Algebra and Discrete Mathematics RESEARCH ARTICLE
Volume 13 **(2012)**. Number 2. pp. 289 – 298

On factorizations of limited solubly ω-saturated formations

Vadim M. Selkin

Communicated by L. A. Shemetkov

Abstract. If $\mathfrak{F} = \mathfrak{F}_1 \ldots \mathfrak{F}_t$ is the product of the formations $\mathfrak{F}_1, \ldots, \mathfrak{F}_t$ and $\mathfrak{F} \neq \mathfrak{F}_1 \ldots \mathfrak{F}_{i-1}\mathfrak{F}_{i+1} \ldots \mathfrak{F}_t$ for all $i = 1, \ldots, t$, then we call this product a non-cancellative factorization of the formation $\mathfrak{F}$. In this paper we gives a description of factorizable limited solubly ω-saturated formations.

Introduction

All groups considered are finite.

We will use $C^p(G)$ to denote the intersection of all centralizers of Abelian p-chief factors of the group G [1] (we note that $C^p(G) = G$ if G has no such chief factors). Let $\mathfrak{X}$ be a set of groups. Then we use $\mathrm{Com}(\mathfrak{X})$ to denote the class of all Abelian groups A such that $A \simeq H/K$ for some composition factor H/K of some group $G \in \mathfrak{X}$. Also, we write $\mathrm{Com}(G)$ for the set $\mathrm{Com}(\{G\})$.

Let $\varnothing \neq \omega \subseteq \mathbb{P}$. For every function f of the form

$$f : \omega \cup \{\omega'\} \to \{\text{group formations}\} \qquad (1)$$

we put

$$CF_\omega(f) = \{G \text{ is a group} \mid G/R(G) \cap O_\omega(G) \in f(\omega')$$

2010 MSC: 20D10.

Key words and phrases: factorizations, solubly ω-saturated formation, composition ω-satelitte, one-generated formation.

and

$$G/C^p(G) \in f(p) \text{ for any prime } p \in \omega \cap \pi(\mathrm{Com}(G)\}.$$

Here $R(G)$ denotes the radical of G (i.e. $R(G)$ is the largest normal soluble subgroup of G).

We call $\mathfrak{F}$ a *solubly ω-saturated formation* [2] if $\mathfrak{F} = CF_\omega(f)$ for some function f of the form (1). In this case, we call f a *composition ω-satelitte* of the formation $\mathfrak{F}$.

If

$$\mathfrak{F} = \mathfrak{F}_1 \dots \mathfrak{F}_t \tag{2}$$

is the product of the formations $\mathfrak{F}_1, \dots, \mathfrak{F}_t$ and $\mathfrak{F} \neq \mathfrak{F}_1 \dots \mathfrak{F}_{i-1}\mathfrak{F}_{i+1} \dots \mathfrak{F}_t$ for all $i = 1, \dots, t$, then we call (2) a *non-cancellative factorization* of the formation $\mathfrak{F}$.

A formation $\mathfrak{F}$ is called a one-generated formation if there is a group G such that $\mathfrak{F}$ is the intersection of all formations containing G. A formation $\mathfrak{F}$ is called a one-generated solubly ω-saturated formation if there is a group G such that $\mathfrak{F}$ is the intersection of all solubly ω-saturated formations containing G.

A formation $\mathfrak{F}$ is called limited if $\mathfrak{F}$ is a subformation of some one-genereted formation. Analogously, a solubly ω-saturated formation $\mathfrak{F}$ is called a limited solubly ω-saturated formation if it is a subformation of some one-generation solubly ω-saturated formation. Let $\mathfrak{H}$ be a class of groups.

We use $\mathfrak{H}(\omega')$ to denote the class $\mathrm{form}(A/(R(A) \cap O_\omega(R)) \mid A \in \mathfrak{H})$. and use $\mathfrak{H}(p)$ to denote the class $\mathrm{form}(A/O_p(A) \mid A \in \mathfrak{H})$.

In this paper we prove the following theorem which gives the answer to Problem 21 in [2].

Theorem 1. *The product*

$$\mathfrak{F}_1\mathfrak{F}_2 \dots \mathfrak{F}_t \tag{$*$}$$

is a non-cancellative factorization of some limited solubly ω-saturated formation $\mathfrak{F}$ if and only if the following conditions hold:

(1) *$t \leq 3$ and every factor in (*) is a nonidentity formation;*

(2) *$\mathfrak{F}_1$ is a one-generated ω-saturated subformation in $\mathfrak{N}_\omega\mathfrak{N}$ and $\pi(\mathrm{Com}(\mathfrak{F})) \cap \omega \subseteq \pi(\mathfrak{F}_1)$;*

