Algebra and Discrete Mathematics RESEARCH ARTICLE
Volume 13 **(2012)**. Number 2. pp. 237 – 272

The spectral measure of the Markov operator related to 3-generated 2-group of intermediate growth and its Jacobi parameters[1]

R. I. Grigorchuk and Ya. S. Krylyuk

Abstract. It is shown that the KNS-spectral measure of the typical Schreier graph of the action of 3-generated 2-group of intermediate growth constructed by the first author in 1980 on the boundary of binary rooted tree coincides with the Kesten's spectral measure, and coincides (up to affine transformation of $\mathbb{R}$) with the density of states of the corresponding diatomic linear chain.

Jacoby matrix associated with Markov operator of simple random walk on these graphs is computed. It shown shown that KNS and Kesten's spectral measures of the Schreier graph based on the orbit of the point 1^∞ are different but have the same support and are absolutely continuous with respect to the Lebesgue measure.

Introduction

The diatomic linear chain is one of the most studied and used models in physics and chemistry [Bri53]. What can relate this model to the torsion group of intermediate growth $\mathcal{G}$, a highly non-commutative object constructed by the first author in [Gri80]. The goal of this article is to describe the relation between these two apparently completely different instances and to draw some consequences.

[1]The first author is supported by NSF grants DMS-0456185 and DMS-06000975. Both authors are supported by Max-Planck Institut of Mathematiks, Bonn.

2010 MSC: 20F, 20P, 37A, 60J, 82D.

Key words and phrases: group of intermediate growth, diatomic linear chain, random walk, spectral measure, Schreier graph, discrete Laplacian.

Let us start with some history. The groups of Burnside type (i.e. finitely generated infinite torsion groups [Gol64, GL02]) and the groups of intermediate (between polynomial and exponential) growth [Gri83, Gri84] had appeared respectively in 60-th and 80-th of the last century, and during decades were considered as exotic groups.

A rich source of such interesting examples is the class of so-called automata groups introduced in 1963 and studied in [Hoř63, Ale72, GNS00b].

Recent developments show that this class together with the closely related classes of self-similar groups and branch groups introduced in [Gri88, Gri00a, Gri00b] (see also the monograph of V. Nekrashevych [Nek05]) play important role in various studies in mathematics and other areas of science.

In this paper we will show how groups of intermediate growth can be used to interpret classical results from the new perspective, and to bring a broader vision to some facts known for a long time. Our investigation is based on results from [BG00a].

Namely we will describe the connection between $\mathcal{G}$ and the diatomic linear chain model [Whe84, Gau84], expressed in the form of a relation between associated Jacobi parameters and spectral measures.

The part of the paper is devoted to a simple introduction to the theory of Schreier graphs related to self-similar groups acting on rooted trees and to their spectral theory. Additional information on this topic can be find in [BG00a, Gri05, Gri11]. Appendix contains the computation of some integrals needed in the proof of the main statements.

While Cayley graphs of finitely generated groups are very popular in combinatorial group theory and its geometric branches, Schreier graphs didn't play a big role (at least in infinite group theory) until recently. Actions of self-similar groups on regular rooted trees demonstrate usefulness of Schreier graphs in different topics of holomorphic dynamics [Nek05], fractal geometry [BGN03], and combinatorics [GŠ06].

One of the interesting ideas related to groups generated by finite automata is to use them for constructing families of expanders. If implemented, it would provide a new construction of expanders, much more effective in practice than the existing constructions. The first step in this direction has been made in [Gri11] where the so called asymptotic expanders are constructed using finite automata. Proving that some finite automata may produce a sequence of expanders has to go through computation (or estimation) of the second after 1 (in the decreasing direction) eigenvalue of the Markov operator of Schreier graphs arising from the action of the group on levels of the regular rooted tree. This is related to the computation of the spectrum, i.e. to the diagonalization of the operator (the involved operators are self-adjoint and therefore diagonalizable).

Another example leading to a spectral problem is the well known combinatorial problem related to Hanoi Towers game on $k \geq 4$ pegs (see [GŠ06] [GŠ07]).

The problem of computation of spectra of operators (or graphs) related to self-similar groups is hard and solved only in a few cases [BG00b, GŻ01, GSŠ07a]. The tested examples are related to finite Mealy type automata generating amenable (in von Neumann sense [vN29])) groups. It is known that amenable infinite groups can not be used for construction of expanders by taking the sequence of finite quotients. Nevertheless even in the amenable case the study of spectra of associated objects is a challenging problem as it could be related to other topics, for instance, to the topics around the Atiyah Problem on L^2-Betti numbers, as shown in [GLSŻ00]. Although this problem has been solved recently [Aus09, Gra09, LW10, PSZ10] there is still a lot of open questions around and one of them is if different groups from the Lampligher group can contribute to the problem.

There is no universal method for computing the spectrum, but there is a general method based on Schur complement transformation of matrices with operator valued entries which works well in certain situations [GSŠ07b, GN07], in particular in the one that we are going to discuss.

Let us consider an easier problem. Namely, using the classical algorithm (say of Hessenberg) let us transform the matrix of the Markov operator to the tridiagonal form and consider it as a Jacobi matrix. We would like to determine what type of Jacobi matrices can appear in this way, what type of orthogonal polynomials correspond to these matrices, and which spectral information (including the information about spectral measures) is obtained in the process. We expect that the Jacobi matrices and orthogonal polynomials arising from self-similar groups to possess some interesting properties.

The goal of this article to follow this plan (at least in part) in the case of the group $\mathcal{G}$. The definition of this group in various forms (including the one given by the action on binary rooted tree) will be given in Section 3. Notions related to graphs and spectral measures will appear in Section 2.

The study of asymptotic properties of graphs and groups is related to the study of spectral measures, first of all of the so called Kesten spectral measure which is the spectral measure of the Markov operator associated with the random walk on the group.

Recent investigations show usefulness of the so called Kesten-von Neumann-Serre (KNS) spectral measure [BG00a]. It is analogous to integrated density of states in mathematical physics and can be rigorously defined for any graph which is the limit of a covering sequence of finite graphs [GŻ01, GŻ04]. Our main result is the following

Theorem 1. *Let $\Gamma = \Gamma(\mathcal{G}, P, \gamma)$ be a Schreier graph of the group $\mathcal{G}$, where γ is any point of the boundary of the binary rooted tree on which $\mathcal{G}$ acts, except in the orbit of the point 1^∞, and let $P = st_{\mathcal{G}}(\gamma)$. Then*

- *The KNS spectral measure of Γ coincides with the Kesten's spectral measure, and coincides (up to affine transformation of $\mathbb{R}$) with the density of states of the corresponding diatomic linear chain.*
- *The Jacobi matrix associated to the Markov operator and the initial vector δ_γ given by the δ-function at vertex γ is*

$$J = \begin{pmatrix} a_1 & b_1 & 0 & 0 & 0 & 0 & \dots \\ b_1 & a_2 & b_2 & 0 & 0 & 0 & \dots \\ 0 & b_2 & a_3 & b_3 & 0 & 0 & \dots \\ & & & \dots & & & \end{pmatrix},$$

where the diagonal elements $a_n = 1$ for $n = 1, 2, 3, \dots$ and

$$b_1 = \sqrt{5},$$

$$b_n = \begin{cases} \sqrt{\dfrac{2^n + 4}{2^n + 1}} & \text{if } n \text{ is even,} \\ \sqrt{\dfrac{2^{n+1} + 1}{2^{n-1} + 1}} & \text{if } n \text{ is odd, } n \geq 3. \end{cases}$$

The coincidence of spectral measures is not an absolute phenomenon due to the following result.

