}a + e_{13}b$  and  $g(e_{12}a + e_{13}b + e_{23}c) = (e_{13} + e_{23})c$ . For  $m = e_{23}1, m' = e_{12}1 \in M, fm = 0$  and  $gm' = 0$ . But no nonzero elements of  $S$  annihilate  $m + m'$  since  $(m + m')R = M$ . Therefore  $T({}_S M)$  is not a submodule of the modules  ${}_S M$  and  $M_R$ .

In the symmetric case we have the following.

**Proposition 2.22.** *If  $M$  is a symmetric module and  $S$  is a domain, then  $T({}_S M)$  is a left  $S$ -submodule of  $M$ .*

*Proof.* Let  $m_1, m_2 \in T({}_S M)$ . There exist nonzero  $f_1, f_2 \in S$  with  $f_1 m_1 = 0$  and  $f_2 m_2 = 0$ . Then  $f_1 f_2 m_2 = 0$ . By hypothesis,  $0 = f_2 f_1 m_1 = f_1 f_2 m_1$ . Since  $S$  is a domain,  $f_1 f_2 \neq 0$  and so  $f_1 f_2 (m_1 - m_2) = 0$  or  $m_1 - m_2 \in T({}_S M)$ . If  $g \in S$ , then  $g f_1 m_1 = 0$ . Since  $M$  is symmetric,  $g f_1 m_1 = 0$  implies  $f_1 g m_1 = 0$ . Hence  $g m_1 \in T({}_S M)$  and so  $T({}_S M)$  is a left  $S$ -submodule of  $M$ . □

**Theorem 2.23.** *Let  $M$  be an  $R$ -module with  $S$  a domain. Then  $M$  is a symmetric module if and only if  $T({}_S M)$  is a symmetric submodule of  $M$ .*

*Proof.* Assume that  $M$  is a symmetric module and  $m \in T({}_S M)$ . There exists a nonzero  $f \in S$  with  $f m = 0$ . For any  $r \in R, f(mr) = (fm)r = 0$ . So  $mr \in T({}_S M)$ . Therefore  $T({}_S M)$  is an  $R$ -submodule of  $M$ . Let  $f, g \in S$  and  $m \in T({}_S M)$  with  $f g m = 0$ . Since  $M$  is symmetric,  $g f m = 0$  and so  $T({}_S M)$  is a symmetric submodule of  $M$ .

Conversely, let  $m \in M$  and  $f, g$  be nonzero elements of  $S$  with  $f g m = 0$ . If  $m \in T({}_S M)$ , by the symmetry condition on  $T({}_S M)$ , we have  $g f m = 0$ . If  $m \notin T({}_S M)$ , then  $f g = 0$ . Since  $S$  is a domain, we have a contradiction. Therefore  $M$  is a symmetric module. □

Let  $M$  be an  $R$ -module with  $S = \text{End}_R(M)$  and  $N$  a submodule of  $M$ . The quotient module  $M/N$  is called  *$S$ -symmetric* if  $f g m \in N$  implies  $g f m \in N$  for any  $m \in M$  and  $f, g \in S$ .

**Theorem 2.24.** *Let  $M$  be an  $R$ -module with  $S$  a domain. If  $M$  is symmetric, then the quotient module  $M/T({}_S M)$  is  $S$ -symmetric.*

*Proof.* Let  $m \in M$  and  $f, g \in S$  with  $f g m \in T({}_S M)$ . So there exists nonzero  $h \in S$  such that  $h f g m = 0$ . By Proposition 2.11, we have  $h f g m = h g f m = 0$ . Then  $g f m \in T({}_S M)$ . Hence  $M/T({}_S M)$  is  $S$ -symmetric. □

Recall that a module  $M$  is called *quasi-projective* if it is  $M$ -projective.

**Theorem 2.25.** *Let  $M$  be a module and  $N$  a submodule of  $M$ .*

- (1) *If  $M$  is a quasi-projective module and  $M/N$  is  $S$ -symmetric, then  $M/N$  is symmetric as a left  $\text{End}_R(M/N)$ -module.*
- (2) *If  $N$  is a fully invariant submodule of  $M$  and  $M/N$  is symmetric as a left  $\text{End}_R(M/N)$ -module, then  $M/N$  is  $S$ -symmetric.*

*Proof.* (1) Let  $f_1, g_1 \in \text{End}_R(M/N)$  and  $m \in M$  with  $f_1 g_1(m+N) = 0+N$  and  $\pi$  denote the natural projection from  $M$  to  $M/N$ . Since  $M$  is quasi-projective, there exist  $f, g \in S$  such that  $f_1 \pi = \pi f$  and  $g_1 \pi = \pi g$ . Then we have  $0 + N = f_1 g_1(m + N) = f_1 g_1 \pi m = f_1 \pi g m = \pi f g m$ , and so  $f g m \in N$ . Hence  $g f m \in N$  by hypothesis. This implies that  $\pi g f m = g_1 \pi f m = g_1 f_1 \pi m = g_1 f_1(m + N) = 0 + N$ . Therefore  $M/N$  is symmetric as a left  $\text{End}_R(M/N)$ -module.