(3) *If $\mathfrak{F}_1 \not\subseteq \mathfrak{N}_\omega$, then $t = 2$, $\mathfrak{F}_2$ is an Abelian one-generated formation and for all groups $A \in \mathfrak{F}_1$ and $B \in \mathfrak{F}_2$, $(|A/F_\omega(A)|, |B|) = 1$ and $(|A/O_\omega(A)|, |B|) = 1$;*

(4) *If* $\mathfrak{F}_1 \subseteq \mathfrak{N}_\omega$ *and* $t = 3$, *then* $|\pi(\mathfrak{F}_1| > 1$, $\mathfrak{F}_3$ *is a one-generated Abelian formation and for every* $p \in \pi(\mathfrak{F}_1)$, *the formation* $\mathfrak{F}_2(p)$ *is one-generated nilpotent and for all groups* $A \in \mathfrak{F}_2$ *and* $B \in \mathfrak{F}_3$, $\pi(A/O_p(A)) \cap \pi(B) = \varnothing$;

(5) *If* $\mathfrak{F}_1 \subseteq \mathfrak{N}_\omega$, $t = 2$ *and* $|\pi(\mathfrak{F}_1)| > 1$, *then* $\mathfrak{F}_2(\omega')$, $\mathfrak{F}_2$ *are limited formations;*

(6) *If* $\mathfrak{F}_1 = \mathfrak{N}_p$ *for some prime* p, *then* $\mathfrak{F}_2(\omega')$ *and* $\mathfrak{F}_2(p)$ *(if* $p \in \omega$*) are limited formations,* $\mathfrak{F}_2 \not\subseteq \mathfrak{F}_1$, *and there is a group* $B \in \mathfrak{F}_2$ *such that for all groups* $A \in \mathfrak{F}_1$, *the* $\mathfrak{F}_2$*-residual* $T^{\mathfrak{F}_2}$ *of the wreath product* $T = A \wr B$ *is contained subdirectly in the base group of* T.

All unexplained natation and terminology are standard. The reader is referred to [3], [4], [5] if necessary.

1. Proof of Theorem

Proof. Suppose that $\mathfrak{F} = \mathfrak{F}_1\mathfrak{F}_2 \ldots \mathfrak{F}_t$ and the conditions (1)–(6) hold. Our first step is to show that $\mathfrak{F}$ is a limited solubly ω-saturated formation. In view of Lemma 5 [2] and Theorem 1 [6], we only need to show that $f(a)$ is a limited formation for all $a \in \pi(Com(\mathfrak{F})) \cap \omega \cup \{\omega'\}$, where f is the smallest composition ω-satelitte of the formation $\mathfrak{F}$, and $|\pi(Com(\mathfrak{F})) \cap \omega| < \infty$.

By (2), we see that $|\pi(Com(\mathfrak{F})) \cap \omega| < \infty$. Let $\mathfrak{H} = \mathfrak{F}_2 \ldots \mathfrak{F}_t$, and m be the smallest composition ω-satelitte of the formation $\mathfrak{F}_1$. Then by Lemma 4.5 [7], we have $\mathfrak{F} = CF_\omega(t)$, where

$$t(a) = \begin{cases} m(p)\mathfrak{H}, & \text{if } a = p \in \pi(\mathrm{Com}(\mathfrak{F}_1)) \cap \omega \\ \varnothing, & \text{if } a = p \in \omega \setminus \pi(\mathrm{Com}(\mathfrak{F}_1)) \\ m(\omega')\mathfrak{H}, & \text{if } a = \omega'. \end{cases}$$

Let $\mathfrak{F}_1 \not\subseteq \mathfrak{N}_\omega$. Then by hypothesis, $\mathfrak{H} = \mathfrak{F}_2$ is a one-generated Abelian formation. Let $p \in \pi(\mathrm{Com}(\mathfrak{F}_1)) \cap \omega$. Since $\mathfrak{F}_1$ is a one-generated solubly ω-saturated formation and $\mathfrak{F}_1 \subseteq \mathfrak{N}_\omega\mathfrak{N}$, then $m(p)$ is a nilpotent one-generated formation. Since for any groups $A \in \mathfrak{F}_1$ and $B \in \mathfrak{F}_2$ we have $(|A/F_p(A)|, |B|) = 1$, then by Lemma 5 [2], $m(p) \cap \mathfrak{H} = (1)$. Then $m(p)\mathfrak{H}$ is soluble formations and it is is a one-generated formation. But $f(p) \subseteq t(p) = m(p)\mathfrak{H}$. Hence $f(p)$ is a limited formation. Analogously, we can show that $f(\omega')$ is a limited formation. Hence, $\mathfrak{F}$ is indeed a limited solubly ω-saturated formation.