Theorem 2. *For the graph Γ and γ belonging to the orbit of 1^∞ the Kesten and KNS spectral measures are different, but have the same support and are absolutely continuous with respect to Lebesgue measure (the density of Kesten spectral measure is explicitly computed in Lemma 6.1).*

We would like to mention that some of results presented in this paper are already known ([BG00a, GŻ04], so the paper can be considered partly as a survey. Also authors are aware that part of the results may be obtained in a shorter way. We preferred to use longer but classical way, based on application of Stieltjes transform.

It would be interesting to check whether in the case of a contracting self-similar group ([Nek05, Gri11]), the uniform the Kesten and KNS spectral measures coincide for almost all points on the boundary of the tree, or whether at least the measures have the same support for all boundary points.

1. Groups acting on rooted trees, automata groups and related representations

In this paper we shall consider only the torsion group $\mathcal{G}$ of intermediate growth defined in [Gri83, Gri84] (see the definition in Section 3), but in order to put our investigation in the general perspective let us briefly review the necessary facts and definitions from the theory of groups acting on regular rooted trees (see, for instance [BG00a, Nek05]). Let Σ be a finite alphabet. The vertex set of the tree $T = T_\Sigma$ is the set of finite sequences over Σ; two sequences are connected by an edge when one can be obtained from the other by right-adjunction of a letter in Σ. The top node is an empty sequence, and the "children" of σ are all the σs, for $s \in \Sigma$. We suppose $\Sigma = \mathbb{Z}/d\mathbb{Z}$, with the operation $\bar{s} = s + 1 \bmod d$. Let a, called the *rooted automorphism* of T_Σ, be the automorphism of T defined by $a(s\sigma) = \bar{s}\sigma$: it acts non-trivially on the first symbol only, and geometrically is realized as a cyclic permutation of the d subtrees just below the root.

For any subgroup $G < \mathcal{A}$ ($\mathcal{A}$ is the group of all automorphisms of the tree T) $st_G(\sigma)$ denotes the subgroup of G consisting of automorphisms that fix the sequence σ, and $st_G(n)$ denotes the subgroup of G consisting of the automorphisms that fix all sequences of length n:

$$st_G(\sigma) = \{g \in G \mid g\sigma = \sigma\}, \quad st_G(n) = \bigcap_{\sigma \in \Sigma^n} st_G(\sigma).$$

The $st_G(n)$ are normal subgroups of finite index in G.

A subgroup $G < \mathcal{A}$ is level-transitive if the action of G on Σ^n is transitive for all $n \in \mathbb{N}$. We shall always implicitly make this assumption.

G acts naturally on the boundary $\partial T = \{0, \dots, d-1\}^{\mathbb{N}}$, and this action preserves the uniform Bernoulli measure ν on the compact space ∂T. We associate the dynamical system $(G, \partial T, \nu)$ to the group G.

This dynamical system is naturally isomorphic to the dynamical system $(G, [0,1], m)$, where m is the Lebesgue measure, and G acts on

$[0,1]$ by measure-preserving transformations in the following way: let $g \in G$, and $\gamma \in [0,1]$ be a d-adic irrational point with base-d expansion $0.\gamma_1\gamma_2\ldots$. Then $g(\gamma) = 0.\delta_1\delta_2\ldots$, where the infinite sequence $(\gamma_1, \gamma_2, \ldots)$ is mapped by g to $(\delta_1, \delta_2, \ldots)$. This defines the action of G on a subset of full measure of $[0,1]$.

Definition 1.1. The infinite sequences $\sigma, \tau \colon \mathbb{N} \to \Sigma$ are *confinal* if there is $N \in \mathbb{N}$ such that $\sigma_n = \tau_n$ for all $n \geq N$.

Confinality is an equivalence relation, and equivalence classes are called *confinality classes.*

A ray e in T is an infinite geodesic starting at the root of T, or equivalently an element of the boundary $\partial T = \Sigma^{\mathbb{N}}$.

Let $G < \mathcal{A}$ and e be a ray. The associated parabolic subgroup is

$P = st_G(e) = \bigcap_{n \geq 0} P_n$, where $P_n = st_G(e_n)$ and e_n is the length-n prefix of e.

Since G acts on the boundary ∂T of the tree by homeomorphisms (with respect to the natural topology on ∂T) and preserves the uniform measure on the boundary, we have a unitary representation π of G in $L^2(\partial T, \nu)$, or equivalently in $L^2([0,1], m)$. Let $\mathcal{H}_n$ be the space of $L^2(\partial T, \nu)$ spanned by the characteristic functions χ_σ supported on the rays e starting by σ, for all $\sigma \in \Sigma^n$. It is of dimension d^n, and can equivalently be seen as spanned by the characteristic functions in $L^2([0,1], m)$ of the intervals of the form $[(i-1)d^{-n}, id^{-n}]$, $1 \leq i \leq d^n$. These $\mathcal{H}_n$ are invariant subspaces, and admit representations $\pi_n = \pi|_{\mathcal{H}_n}$. Since π_{n-1} is a subrepresentation of π_n, we set $\pi_n^\perp = \pi_n \ominus \pi_{n-1}$, so that $\pi = \bigoplus_{n=0}^{\infty} \pi_n^\perp$.

Denote by $\rho_{G/P}$ the quasi-regular representation of G in $\ell^2(G/P)$ given by the left action and, for $n \geq 1$, denote by ρ_{G/P_n} the finite-dimensional representations of G in $\ell^2(G/P_n)$. Since G is level-transitive, the representations π_n and ρ_{G/P_n} are unitary equivalent.

Definition 1.2. Let G be a group generated by a set S and H be a subgroup of G. The Schreier graph $\mathcal{S}(G, H, S)$ is the directed graph on the vertex set G/H, with an edge from gH to sgH withevery $s \in S$ and every $gH \in G/H$. The *base point* of $\mathcal{S}(G, H, S)$ is the coset H.

If a group G acts on a set X and $x \in X$, then the Schreier graph $\Gamma(G, S, st_G(x))$ can be interpreted as the connected component of the graph of action (i.e. the graph whose vertices are the points of orbit of x and two vertices are joined by an edge if one can pass from one vertex to another by the action of a generator). In the case of a transitive action the graph of the action is isomorphic to the Schreier graph.

Let Γ_n be the Schreier graph of the action of a group G on the n-th level of the tree and let Γ be the Schreier graph of the action of G on the boundary with the set of vertices belonging to the orbit of some point $\xi \in \partial T$. Let v_n be a vertex of the ray ξ at level n. We consider Γ_n as a marked graph with a distinguished vertex v_n and Γ as a marked graph with a distinguished vertex ξ. Then Γ_n converge to Γ in the natural topology on the space of marked graphs of given degree (two graphs are close in this topology if they have isomorphic neighborhoods of large radius around distinguished vertices) [GŻ99]. On the language of representations this corresponds to the approximation of the infinite dimensional quasi-regular representation $\rho_{G/P}$ by the finite-dimensional $\rho_{G/P_n}, n \in \mathbb{N}$.

Definition 1.3. The Markov operator M_n on Γ_n is the operator on $\ell^2(\Gamma_n)$ given by

$$(M_n f)(v) = \frac{1}{|S|} \sum_{w \sim v} f(w),$$

where $w \sim v$ means that w is a neighbor of v in Γ_n. This formula defines a Markov operator M also on Γ and more generally on any regular locally finite graph.

We have parallel definitions of Hecke type operators $\widetilde{M}_n$ and $\widetilde{M}$ for the quasi-regular representations ρ_{G/P_n} and $\rho_{G/P}$ in the Hilbert spaces $\ell^2(G/P_n)$ and $\ell^2(G/P)$ respectively. For example, $\widetilde{M}_n$ is defined on $\ell^2(G/P_n)$ by formula

$$(\widetilde{M}_n f)(x) = \frac{1}{|S|} \sum_{s \in S} (\rho_{G/P_n}(s) f)(x) \quad \text{for} \quad x \in \ell^2(G/P_n).$$

It is clear that in the case of a level transitive action we can identify $\widetilde{M}_n$ with M_n and $\widetilde{M}$ with M as they are symmetric.