(2) Let  $f, g \in S$  and  $m \in M$  with  $f g m \in N$  and  $\pi$  denote the natural projection from  $M$  to  $M/N$ . Since  $N$  is fully invariant, there exist  $\bar{f}, \bar{g} \in \text{End}_R(M/N)$  such that  $\bar{f} \pi = \pi f$  and  $\bar{g} \pi = \pi g$ . It follows that  $\bar{f} \bar{g}(m + N) = \bar{0}$ , and so  $\bar{g} \bar{f}(m + N) = \bar{0}$ . Therefore  $g f m \in N$ .  $\square$

Proposition 2.26 follows from [1, Theorem 2.14] and [4, Theorem 2.25].

**Proposition 2.26.** *If  $M$  is a principally projective module, then the following conditions are equivalent.*

- (1)  *$M$  is a rigid module.*
- (2)  *$M$  is a reduced module.*
- (3)  *$M$  is a symmetric module.*
- (4)  *$M$  is a semicommutative module.*
- (5)  *$M$  is an abelian module.*

**Remark 2.27.** It follows from Theorem 2.14 of [1], every reduced module is semicommutative, and every semicommutative module is abelian. The converses hold for principally projective modules. Note that for a prime integer  $p$  the cyclic group  $M$  of  $p^2$  elements is a  $\mathbb{Z}$ -module for which  $S = \mathbb{Z}_{p^2}$ . The module  $M$  is neither reduced nor principally projective although it is semicommutative.

Every symmetric module has a symmetric endomorphism ring. However, despite all our efforts we have not succeeded in answering positively the following question for an arbitrary module.

**Question.** Is any module symmetric if its endomorphism ring is symmetric?

The answer is positive for simple modules, vector spaces and the modules which satisfy the conditions in Theorem 2.4. But if the answer is negative for an arbitrary module, then what is the counterexample?

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## References

- [1] N. Agayev, G. Güngöroğlu, A. Harmanci and S. Halicioğlu, *Abelian modules*, Acta Math. Univ. Comenianae 78(2)(2009), 235-244.
- [2] N. Agayev, S. Halicioğlu and A. Harmanci, *On symmetric modules*, Riv. Mat. Univ. Parma 8(2)(2009), 91-99.
- [3] N. Agayev, S. Halicioğlu and A. Harmanci, *On reduced modules*, Commun. Fac. Sci. Univ. Ank. Series A1 58(1)(2009), 9-16.
- [4] N. Agayev, S. Halicioğlu and A. Harmanci, *On Rickart modules*, Bull. Iran. Math. Soc. 38(2)(2012), 433-445.
- [5] D. D. Anderson and V. Camillo, *Semigroups and rings whose zero products commute*, Comm. Algebra 27(6)(1999), 2847-2852.
- [6] A. W. Chatters and C. R. Hajarnavis, *Rings with chain conditions*, Pitman, Boston, 1980.
- [7] M. W. Evans, *On commutative p.p. rings*, Pacific J. Math. 41(1972), 687-697.
- [8] A. Ghorbani and M. R. Vedadi, *Epi-retractable modules and some applications*, Bull. Iran. Math. Soc. 35(1)(2009), 155-166.
- [9] K. R. Goodearl and A. K. Boyle, *Dimension theory for nonsingular injective modules*, Memoirs Amer. Math. Soc. 7(177), 1976.
- [10] A. Hattori, *A foundation of the torsion theory over general rings*, Nagoya Math. J. 17(1960), 147-158.
- [11] C. Y. Hong, N. K. Kim and T. K. Kwak, *Ore extensions of Baer and p.p.-rings*, J. Pure Appl. Algebra 151(3)(2000), 215-226.
- [12] I. Kaplansky, *Rings of operators*, Math. Lecture Note Series, Benjamin, New York, 1965.
- [13] J. Krempa, *Some examples of reduced rings*, Algebra Colloq. 3(4)(1996), 289-300.
- [14] T. Y. Lam, *Exercises in classical ring theory*, Springer-Verlag, New York, 1995.
- [15] J. Lambek, *On the representation of modules by sheaves of factor modules*, Canad. Math. Bull. 14 (3)(1971), 359-368.

- [16] T. K. Lee and Y. Zhou, *Reduced modules*, Rings, modules, algebras and abelian groups, 365-377, Lecture Notes in Pure and Appl. Math., 236, Dekker, New York, (2004).
- [17] B. L. Osofsky, *A counter-example to a lemma of Skornjakov*, Pacific J. Math. 15(1965), 985-987.
- [18] R. Raphael, *Some remarks on regular and strongly regular rings*, Canad. Math. Bull. 17(5)(1974/75), 709-712.
- [19] S. T. Rizvi and C. S. Roman, *Baer and quasi-Baer modules*, Comm. Algebra 32(2004), 103-123.
- [20] S. T. Rizvi and C. S. Roman, *On direct sums of Baer modules*, J. Algebra 321(2009), 682-696.
- [21] J. E. Roos, *Sur les categories spectrales localement distributives*, C. R. Acad. Sci. Paris 265(1967), 14-17.
- [22] B. Ungor, N. Agayev, S. Halicioglu and A. Harmanci, *On principally quasi-Baer modules*, Albanian J. Math. 5(3)(2011), 165-173.
- [23] J. M. Zelmanowitz, *Regular modules*, Trans. Amer. Math. Soc. 163(1972), 341-355.

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