Let $\mathfrak{F}_1 \subseteq \mathfrak{N}_\omega$. In this case, by Theorem 2 [6], we only need to show that if $t = 3$, then $\mathfrak{H}(\omega')$ and $\mathfrak{H}(p)$ (for all $p \in \pi(\mathfrak{F}_1)$) are limited formations and $\mathfrak{H}$ is a one-generated formation if $|\pi(\mathfrak{F})| > 1$.

Let $p \in \pi(\mathfrak{F}_1) \cap \omega$. Consider the formation $\mathfrak{F}_2(p)\mathfrak{F}_3$. In view of Condition (4) we have $\pi(\mathfrak{F}_2(p)) \cap \pi(\mathfrak{F}_3) = \varnothing$. Besides $\mathfrak{F}_2(p)$ is a nilpotent one-generated formation. In these cases, the product $\mathfrak{F}_2(p)\mathfrak{F}_3$ is a one-generated formation. But evidently $\mathfrak{F}_2(p)\mathfrak{F}_3$ is a soluble formation, and every subformation of $\mathfrak{F}_2(p)\mathfrak{F}_3$ is also one-generated. Thus, in order to prove that $(\mathfrak{F}_2\mathfrak{F}_3)(p)$ is a limited formation, we only need to show that it is a subformation in $\mathfrak{F}_2(p)\mathfrak{F}_3$. Let $A \in \mathfrak{F}_2\mathfrak{F}_3$. Then $A^{\mathfrak{F}_3} \in \mathfrak{F}_2$. Hence $O_p(A^{\mathfrak{F}_3})$ is a characteristic subgroup of $A^{\mathfrak{F}_3}$ such that

$$A^{\mathfrak{F}_3}/O_p(A^{\mathfrak{F}_3}) \in \mathfrak{F}_2(p).$$

But $A^{\mathfrak{F}_3}/O_p(A^{\mathfrak{F}_3}) = (A/O_p(A^{\mathfrak{F}_3}))^{\mathfrak{F}_3}$, we have $A/O_p(A^{\mathfrak{F}_3}) \in \mathfrak{F}_2(p)\mathfrak{F}_3$, and so $A/O_p(A) \in \mathfrak{F}_2(p)\mathfrak{F}_3$. Thus $(\mathfrak{F}_2\mathfrak{F}_3)(p) \subseteq \mathfrak{F}_2(p)\mathfrak{F}_3$. This shows that the formation $(\mathfrak{F}_2\mathfrak{F}_3)(p)$ is limited.

Assume that $|\pi(\mathfrak{F})| > 1$ and let $p, q \in \pi(\mathfrak{F}_1)$. Let $A \in \mathfrak{F}_2$ and $B \in \mathfrak{F}_3$. Since $|A/O_p(A)|, |B|) = 1$ and $(|A/O_q(A)|, |B|) = 1$, we have $(|A|, |B|) = 1$. This shows that the exponents of the formations $\mathfrak{F}_2$ and $\mathfrak{F}_3$ are coprime. Same as above, one can show that $\mathfrak{F}_2$ is a nilpotent formation. Hence by Theorem 1 [8], $\mathfrak{F}_2\mathfrak{F}_3$ is a limited formation.

Consider the formation $\mathfrak{F}_2(\omega')\mathfrak{F}_3$. Clearly, $\mathfrak{F}_2(\omega')$ is a one-generated nilpotent formation and $\pi(\mathfrak{F}_2(\omega')) \cap \pi(\mathfrak{F}_3) = \varnothing$. Hence $\mathfrak{F}_2(\omega')\mathfrak{F}_3$ is a soluble one-generated formation. Now, in order to prove that $(\mathfrak{F}_2\mathfrak{F}_3)(\omega')$ is a limited formation, we only need to show that $(\mathfrak{F}_2\mathfrak{F}_3)(\omega') \subseteq \mathfrak{F}_2(\omega')\mathfrak{F}_3$. Let $A \in \mathfrak{F}_2\mathfrak{F}_3$. Since $O_\omega(A^{\mathfrak{F}_3})$ is a characteristic subgroup of $A^{\mathfrak{F}_3}$ such that $A^{\mathfrak{F}_3}/(O_\omega(A^{\mathfrak{F}_3}) \in \mathfrak{F}_2(\omega')$, and so $A/O_\omega(A) \in \mathfrak{F}_2(\omega')\mathfrak{F}_3$. Hence $(\mathfrak{F}_2\mathfrak{F}_3)(\omega') \subseteq \mathfrak{F}_2(\omega')\mathfrak{F}_3$.