The study of the Markov operator $\widetilde{M}$ with a distinguished vector δ_P (i.e. the study of the corresponding spectral measure, the Jacobi matrix, etc.) is closely related to the study of the simple random walk on Γ.

This allows to employ the convergence of the marked graphs $\{\Gamma_n\}$ to Γ in the space of marked graphs in order to find the spectral measure μ of Γ (i.e. the spectral measure of $\widetilde{M}$, with initial vector δ_P) and the corresponding Jacobi matrix J for $(\widetilde{M}, \delta_P)$).

We shall prove that in the case of group $\mathcal{G}$, the measure μ coincides with the KNS-measure calculated in [BG00a]. Generally, it is a very interesting question what kind of measures and what kind of Jacobi matrices and the corresponding orthogonal polynomials stand behind a group acting on a regular rooted tree, in particular, an automaton group. Computer experiments of drawing histograms of spectra associated with automata, even with a small number of states, show that corresponding distributions and their supports may have complicated topological and measurable structure and there is a lot of open questions in this area [BGK+07].

2. Spectral measures

Let Γ be a graph and M be its Markov operator. For any two vertices x, y and $n \in \mathbb{N}$ let $p^n_{x,y}$ be the probability that a simple random walk starting at x will be at y after n steps. Recall that if δ_v is the characteristic vector of the vertex v, then

$$p^n_{x,y} = \langle M^n \delta_x | \delta_y \rangle.$$

Spectral measures (not necessary positive) are given by their distributions (or spectral functions) $\sigma_{x,y}$ via the moments

$$p^n_{x,y} = \int_{-1}^{1} \lambda^n \, d\sigma_{x,y}(\lambda) \qquad \forall n \in \mathbb{N},$$

or, equivalently,

$$\sigma_{x,y}(\lambda) = \langle M(\lambda)\delta_x | \delta_y \rangle,$$

where $M(\lambda)$ is the spectral decomposition of M. In case $x = y$ the distribution determines a positive probabilistic measure. Set $\sigma_x = \sigma_{x,x}$. The corresponding measure $d\sigma_x(\lambda)$ is called the *Kesten* spectral measures as it was introduced in [Kes59] in the situation of Markov operators corresponding to symmetric random walks on groups. (In the case of the uniform distribution on a finite generating set Kesten measure is the spectral measure of the Markov operator on the corresponding Cayley graph of the group). We will usually identify spectral functions and corresponding measures, but will be more careful when using them in integral expressions.

It is natural to expect that if Γ is infinite, regular and connected, then all the measures σ_x are equivalent, and in particular, have the same support, but there is no proof of this fact as far as we know.

One of approaches to study infinite graphs is to approximate them by finite graphs and to use limit type theorems.

The traditional way is to approximate infinite graphs by ascending sequence of finite subgraphs. For instance in the case of an amenable graph one can use the approximation by a sequence of Fölner sets. Recent studies show the usefulness of another approach when the infinite graph is the limit of a covering sequence of finite graphs. This is the case of Schreier graph of the group $\mathcal{G}$ and other groups acting on rooted trees. In the algebraic language this corresponds to the approximation of the infinite group by its finite quotients. To be more precise, consider the family of subgroups

$$H \leq G, H = \bigcap_{n=1}^{\infty} H_n,$$

where H_n is a descending sequence of subgroups of finite index in G. Then the sequence of finite graphs $\mathcal{T}_n = S(G, H_n, S)$ is a covering sequence (in the sense that each next graph covers the previous one) which converges to the graph $\mathcal{T} = S(G, H, S)$ in the space of regular marked graphs of degree $|S|$, and the following statement holds.

Proposition 2.1 ([GŻ99]). *Let σ_n and σ be the Kesten measures of the Schreier graphs $\mathcal{T}_n$ and $\mathcal{T}$. Then $\sigma_n(\lambda) \to \sigma(\lambda)$ weakly.*

On the other hand, we have the counting measure τ_n on each finite Schreier graph $\mathcal{T}_n$, counting the average number of eigenvalues in any given interval. The measures τ_n converge to some measure τ_* which was called in [GŻ04] the KNS spectral measure on $\mathcal{T}$. This is analogous to the notion of integrated density of states often used in mathematical physics literature. The existence of the limit measure τ_* is the corollary of one of results of Serre [Ser97].

Problem 1. *Under what conditions on marked graph $\mathcal{T}$ which is the limit of covering sequence $\mathcal{T}_n$ of finite marked graphs the $KNS-$spectral measure τ_* coincides with the Kesten spectral measure?*

As we shall show this is the case for the two ended Shreier graphs of 3-generated 2-group $\mathcal{G}$ of intermediate growth with the standard set of generators $S = \{a, b, c, d\}$.

In this case the corresponding KNS measure is a continuous measure, given, up to multiplication by 4 and shift by 1 along $\mathbb{R}$ is given by the following formula:

$$\tau_*(d\lambda) = \frac{|\lambda|\ d\lambda}{\pi\sqrt{(\lambda+3)(\lambda+1)(\lambda-1)(3-\lambda)}}, \quad \lambda \in [-3,-1] \cup [1,3]. \quad (2.1)$$

The measure τ_* was computed in [BG00a] in the following way. The Markov operator M on the infinite Schreier graph Γ is replaced by the operator $4M$ and is included in a two parametric family of operators $Q(\lambda, \mu)$. These operators are in certain sense approximated by operators $Q_n(\lambda, \mu)$ in 2^n dimensional space and include, for the value $\lambda = -1, \mu = -1$, the operator $4M_n$ where M_n is the Markov operator on the n-th approximation Γ_n of the graph Γ by the Shreier graphs of the action of $\mathcal{G}$ on the n-th level of the tree, $n = 1, \dots$. The operator $4M_n$ is represented by the following matrix

$$\begin{bmatrix} 3 & 1 & 0 & 0 & 0 & \dots & 0 \\ 1 & 1 & 2 & 0 & 0 & \dots & 0 \\ 0 & 2 & 1 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & 1 & 2 & \dots & 0 \\ \hdashline 0 & 0 & 0 & \dots & 1 & 2 & 0 \\ 0 & 0 & 0 & \dots & 2 & 1 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 3 \end{bmatrix}$$

of size 2^n in a suitable basis (related to nonstandard ordering of vertices of the n-th level of the tree discussed in Section 3). A tricky computation based on the existence of a recursion between the spectrums of $Q_n(\lambda, \mu), n = 1, 2, \dots$ which involves a two dimensional rational map and the semiconjugacy of this map to the Chebyshev-von Neumann-Ulam map $x \to 2x^2 - 1$ results in that the spectrum of the above matrix is

$$\{1 \pm \sqrt{5 - 4\cos(\theta)} : \theta \in 2\pi\mathbb{Z}/2^n\} - \{0, 2\}.$$

When $n \to \infty$ the distribution of values of θ tends to the uniform distribution and hence the distribution of eigenvalues of the above matrix tends to the image of the uniform distribution under the functions $\{1 \pm \sqrt{5 - 4\cos(\theta)}\}$, that is to the distribution supported in $[-2, 0] \cup [2, 4]$ with density

$$\frac{|x - 1|}{\pi\sqrt{x(x-2)(x+2)(4-x)}}.$$

The shift of $\mathbb{R}$ by 1 (which corresponds to transform $4M \to 4M - 1$) gives the above measure 2.1.

Let us compare the way in which this measure appears in diatomic linear chain on one hand and in association with $\mathcal{G}$ on the other hand. The reader is advised to consult [Whe84] and [Gau84] for details on used notions and formulas.

The allowed frequencies $\omega_\pm$ of the one-dimensional diatomic linear chain are given by

$$\frac{\omega_\pm}{\gamma} = \left(\frac{1}{m} + \frac{1}{M}\right) \pm \left[\left(\frac{1}{m} + \frac{1}{M}\right)^2 - \frac{4\sin^2\theta}{mM}\right]^{1/2},$$

where δ is the harmonic force constant, and m and M are the masses of the two kinds of particles which alternate on the chain.

Setting $x = (\omega/\omega_m)^2$ where $\omega_m = \{2\delta[1/m + 1/M)]\}^{1/2}$ gives

$$x = \frac{1}{2}\{1 \pm [1 - y\sin^2\theta]^{1/2}\},$$

where

$$y = \frac{4mM}{(m+M)^2}.$$

The allowed values of x lie in the union of two intervals $0 \le x \le r/(1+r)$ and $1/(1+r) \le x \le 1$ where $r = m/M$.

The values of θ are uniformly distributed on $(0, \pi/2)$ and therefore x is distributed in the union of the above intervals with the law given by the density

$$\frac{|1-2x|}{2\pi\sqrt{x(1-x)[r/(1+r)-x][(1+r)^{-1}-x]}}.$$

In the case $m = 1, M = 2$ the parameter $r = 1/2$. The linear transform $z = 6x - 2$ leads to the distribution 2.1.

The same type of argument shows that there is a relation between the diatomic linear chain to the model of Markov operator related to $\mathcal{G}$ and Γ in the case of arbitrary value of the parameter r.

3. The group $\mathcal{G}$ and its Schreier graphs

In the sequel we shall need only information about the Schreier graphs Γ_n and Γ related to group $\mathcal{G}$, but for completeness lets us recall the original definition from [Gri00a] of this group as well as the realization of it by a finite automaton. Also we will describe the corresponding action on a 2-regular rooted tree T_2.

The vertices of this tree can be naturally identified with the words over the alphabet $\Sigma = \{0, 1\}$ (the words of length n correspond to the vertices of level n). One can view the tree as embedded into a plane (with the root at the top) and with the lexicographic ordering of vertices of n-th level given by $0 < 1$.

Then a is the automorphism permuting the top two branches of T_2. Let define b recursively as the automorphism which acts as a on the left branch and as c on the right, where c is the automorphism which acts as a on the left branch and as d on the right, and finally d is the automorphism which acts as 1 on the left branch and as b on the right. In formula,

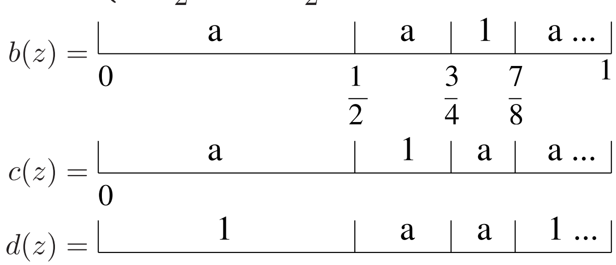
$$\begin{aligned} b(0x\sigma) &= 0\bar{x}\sigma, & b(1\sigma) &= 1c(\sigma),\\ c(0x\sigma) &= 0\bar{x}\sigma, & c(1\sigma) &= 1d(\sigma),\\ d(0x\sigma) &= 0x\sigma, & d(1\sigma) &= 1b(\sigma), \end{aligned}$$

where σ represents any finite binary sequence. $\mathcal{G}$ is the group of automorphisms of the tree T_2 generated by $\{a, b, c, d\}$. It is readily checked that these generators are of order 2 and that $\{1, b, c, d\}$ constitute a group isomorphic to the Klein group. Therefore any of the generators $\{b, c, d\}$ can be omitted from generating set (but we prefer to keep all of them). One can visualize the definition of generators by Figure 1, where e represents the trivial action below a vertex and ϵ represents a switch.

Originally the group $\mathcal{G}$ was defined in [Gri80] by its action on the interval $[0, 1]$ from which rational dyadic points are removed. This definition is presented by Figure 2 where P stands for permutation of halves of the interval and I represents the identity transformation.

More precisely, the transformations a, b, c, d) act as

$$a(z) = \begin{cases} z + \frac{1}{2} & \text{if } z < \frac{1}{2} \\ z - \frac{1}{2} & \text{if } z \geq \frac{1}{2} \end{cases}$$



Here the subintervals represent the whole interval $[0, 1]$, labeled with either a or 1 (the identity transformation) which act on the described subintervals in a similar way as a or id act on $[0, 1]$.

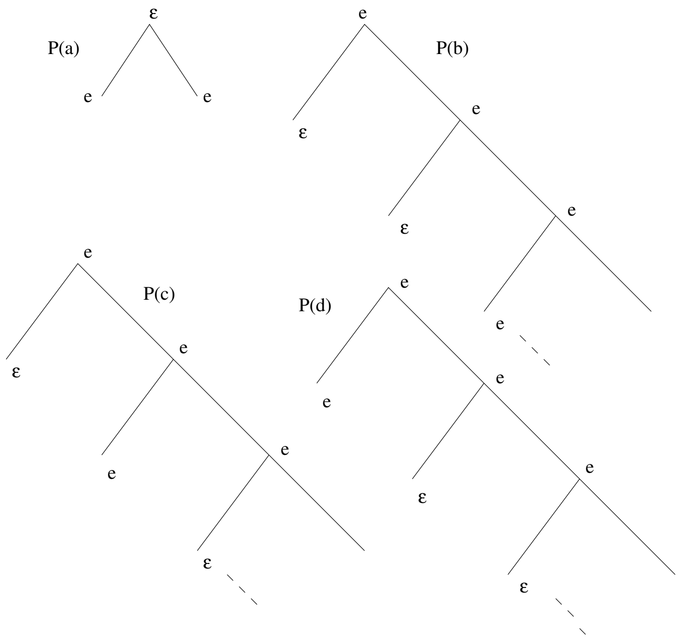


Figure 1. Action on the tree

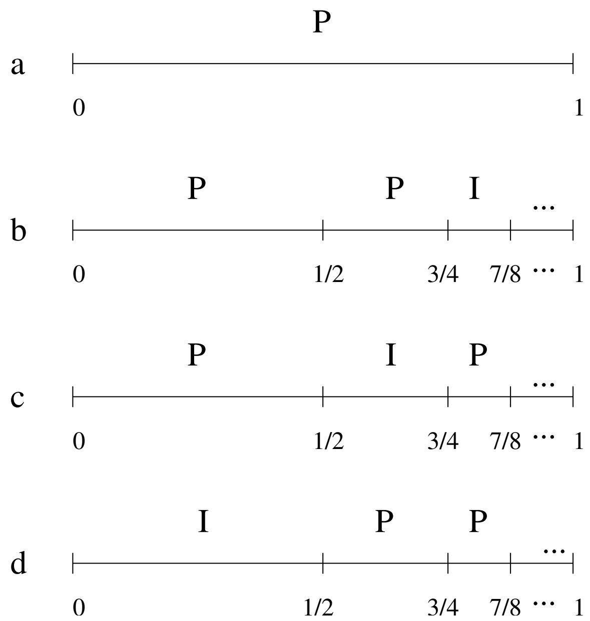


Figure 2. Action on the interval $[0, 1]$

Also $\mathcal{G}$ can be defined as a group generated by the automaton given by Figure 3 (the identity and non-identity elements of the symmetric group on two symbols are represented as 1 and ε and are used to label states of the automaton). For more on automata groups (called also self-similar groups) see [GŠ07, GNS00a, BGK+07, Nek05].