We still need to show that the factorization (*) is non-cancellative. For this purpose, we first take $t = 2$. Assume that $\mathfrak{F}_1 \not\subseteq \mathfrak{N}_\omega$. Then by Conditions (3), $\mathfrak{F}_2$ is an Abelian formation, and so by Lemma 5.1 [7], $\mathfrak{F} \neq \mathfrak{F}_2$. Suppose that $\mathfrak{F} = \mathfrak{F}_1$. And let A be a group with minimal order in $\mathfrak{F}_1 \setminus \mathfrak{N}_\omega$. Let R be the monolith of A. Then $R = A^{\mathfrak{N}_\omega}$. Clearly, we have $R \not\subseteq \Phi(A)$. Let B be a simple group in $\mathfrak{F}_2$ and $T = A \wr B = [K]B$ where K is the base group of T. Since $\mathfrak{F}_2$ is Abelian, we have $B = C_p$. Assume that the $\mathfrak{F}_2$-residual $T^{\mathfrak{F}_2}$ of the wreath product $T = A \wr B$ is not contained subdirectly in the base group of T. Let $\pi(T^{\mathfrak{F}_2})$ is a projection of $T^{\mathfrak{F}_2}$ in A_1, where A_1 is the first copy of A in K. Then $N/\pi(T^{\mathfrak{F}_2}) \wr B$ is a homomorphic image of the group $T/T^{\mathfrak{F}_2}$. By (3), $|N/O_\omega(N), |A|) = 1$. This contradiction shows that the $\mathfrak{F}_2$-residual $T^{\mathfrak{F}_2}$ of the wreath product $T = A \wr B$ is contained subdirectly in the base group of T. Hence $T^{\mathfrak{F}_2} \simeq A \in \mathfrak{F}_1$. So

$$T \in \mathfrak{F} = \mathfrak{F}_1 \subseteq \mathfrak{N}_\omega \mathfrak{N}.$$

It is clear that $R = F(A)$ and by Lemma 3.1.9 [5], we deduce that

$$L = R^\natural = \prod_{b \in B} R_1^b = F(T)$$

is the monolith of T, where R_1 is the monolith of the first copy A in K. Since $T \in \mathfrak{N}_\omega \mathfrak{N}$, we have $T^{\mathfrak{N}} \in \mathfrak{N}_\omega$, i.e. $T^{\mathfrak{N}} \subseteq L$. Let R be an ω'-group. Hence L is an ω'-group, and so $O_\omega(T) = 1$. Since $T \in \mathfrak{N}_\omega \mathfrak{N}$, T must be a nilpotent group. But $F(T) = L \neq T$, which is a contradiction. Hence R is a p-group, for some $p \in \omega$. However, since $A \notin \mathfrak{N}_\omega$, we have $R = F_\omega(A)$ and consequently, $(|A/R|, |B|) = 1$. Let B be a q-group. Then B is a Sylow q-subgroup of $T_1 = (A/R) \wr B$. By [[1], A, (18.2)], we have $T_1 \simeq T/L$. This proves that T_1 is a nipotent group. Thus, $B \trianglelefteq T$, and so $B \cap K_1 \neq 1$, where K_1 is the base group of T_1, a contradiction. This shows that $\mathfrak{F}_1 \neq \mathfrak{F} \neq \mathfrak{F}_2$.

Now we assume that $\mathfrak{F}_1 \subseteq \mathfrak{N}_\omega$. Let $|\pi(\mathfrak{F}_1)| > 1$ and p, q be different primes in $\pi(\mathfrak{F}_1)$. Also we let B be a group such that $\mathfrak{F}_2 \subseteq \mathrm{form} B$. Since $\mathfrak{F}_1$ is an ω-local formation, we have $\mathfrak{N}_{\{p,q\}} \subseteq \mathfrak{F}_1$. Hence $\mathfrak{F} \neq \mathfrak{F}_2$ by Lemma 3.1.5 [5]. In view of Lemma 5.1 [7], we conclude that $\mathfrak{F} \neq \mathfrak{F}_1$.

Let $\pi(\mathfrak{F}_1) = \{p\}$ for some $p \in \omega$. Then $\mathfrak{F}_1 = \mathfrak{N}_p$. Let B be a group in $\mathfrak{F}_2$ such that for every group $A \in \mathfrak{F}_1$ the $\mathfrak{F}_2$-residual $T^{\mathfrak{F}_2}$ of the wreath product $T = A \wr B$ is contained subdirectly in the base group of T. Assume that $\mathfrak{F} = \mathfrak{F}_2 = \mathfrak{N}_p \mathfrak{F}_2$ and let A be a non-identity group in $\mathfrak{N}_p$. If $T = A \wr B$, then $T \in \mathfrak{F} = \mathfrak{F}_2$, and so $T^{\mathfrak{F}_2} = 1$ is not contained subdirectly in the base group of T. This contradiction shows that $\mathfrak{F} \neq \mathfrak{F}_2$. And since, by Condition (6), $\mathfrak{F}_2 \nsubseteq \mathfrak{F}_1$, then we have $\mathfrak{F} \neq \mathfrak{F}_1$. This shows that the factorization (*) is indeed non-cancellative.

Now let $t = 3$. Consider the case $\pi(\mathfrak{F}_1) = \{p\}$. By (6) $\mathfrak{F}_2 \nsubseteq \mathfrak{N}_p$. Let A be a group of minimal order in $\mathfrak{F}_2 \setminus \mathfrak{N}_p$. Then $O_p(A) = 1$. Thus, if $B \in \mathfrak{F}_3$, we have $(|A|, |B|) = 1$. Let $T = Z_p \wr (A \wr B)$. Evidently, $T \in \mathfrak{F}$ and T is not a metanilpotent group. Hence $\mathfrak{F} \nsubseteq \mathfrak{N}^2$. Since the formations $\mathfrak{F}_1\mathfrak{F}_2$, $\mathfrak{F}_1\mathfrak{F}_3$ are all metanilpotent, $\mathfrak{F} \neq \mathfrak{F}_1\mathfrak{F}_2, \mathfrak{F}_1\mathfrak{F}_3$. By (6) we can let B be a group in $\mathfrak{F}_2$ such that for all non-identity groups $A \in \mathfrak{F}_1$ the $\mathfrak{F}_2$-residual $D^{\mathfrak{F}_2}$ of the group $D = A \wr B$ is contained subdirectly in the base group of the wreath product D. Let C be a non-identity group in $\mathfrak{F}_3$. Assume that $\mathfrak{F}_2\mathfrak{F}_3 = \mathfrak{F}_1\mathfrak{F}_2\mathfrak{F}_3$ and $T = D \wr C = [K]C$, where K is the base group of D. Then since $\mathfrak{F}_3$ is Abelian formation, we see that $T^{\mathfrak{F}_3}$ is contained subdirectly in K, and so $D \in \mathfrak{F}_2$, that is, $D^{\mathfrak{F}_2} = 1$, a contradiction. Hence $\mathfrak{F} \neq \mathfrak{F}_2\mathfrak{F}_3$. Now assume $|\pi(\mathfrak{F}_1)| > 1$. Let $\{p, q\} \subseteq \pi(\mathfrak{F}_1)$. By (4), $\mathfrak{F}_2(p)$ and $\mathfrak{F}_2(q)$ are a one-generated nilpotent formations. Hence, $\mathfrak{F}_2$ is also a one-generated nilpotent formation by Lemma 4.6 [7]. Therefore $\mathfrak{F} \neq \mathfrak{F}_2\mathfrak{F}_3$ by Lemma 5.1 [7]. For the case $|\pi(\mathfrak{F}_1)| > 1$, the proof is similar.

Assume that

$$\mathfrak{F} = \mathfrak{F}_1\mathfrak{F}_2 \dots \mathfrak{F}_t \subseteq c_\omega\mathrm{form}(G) = \mathfrak{F}^*$$

for some group G. Let f be the smallest composition ω-satelitte of the formation $\mathfrak{F}$, f^* be the smallest composition ω-satelitte of the formation $\mathfrak{F}^*$. We will show that the factors of the non-cancellative factorization (*) all satisfy Conditions (1)–(6).

It is clear that every factor in (*) is a non-identity formation. In additions, by Lemma 5.3 [7], we have $t \leq 3$. Thus, (1) is true.

By Theorem 1 [9] we have $\mathfrak{F}_1 \subseteq \mathfrak{N}_\omega\mathfrak{N}$, hence $\mathfrak{F}_1$ is a hereditary formation. Then $\mathfrak{F}_1 \subseteq \mathfrak{F} \subseteq \mathfrak{F}^*$. By Lemma 3.5 [7] and Proposition A [7] we see that $\mathfrak{F}_1$ is an ω-saturated soluble formation. But by Lemma 4 [10] the set of all ω-saturated subformations of the formation $\mathfrak{F}_1$ is finite. Hence, there is a chain

$$(1) = \mathfrak{M}_0 \subseteq \mathfrak{M}_1 \subseteq \dots \subseteq \mathfrak{M}_{n-1} \subseteq \mathfrak{M}_n = \mathfrak{F}_1,$$

where $\mathfrak{M}_i$ is a maximal ω-saturated subformation of $\mathfrak{M}_{i+1}$, $i = 0, 1, \dots, n-1$. Let $A_i \in \mathfrak{M}_i \backslash \mathfrak{M}_{i-1}$, $i = 1, 2, \dots, n$. Assume that $l_\omega\mathrm{form}(\mathfrak{M}_{i-1} \cup \{A_i\}) \neq \mathfrak{M}_i$. Then

$$\mathfrak{M}_{i-1} \subseteq l_\omega\mathrm{form}(\mathfrak{M}_{i-1} \cup \{A_i\}) \subseteq \mathfrak{M}_i.$$

But $\mathfrak{M}_{i-1}$ is a maximal ω-saturated subformation of $\mathfrak{M}_i$. This contradiction shows that $l_\omega\mathrm{form}(\mathfrak{M}_{i-1} \cup \{A_i\}) = \mathfrak{M}_i$, and so

$$\mathfrak{F}_1 = l_\omega\mathrm{form}(A_1, \dots, A_n) = l_\omega\mathrm{form}(A_1 \times \dots \times A_n)$$

is a one-generated ω-saturated formation.