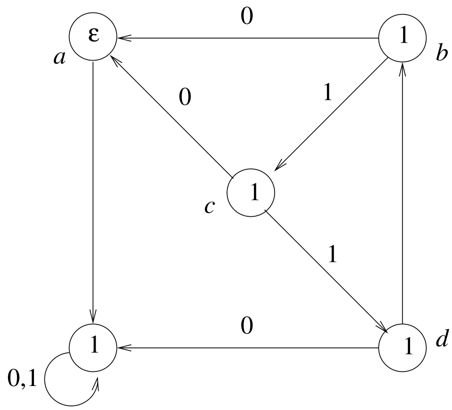


Figure 3. The automaton for $\mathcal{G}$.

The action of $\mathcal{G}$ on the interval preserves the Lebesgue measure m and the action on ∂T_2 preserves the uniform measure ν (which is the Bernoulli $\{1/2, 1/2\}$ measure). It is easy to see that there is a natural isomorphism of two dynamical systems $(\mathcal{G}, [0,1], m)$ and $(\mathcal{G}, \partial T_2, \nu)$ given by the presentation of 2−adic irrational points by corresponding binary sequences (which in turn are identified with points of the boundary).

Now we are going to summarize the known information about the Schreier graphs Γ_n and the orbit graph $\Gamma = \Gamma_\zeta$ of the action of $\mathcal{G}$ on the orbit of $\zeta \in \partial T_2$.

The sequence Γ_n is the sequence of substitutional graphs i.e. can be described by the initial graph called the axiom (in our case it is the graph given by Figure 5 and represents the action of $\mathcal{G}$ on the first level of the tree) and by the substitutional rule that allows to get Γ_{n+1} from Γ_n (in our case the substitutional rule is presented by Figures 6 and 7).

The application of one (respectively two) times of the rule to the axiom gives the Schreier graphs for levels 2 (respectively 3) represented on the bottom of Figure 4 (it takes some time to realize that the graphs of the action of $\mathcal{G}$ on levels of the tree represented in the top part of Figure 4 are isomorphic to the corresponding graphs given in the bottom

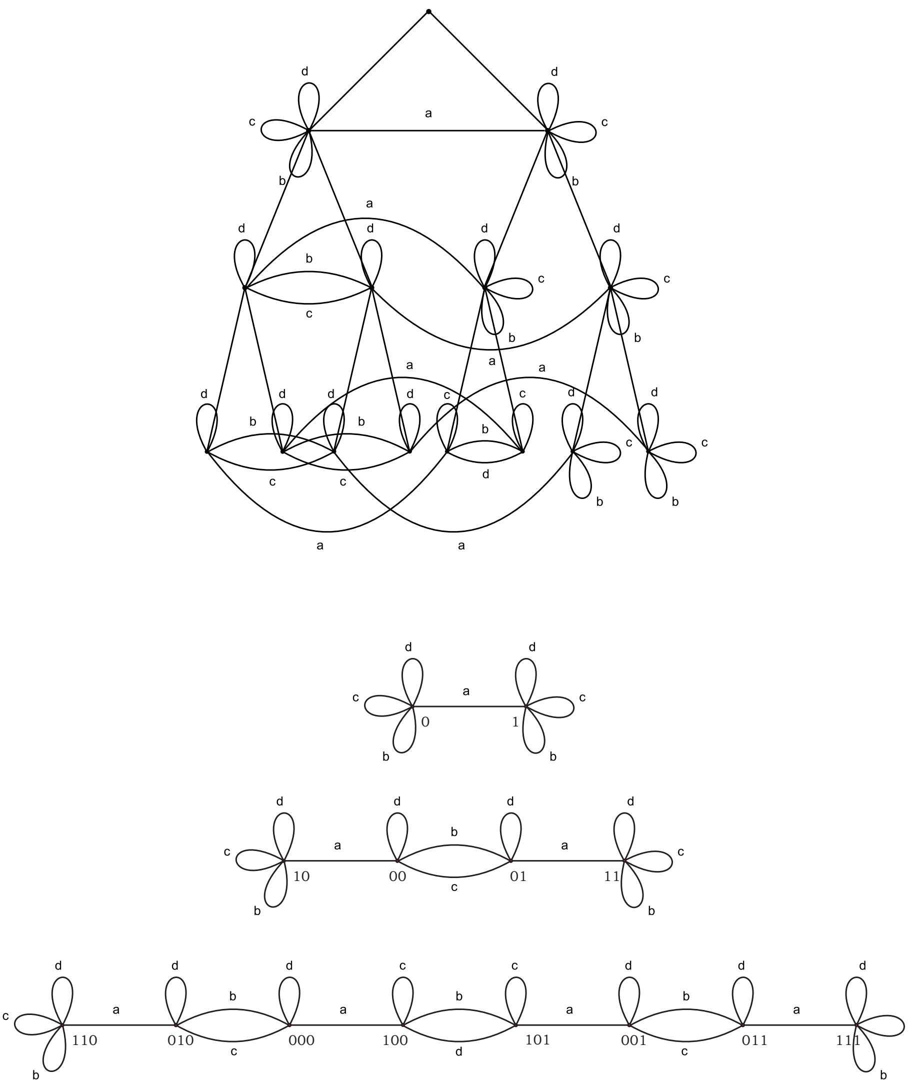


Figure 4.

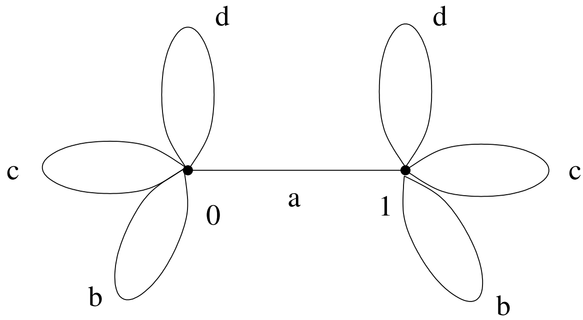


Figure 5. Axiom.

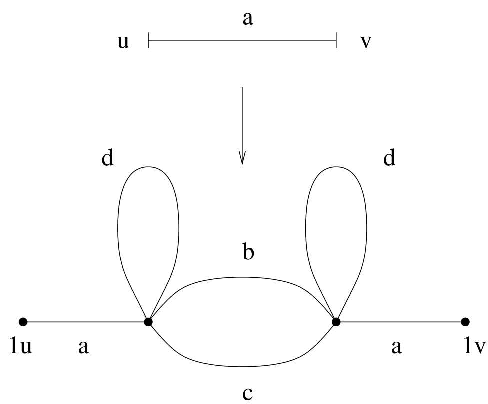


Figure 6. Rule 1.

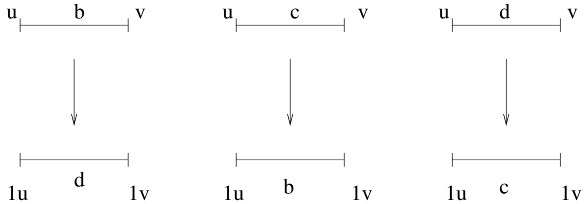


Figure 7. Rule 2.

part (see the more precise statement about the isomorphism below). The iteration of this process allows to build Γ_n for any $n = 1, 2....$

The vertices of Γ_n are naturally identified with Σ^n where $\Sigma = \{0, 1\}$ is supplied with the lexicographic ordering:

$$00\ldots0, 00\ldots1, \ldots, 11\ldots1$$

(the ordering of the vertices on n^{th} level of T_2 from the left to the right). On the other hand, in order to realize Γ_n as a chain with one loop at each vertex (as was suggested in [BG00a]) we need to consider another ordering. This new ordering, to which we shall refer as the non-standard ordering, is defined by induction ; if the ordering of Σ^{i-1} is $(\sigma_1, \ldots, \sigma_{2^{i-1}})$, the ordering of Σ^i is

$$(1\sigma_1, 0\sigma_1, 0\sigma_2, 1\sigma_2, 1\sigma_3, 0\sigma_3, \ldots, 0\sigma_{2^{i-1}}, 1\sigma_{2^{i-1}}).$$

The non-standard ordering gives the possibility to draw the figure of the Schreier graph in the form of a chain as is shown in Figure 4.