Assume that p is prime such that $p \in \pi(\mathrm{Com}(\mathfrak{F})) \cap \omega$ and $p \notin \pi(\mathfrak{F}_1)$. Then $\mathfrak{N}_p \subseteq \mathfrak{F}$ and $Z_p \notin \mathfrak{F}_1$. Let Z_q be a simple group in $\mathfrak{F}_1$ such that $q \in \pi(\mathrm{Com}(\mathfrak{F}_1)) \cap \omega$. Then $q \neq p$. Let B be a cycle group of order p^m, where $m = |G|$. And let $D = Z_q \wr B = [K]B$, where K is the base of the wraeth product D. In is clear that $D \in \mathfrak{F}$, and hence

$$D/C^q(D) = D/K \simeq B \in f(q).$$

By Lemma 5 [2] we see

$$f(q) \subseteq f^*(q) = \mathrm{form}(G/C^q(G)),$$

which is a contradition to Lemma 3.1.5 [5]. Hence $\pi(\mathrm{Com}(\mathfrak{F})) \cap \omega \subseteq \pi(\mathfrak{F}_1)$. This shows that (2) holds.

Assume that $\mathfrak{F}_1 \not\subseteq \mathfrak{N}_\omega$. Let $\mathfrak{H} = \mathfrak{F}_2 \ldots \mathfrak{F}_t$ and A be a group of minimal order in $\mathfrak{F}_1 \setminus \mathfrak{N}_\omega$. Let R be the monolith of A. Then $R = A^{\mathfrak{N}_\omega} = C_A(R)$ and $R \not\subseteq \Phi(A)$. If R is a p-group, then $R = O_p(A) = F_p(A)$. By (2), A/R is a nilpotent group. But by Lemma 1.7.11 [11], we have $O_p(A/C_A(R)) = 1$, and so $p \notin \pi(A/R)$. Assume that $R = A$. Then, we have $|R| = p$ and $R = A^{\mathfrak{N}_\omega}$. We can deduce that $p \notin \omega$. Hence $\mathfrak{H}$ is an Abelian formation by Lemma 3.1 [7]. Let $R \neq A$ and let $R \leq M \leq A$, where M is a maximal subgroup of A. Then A/M is a simple group with $|A/M| \neq p$. Let m be the smallest local ω-satelitte of $\mathfrak{F}_1$. Since $A \in \mathfrak{F}_1$, we have $A/F_p(A) = A/R \in m(p)$, and so $A/M \in m(p)$. Now, by using Lemma 5.2 [7], we see that $t = 2$ and that $\mathfrak{F}_2$ is an Abelian formation. It follows that $\mathfrak{F} = \mathfrak{F}_1\mathfrak{F}_2$ is a solubly ω-saturated soluble formation and hence, in this case, $\mathfrak{F}$ is an ω-saturated formation.

Now we prove that $\mathfrak{F}_2$ is a one-generated formation. Assume that $\pi(\mathfrak{F}_1) \cap \omega = \{p\}$. Since $\mathfrak{F}_1 \not\subseteq \mathfrak{N}_\omega$, we may choose in $\mathfrak{F}_1$ a group A of prime order $q \notin \omega$. Let $B \in \mathfrak{F}_2$ and $T = A \wr B$. Then, it is clear that $T \in \mathfrak{F}$ and that $O_\omega(T) = O_p(T) = 1$. Hence by Lemma 5 [2], we have

$$T \simeq T/O_\omega(T) \in f(\omega') \subseteq f^*(\omega') = \mathrm{form}(G/O_\omega(G)).$$

Hence $\mathfrak{F}_2 \subseteq \mathrm{from}(G/O_\omega(G))$. Since the formation $\mathfrak{F}_2$ is soluble, $\mathfrak{F}_2$ is a one-generated formation. Now let $|\pi(\mathfrak{F}_1) \cap \omega| > 1$ and let p, q be two different primes in $\pi(\mathfrak{F}_1) \cap \omega$. Let $B \in \mathfrak{F}_2$. Then by Lemma 4.5 [7], we have $\mathfrak{F} = CF(t)$, where

$$t(a) = \begin{cases} m(p)\mathfrak{F}_2 & \text{if } a = p \in \pi(\mathrm{Com}(\mathfrak{F}_1)) \cap \omega; \\ \varnothing & \text{if } a = p \in \omega \setminus \pi(\mathrm{Com}(\mathfrak{F}_1)); \\ m(\omega')\mathfrak{F}_2 & \text{if } a = \omega'. \end{cases}$$

We note that the formation function m is an inner composition ω-satelitte of $\mathfrak{F}_1$. Now, using Lemma 5 [2], we deduce that

$$B/O_p(B) \in f(p) \subseteq f^*(p) = \\ = \mathrm{form}(G/C^p(G)) \subseteq \mathrm{form}(G/C^p(G) \times (G/C^q(G))$$

and

$$B/O_q(B) \in f(q) \subseteq f^*(q) = \\ = \mathrm{form}(G/C^q(G)) \subseteq \mathrm{form}(G/C^p(G) \times (G/C^q(G)).$$

Hence

$$B \simeq B/(O_p(B) \cap O_q(B)) \in \mathrm{form}((G/C^p(G)) \times (G/C^q(G))).$$

This shows that $\mathfrak{F}_2 \subseteq \mathrm{form}(G/C^p(G) \times (G/C^q(G))$ and so $\mathfrak{F}_2$ is a one-generated formation.

Let $A \in \mathfrak{F}_1$ and $B \in \mathfrak{F}_2$. Then by Lemma 3.1.5 [5] and Lemma 3.1.7 [5] we can show that $(|A/F_\omega(A)|, |B|) = 1$ and $(|A/O_\omega(A)|, |B|) = 1$.

This proves the condition (3).

Next we assume that $\mathfrak{F}_1 \subseteq \mathfrak{N}_\omega$ and $t = 3$. First consider the case where $|\pi(\mathfrak{F}_1)| > 1$. We claim that $\mathfrak{F}_2(p)$ is a nilpotent formation, for all $p \in \omega$. In fact, if the claim is not true, then we can let A be a non-nilpotent group of smallest order in $\mathfrak{F}_2(p)$. Let R be the monolith of A. And let q be a prime in $\pi(\mathfrak{F}_1)$ such that $R \not\subseteq O_q(A)$. Let $T = Z_q \wr A$, where Z_q is a group of order q. Then

$$T \in \mathfrak{F}_1(\mathfrak{F}_2(p)) \subseteq \mathfrak{F}_1\mathfrak{F}_2 \subseteq \mathfrak{N}_\omega\mathfrak{N}.$$

Also, it is not difficult to show that $F(T) = K$, where K is the base group of T. Clearly, $T/K \simeq A$ is not nilpotent, and so $T \notin \mathfrak{N}_\omega\mathfrak{N}$. This contradiction shows that $\mathfrak{F}_2(p)$ is a nilpotent formation, and our claim is established. If $\mathfrak{F}_1\mathfrak{F}_2 \subseteq \mathfrak{N}_\omega$, then by Lemma 5.1 [7], we have $\mathfrak{F}_1\mathfrak{F}_2 = \mathfrak{F}_1 = \mathfrak{N}_p$ which is impossible. Hence $\mathfrak{F}_1\mathfrak{F}_2 \not\subseteq \mathfrak{N}_\omega$ and so by (3), $\mathfrak{F}_3$ is an Abelian one-generated formation. Let $p \in \pi(\mathfrak{F}_1)$, $A \in \mathfrak{F}_2$, $T_0 = A/O_p(A)$ and $B \in \mathfrak{F}_3$. Assume that there is prime q such that $q \in \pi(T_0) \cap \pi(B)$. Let Z_q be a group of order q. Let $T = Z_q \wr (Z_1 \times \ldots \times Z_n)$, where $Z_1 \simeq Z_2 \simeq \ldots \simeq Z_n \simeq Z_q$ and $n = |G|$. Form $Y = T_0 \wr (B_1 \times \ldots \times B_n)$, where $B_i \simeq B$. Then, by Lemma 3.1.7 [5], the nilpotent class $c(T)$ of the group T is at least $n+1$. If $q \notin \omega$, then by [[1], A, (18,2)] and by Lemma 5 [2],

$$T/O_\omega(T) \simeq T \simeq E \leq Y/O_\omega(Y) \in \mathrm{form}(G/O_\omega(G)).$$

This clearly contradicts to Lemma 3.1.5 [5]. Let $q \in \omega$. Then, in view of (2), we have $q \in \pi(\mathfrak{F}_1) \setminus \{p\}$. Let Z_p be a group of order p and $D = Z_p \wr T = [K]T$, where K is the base group of D. Clearly $D \in \mathfrak{F}_1\mathfrak{F}_2\mathfrak{F}_3 = \mathfrak{F}$, and so by Lemma 5 [2], we have

$$D/C^p(D) = D/K \simeq T \in \mathrm{form}(G/C^p(G)),$$

again, this contradicts to Lemma 3.1.5 [5]. Thus, for all groups $A \in \mathfrak{F}_2$ and $B \in \mathfrak{F}_3$, we have $\pi(A/O_p(A)) \cap \pi(B) = \varnothing$.