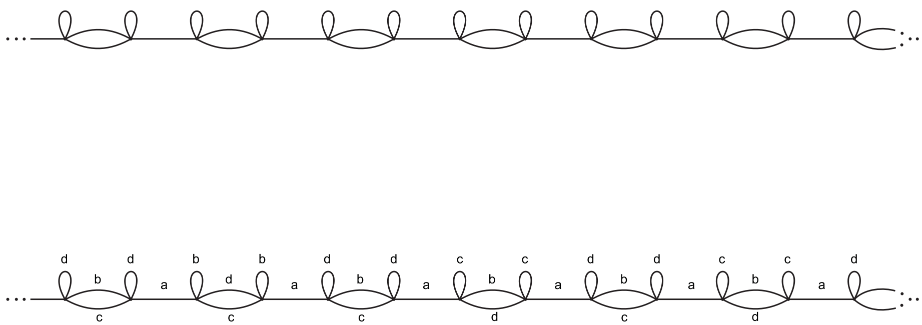


Figure 8. Two ends.

Now let $\gamma \in \partial T_2$ be any point of the boundary and v_n be a vertex belonging to the path γ and to the n-th level of the tree. Then, as mentioned above the sequence of marked graphs (Γ_n, v_n) converges in the space of marked labeled graphs to the infinite marked graph $(\Gamma(\gamma), \gamma)$ whose vertices are points of the orbit of γ and γ is the distinguished vertex of the graph.

It is known (and easy to see) that the partition of the action of $\mathcal{G}$ on ∂T_2 in orbits is the confinality partition described in Definition 1.1. Another fact is that the infinite graphs Γ_γ are all isomorphic to the graph represented in Figures either 8 or 9. In other words, these are graphs looking as a two-periodic chain with one or two ends and with a loop at each vertex (except for the left vertex in one ended case, which has three loops). More precisely, if γ is not in the orbit of point 1^∞ then Γ_γ is isomorphic to the two ended chain, while for the points in the orbit of 1^∞ the graph is the one ended chain.

The labeling of the graph depends on γ. For the case of 0^∞ it is presented in the bottom of the Figure 8 and the case of 1^∞ is presented inFigure 9.

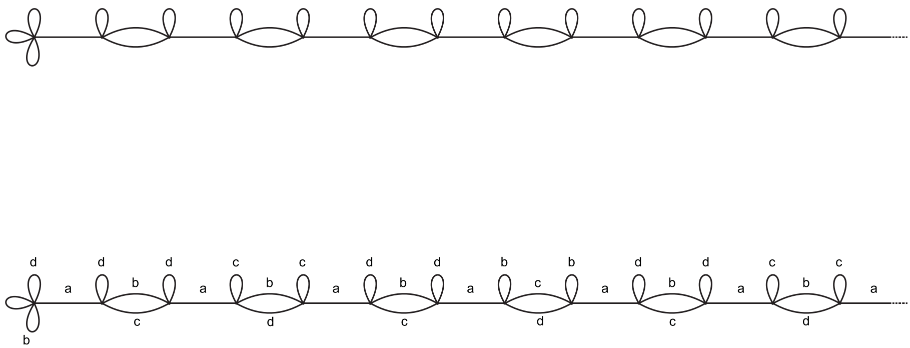


Figure 9. One end.

For the rest of the text we will choose the distinguished vertex $o = 0^\infty$ and will denote by e_n its n-th vertex 0^n. The vertex $e_3 = 000$ is on the 3-rd place according to the non-standard ordering. Generally, e_n will occupy $K(n)$ – th place according to the non-standard ordering, where

$$K(n) = \begin{cases} \frac{1}{3}(2^{n+1} + 2) & \text{if } n \text{ is odd,} \\ \frac{1}{3}(2^{n+1} + 1) & \text{if } n \text{ is even.} \end{cases}$$

Observe that $K(n)$ asymptotically behaves as $\frac{2}{3}m$, where $m = 2^n$ and $n \to \infty$. Therefore the limit of graphs (Γ_n, e_n) is the two ended chain as was already mentioned.

4. Calculation of the Green function for Γ

In this Section we shall compute the Green function of the simple random walk on the two ended Schreier graph Γ of the group $\mathcal{G}$ (more precisely on its modification obtained by deletion of loops). This is not difficult and surely the expression for the function exists somewhere in literature, but we were unable to find a reference. The computation of the

Green function can be provided by standard probabilistic methods but instead we will use a small "trick" coming from the theory of orthogonal polynomials and moments problem. The matrix $M_n^0 = 4M_n - 1$, where $4M_n$ is the “adjacency” matrix for Γ_n, has the size 2^n and has the following tridiagonal form:

$$M_n^0 = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 2 & 0 & 0 & \dots & 0 \\ 0 & 2 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & 2 & \dots & 0 \\ \hdotsfor{7} \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 2 \end{bmatrix}.$$

M_n^0 can be interpreted as the “adjacency” matrix of the graph Γ_n^0, when we remove from Γ_n one loop out each vertex. Graph Γ_n^0 is a 3-regular graph and $\frac{1}{3}M_n^0$ can be interpreted as the Markov operator on the graph Γ_n^0. Since the marked vertex $e_n = 0^n$ in the graph Γ_n^0 is on the $K(n) \approx \frac{2}{3}2^n$ place, Γ_n^0 converge to the infinite directed graph Γ^0, which determines the Markov chain X given in Figure 10.

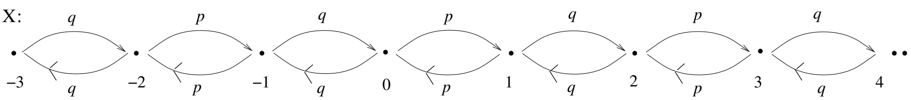


Figure 10. The Markov chain X, corresponding to Γ^0.

Here $p = 2/3, q = \frac{1}{3}$. In order to calculate the Green function (also called random walk generating function) $\varphi_X(t)$ of this Markov chain,

$$\varphi_X(t) = \sum_{n=0}^{\infty} P_{1,1}^n t^n,$$

($P_{1,1}^n$ is the probability of return to the initial point 1 of random walk after n steps) we notice that $\varphi_X(t)$ is the even part of the walk generating function $\varphi_Y(t) = \Sigma Q_{1,1}^n t^n$ of the following semi-infinite Markov chain Y given in Figure 11 below.

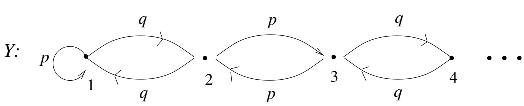


Figure 11. The Markov chain Y.

Lemma 4.1. *There is one to one correspondence between the even length paths on X starting at 1 and ending at 1 and even length paths on Y starting at 1 and ending at 1. The length and the transition probabilities under this correspondence are preserved.*

In order to get this correspondence one imagines the mirror "perpendicular" to our chain (assuming that states are at integer points on the x-axis) at $x = \frac{1}{2}$ and use the reflection of those parts of the trajectory that stay to the left of 1.