Now we claim that $\mathfrak{F}_2(p)$ is a one-generated nilpotent formation. Indeed, by (2), $\mathfrak{F}_1\mathfrak{F}_2$ is a one-generated ω-saturated formation. By using Proposition 4.7 [7] we see that $\mathfrak{F}_2$ or $\mathfrak{F}_2(p)$ is a one-generated formation. But $\mathfrak{F}_2$ is a soluble formation and $\mathfrak{F}_2(p) \subseteq \mathfrak{F}_2$. Hence $\mathfrak{F}_2(p)$ is a one-generated formation in view of Theorem VII.1.7 [1].

Now consider $\mathfrak{F}_1 = \mathfrak{N}_p$, for some $p \in \omega$. By Proposition 4.7 [7], $\mathfrak{F}_2(p)$ is a one-generated formation since $\mathfrak{F}_1\mathfrak{F}_2$ is a one-generated solubly ω-saturated formation. Thus, condition (4) holds.

Condition (5) and the first two conditions of (6) follow directly from Proposition 4.7 [7]. It is clear that $\mathfrak{F}_2 \not\subseteq \mathfrak{F}_1$. Now we assume that for every group $B \in \mathfrak{F}_2$ there is a group $A \in \mathfrak{F}_1$ such that the $\mathfrak{F}_2$-residual $T^{\mathfrak{F}_2}$ of the wreath product $T = A \wr B$ is not contained subdirectly in the base group of T. And let B be a group of minimal order in $\mathfrak{N}_p\mathfrak{F}_2 \setminus \mathfrak{F}_2$. Then the group B is monolithic and its monolith $R = B^{\mathfrak{F}_2}$. Now let $T = A \wr (B/R)$, where A is a group in $\mathfrak{F}_1$ such that the $\mathfrak{F}_2$-residual $T^{\mathfrak{F}_2}$ of T is not contained subdirectly in the base group K of T. But then the formation $\mathfrak{F}_2$ contains the group $Z_p \wr (B/R)$, where $|Z_p| = p$. Evidently, R is an elementary Abelian p-group, so by Lemma 3.5.2 [5], $B \in \mathfrak{F}_2$. This contradiction shows that $\mathfrak{N}_p\mathfrak{F}_2 \subseteq \mathfrak{F}_2$. Hence $\mathfrak{N}_p\mathfrak{F}_2 = \mathfrak{F}_2 = \mathfrak{F}$. This contradiction shows that condition (6) holds. Thus, the theorem is proved. □

References

[1] K. Doerk and T. Hawkes, Finite Soluble Group, Walter de gruyter, Berlin–New York, 1992.

[2] A.N.Skiba and L.A.Shemetkov, Multiply $\mathfrak{L}$-composition formations of finite groups, Ukrainsk. Math. Zh., **52**(6), 2000, 783–797.

[3] L.A.Shemetkov, Formations of Finite Groups, Nauka, Moscow, 1978.

[4] L.A.Shemetkov, A.N.Skiba, Formations of Algebraic Systems, Nauka, Moscow, 1989.

[5] A.N.Skiba, Algebra of Formations, Minsk, Belaruskaja Navuka, 1997.

[6] V.M.Selkin, Criterions of a limitation of solubly ω-saturaed formations of finite groups, Vesti NAN Belarusi, 2012, 2, p.131-136.

[7] W.Go, V.M.Selkin, K.P.Sham Factorization theory of one-generated Bear ω-local formations, Communications in Algebra, Volume 35, Issue 9 September 2007, p. 2901-2931

[8] A.N.Skiba, On product of formations. Algebra and Logic, 22(5), 1983, 574–583.

[9] V.M.Selkin, About one application of the theory of minimal τ-closed ω-saturated non $\mathfrak{H}$-formations, Vesnik VSU, v3, 2012, 46-52.

[10] V.M.Selkin, On one criterion of product of nonidentity formations, Proc. GGU, v2, 2012, p.23-27.

[11] Wenbin Guo, The Theory of Classes of Groups, Science Press-Kluwer Academic Publishers, Beijing-New York-Dordrecht-Boston-London, 2000.

Contact information

V. Selkin Department of Mathematics, Francisk Skorina Gomel State University, Sovetskya Str., 104, Gomel, 246019, Belarus
E-Mail: vselkin@gsu.by

Received by the editors: 25.02.2012
and in final form 25.02.2012.