Lemma 4.1 implies that

$$\varphi_X(t) = \frac{\varphi_Y(t) + \varphi_Y(-t)}{2}. \tag{4.1}$$

Now, in order to calculate $\varphi_Y(t)$, we shall calculate the moment generating function

$$m_Y(z) = \frac{\tilde{m}_0}{z} + \frac{\tilde{m}_1}{z^2} + \cdots$$

of the matrix, which is 3-times the matrix of transition probabilities of the Markov chain Y, i.e. of the following infinite tridiagonal matrix M^0,

$$M^0 = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & \dots \\ 1 & 0 & 2 & 0 & 0 & \dots \\ 0 & 2 & 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & 0 & 2 & \dots \\ & \dots & \dots & \dots & \dots & \end{bmatrix},$$

where $\tilde{m}_n$ is the $(1,1)$ position matrix element of the $n-$th power of matrix M^0 (we assume that the rows and columns of M^0 are numerated by numbers $1, 2, \dots$ i.e. by states of the Markov chain Y). Using the well known correspondence between the moment generating function of the Jacobi matrices and the continued fraction (see, for example, [Akh65]) we have asymptotic expansion

$$m_Y(z) = \cfrac{1}{z - 2 - \cfrac{1}{z - \cfrac{4}{z - \cfrac{1}{z - \cfrac{4}{z - \dots}}}}}. \tag{4.2}$$

Consider $\psi(z) = \cfrac{1}{z - \cfrac{4}{z - \cfrac{1}{z - \cfrac{4}{z - \dots}}}}$.

Obviously,

$$\psi(z) = \cfrac{1}{z - \cfrac{4}{z - \psi(z)}}, \tag{4.3}$$

which immediately implies that $\psi(z)$ is a root of quadratic equation

$$z[\psi(z)]^2 - (z^2 - 3)\psi(z) + z = 0. \tag{4.4}$$

i.e

$$\psi(z) = \frac{z^2 - 3 - \sqrt{z^4 - 10z^2 + 9}}{2z} \tag{4.5}$$

with the appropriate choice of a branch of the square root function.

Therefore,

$$\begin{aligned} m_Y(z) &= \frac{1}{z - 2 - \psi(z)} = \frac{1}{z - 2 - \frac{z^2 - 3 - \sqrt{z^4 - 10z^2 + 9}}{2z}} \\ &= \frac{2z}{z^2 - 4z + 3 + \sqrt{z^4 - 10z^2 + 9}}. \end{aligned}$$

We will omit the index Y and write $m(z)$. Next we can calculate

$$\varphi_{M^0}(t) = \widetilde{m_0} + \widetilde{m_1}t + \widetilde{m_2}t^2 + \cdots .$$

Since

$$\begin{aligned} \varphi_{M^0}(t) &= \frac{1}{t} m\left(\frac{1}{t}\right) \\ &= \frac{\frac{2}{t^2}}{\frac{1}{t^2} - \frac{4}{t} + 3 + \sqrt{\frac{1}{t^4} - \frac{10}{t^2} + 9}} \\ &= -\frac{1}{4t}\left(1 - \sqrt{\left(\frac{1+3t}{1-3t}\right)\left(\frac{1+t}{1-t}\right)}\right), \end{aligned}$$

we get,

$$\varphi_{M^0}(t) = -\frac{1}{4t}\left(1 - \sqrt{\left(\frac{1+3t}{1-3t}\right)\left(\frac{1+t}{1-t}\right)}\right). \tag{4.6}$$

The walk generating function of the paths on Y starting at 1 and ending at $1, \varphi_Y(t)$ is related to $\varphi_{M^0}(t)$ by equation $\varphi_{M^0}(t) = \varphi_Y(3t)$. Now we are ready to calculate the walk generating function for $X, \varphi_X(t)$. Recall that $\varphi_X(t)$ is the even part of $\varphi_Y(t)$. Let $\varphi(t)$ is the even part of $\varphi_{M^0}(t)$. (It is clear that $\varphi(t) = \varphi_X(3t)$.) We have

$$\begin{aligned}\varphi(t) &= \frac{\varphi_{M^0}(t) + \varphi_{M^0}(-t)}{2} \\ &= \frac{1}{2}\left\{-\frac{1}{4t}\left[1 - \sqrt{\left(\frac{1+3t}{1-3t}\right)\left(\frac{1+t}{1-t}\right)}\right]\right. \\ &\left. + \frac{1}{4t}\left[1 - \sqrt{\left(\frac{1-3t}{1+3t}\right)\left(\frac{1-t}{1+t}\right)}\right]\right\} \\ &= \frac{1}{\sqrt{(1-3t)(1+3t)(1-t)(1+t)}}.\end{aligned}$$

Hence,

$$\varphi(t) = \varphi_X(3t) = \frac{1}{\sqrt{(1-3t)(1+3t)(1-t)(1+t)}}. \tag{4.7}$$

and we are done with the computation.

5. Proof of the result

Define $m_1(z)$ from the equation

$$m_1(z) = \frac{1}{z}\varphi\left(\frac{1}{z}\right). \tag{5.1}$$

i.e

$$m_1(z) = \frac{z}{\sqrt{(z+3)(z-3)(z-1)(z+1)}}. \tag{5.2}$$

Remark 1. There is a shortcut to obtain $m_1(z)$ from $m(z)$ directly. It is clear that $m_1(z)$ is the odd part of $m(z)$. Hence, $m_1(z) = \frac{m(z)-m(-z)}{2}$. We shall obtain the same formula for $m_1(z)$.

Recall that for any measure $d\sigma(x)$ on $\mathbb{R}$ with a compact support its Stieltjes transform $S(z)$ is defined by

$$S(z) = \int\limits_{\mathbb{R}} \frac{d\sigma(x)}{z-x}, \qquad x \in \mathbb{C}.$$

From what we have discussed follows that

$$m_1(z) = \frac{m_0}{z} + \frac{m_1}{z^2} + \cdots$$

is the Stieltjes transform of the spectral measure of the operator $4M - 1$ and initial vector $\delta_\gamma, \gamma = 0^\infty$ where M is the Markov operator of simple random walk on (two ended) graph Γ.

Theorem 1 will follow from the next two propositions. Observe first of all that in two-ended case (i.e. when $\gamma \notin Orbit(1^\infty)$) the Kesten spectral measure does not depend on γ as the change of γ corresponds (up to isomorphism of graphs) to the change of initial point in Markov chain X, but this does not lead to the change of transition probabilities and therefore does not effect the measure. Therefore, we can choose $\gamma = 0^\infty$.

Proposition 5.1. *The KNS-measure for the Markov operator M with initial vector δ_γ, $\gamma = 0^\infty$, coincides with the Kesten measure of the marked Schreier graph (Γ, γ).*

Proof. It follows immediately from the equation 5.2, the calculation of the KNS-measure in [BG00a] (the measure is presented by formula 2.1), the fact that the coincidence of Stieltjes transforms implies the coincidence of measures, and the Lemma 5.1 the proof of which is postponed to the Appendix.

□

Lemma 5.1. *Let ρ be a measure with support $B = [-3, -1] \cup [1, 3]$ given by the following density function*

$$p(x) = \frac{|x|}{\pi\sqrt{(x+3)(x+1)(x-1)(3-x)}}, \qquad x \in B.$$

Then the Stieltjes transform of ρ is $f(z) = \frac{z}{\sqrt{(z+3)(z-3)(z+1)(z-1)}}$.

Proposition 5.2. *The Jacobi matrix J^0 of the operator $4M - 1$ for the Schreier graph of the group $\mathcal{G}$ and initial vector δ_γ, $\gamma = 0^\infty$ has the following tridiagonal form*

$$J^0 = \begin{bmatrix} 0 & b_1 & 0 & 0 & \dots \\ b_1 & 0 & b_2 & 0 & \dots \\ 0 & b_2 & 0 & b_3 & \dots \\ 0 & 0 & b_3 & 0 & \dots \end{bmatrix}$$

where

$$b_1 = \sqrt{5},$$

$$b_n = \begin{cases} \sqrt{\dfrac{2^n+4}{2^n+1}} & \textit{if } n \textit{ is even,} \\ \sqrt{\dfrac{2^{n+1}+1}{2^{n-1}+1}} & \textit{if } n \textit{ is odd, } n \geq 3. \end{cases}$$

Proof. Proposition 5.2 follows from the results of [Gau84], where the Jacobian parameters are calculated for a class of densities, including density

$$d\rho(t) = \frac{\left|t-\frac{1}{2}\right|}{\pi\sqrt{t\left(t-\frac{1}{3}\right)\left(t-\frac{2}{3}\right)(1-t)}}\,dt,$$

which is different from the density (2.1) just by the affine transformation. As was already mentioned in the introduction, this measure is related to the diatomic linear model in chemistry (see [Gau84], [Whe84] and references therein).

To be more precise, let us show how to get the Jacobi parameters from the results of [Gau84]. First of all, let us mention that the Jacobi parameters, Kesten spectral measure, Green function for the random work with beginning at the distinguished point of a Scheier graph, and moments of the Kesten spectral measure determine each other. The graph (Γ, γ) is the limit of the sequence (Γ_n, v_n) where $v_n = 0^n$, the graph Γ_n looks like a chain of length 2^n, and the distinguished point v_n occupies the place $K(n) \approx \frac{2}{3}2^n$ in this chain. It is obvious that the Green functions $G_n(z)$ of (Γ_n, v_n) in some neighborhood of 0 converge pointwise to the Green function $G(z)$ of (Γ, γ) (indeed, we have the stabilization of coefficients for each power of z in the asymptotic expansion of $G(z)$). This, in fact, corresponds to the convergence of Kesten spectral measures of (Γ_n, v_n) to the Kesten spectral measure of (Γ, γ). The same holds for the moments and for the Jacobi parameters. Hence, we may start with the spectral measure given by density (5.2) and compute the corresponding Jacobi parameters. The computation was made in [Gau84] even in more general setting.

Consider the symmetric Jacobi matrix J^0 corresponding to operator $4M-1$ with the elements on upper and lower diagonals $b_1, b_2, \ldots$ (getting zeros along the main diagonal). Our b_i correspond to $\sqrt{\beta_i}$, where β_i are parameters from the Gautschi paper used there to describe the recurrent relations for the orthogonal polynomials. To see this, remind that starting with a symmetric Jacobi matrix

$$\begin{bmatrix} a_1 & b_1 & 0 & 0 & \dots \\ b_1 & a_2 & b_2 & 0 & \dots \\ 0 & b_2 & a_3 & b_3 & \dots \\ 0 & 0 & b_3 & a_4 & \dots \\ & \dots\dots\dots\dots & & & \end{bmatrix}$$

one gets a sequence of orthogonal polynomials $\{p_k(\lambda)\}$ satisfying the recurrent relation

$$b_{k-1}p_{k-2}(\lambda) + a_k p_{k-1}(\lambda) + b_k p_k(\lambda) = \lambda p_{k-1}(\lambda).$$

The corresponding $\{p_k(\lambda)\}$ are polynomials of degree k, but not monic. To be able to use the formulas from [Gau84] we have to replace $\{p_k(\lambda)\}$ by proportional monic polynomials $q_k(\lambda)$. Let $q_k(\lambda) = b_1 b_2 \cdots b_k p_k(\lambda)$. Then we have

$$b_{k-1}\frac{q_{k-2}}{b_1 \dots b_{k-2}} + a_k \frac{q_{k-1}}{b_1 \dots b_{k-1}} + b_k \frac{q_k}{b_1 \dots b_{k-1}} = \lambda \frac{q_{k-1}}{b_1 \dots b_{k-1}}.$$

After multiplication by $b_1 \dots b_{k-1}$ we obtain

$$b_{k-2}^2 q_{k-2} + a_k q_{k-1} + q_k = \lambda q_{k-1},$$

which is equivalent to

$$q_k = (\lambda - a_k)q_{k-1} - b_{k-1}^2 q_{k-2}.$$

As Gautschi's paper deals with monic orthogonal polynomials and uses the above recurrent formula, we have to take the square root of parameters β_i as was mentioned above.

Again returning to notations of Gautschi paper, we notice that in our case $r = 1/2$ for the two-sided Markov chain represented by Figure 3,

$$\xi = \frac{1-r}{1+r} = \frac{1}{3}$$

(the parameter ξ appears on page 473 of [Gau84]), and

$$\eta = \frac{1-\xi}{1+\xi} = \frac{1}{2}$$

(the definition of η is just before the formula (5.6) on page 480 of [Gau84]). Now, we use the formulas (5.6) and (5.7) from [Gau84] to find β_{2k} and β_{2k+1}, respectively. We have

$$\beta_{2k} = \frac{1}{4}\left(1 - \frac{1}{3}\right)^2 1 = \left(\left(\frac{1}{2}\right)^{2k}\right) / \left(1 + \left(\frac{1}{2}\right)^{2k}\right) = \frac{1}{9}\left(\frac{2^{2k}+4}{2^{2k}+1}\right),$$

$$\beta_1 = \frac{1}{2}\left(1 + \left(\frac{1}{3}\right)^2\right) = \frac{5}{9},$$

$$\beta_{2k+1} = \frac{1}{4}\left(1 + \left(\frac{1}{3}\right)^2\right)\left(1 + \left(\frac{1}{2}\right)^{2k+2}\right) / \left(1 + \left(\frac{1}{2}\right)^{2k}\right) = \frac{1}{9}\left(\frac{2^{2k+2}+1}{2^{2k}+1}\right)$$

which completes the proof of Proposition 5.2 and therefore the proof of Theorem 1. □

Proof. The proof of Theorem 2 is based on the same type arguments that were used for the proof of Theorem 1 and on Lemma 6.1 the proof of which is done in the Appendix. □

6. Appendix

The integrals needed for proofs of the technical lemmas can be computed using the computers software. Nevertheless, in order to be completely rigorous in our arguments, we provide in this Appendix the proof of Lemma 5.1 as well as of Lemma 6.1. We start with the proof of Lemma 5.1.

Proof. Introduce the ε-neighborhood of the set B in w-complex plane

$$B(\varepsilon) = \{w \in \mathbb{C} \mid d(w, B) \leq \varepsilon\} \qquad (0 < \varepsilon < 1).$$

Let $\partial B(\varepsilon) = \gamma = \gamma_1 \cup \gamma_2$ and

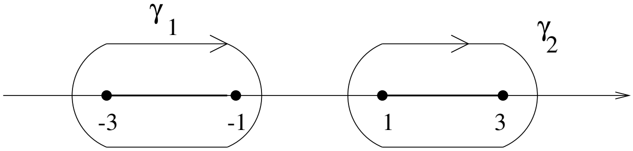


Figure 12.

let $g(z)$ be the Stieltjes transform of ρ:

$$g(z) = \int\limits_B \left(\frac{1}{z-x}\right) \frac{|x|}{\pi\sqrt{(x+3)(x+1)(x-1)(3-x)}}\, dx. \tag{6.1}$$

Introduce an auxiliary path integral $I_\varepsilon(z)$ of the function

$$h(w) = \frac{1}{(z-w)} \frac{w}{\sqrt{(w+3)(w+1)(w-1)(w-3)}},$$

namely

$$I_\varepsilon(z) = \int\limits_\gamma \left(\frac{1}{z-w}\right) \frac{w\, dw}{\sqrt{(w+3)(w+1)(w-1)(w-3)}}. \tag{6.2}$$

In order to calculate $g(z)$ defined by (6.1) we consider limit in (6.2) as ε approaches 0. On one hand, using the right hand side of (6.2) this limit turns out to be proportional to $g(z)$. On the other hand, we can calculate the path integral (6.2) using the residue calculus and then take limit as $\varepsilon \to 0$. We will find the Stieltjes transform of the measure ρ.

In order to calculate $I_\varepsilon(z)$ using the residue calculus, we will apply Cauchy's Integral formula for the open region $\mathcal{D}$ in w-plane with the boundary $\gamma \cup C_R \cup C_r$. Here C_R is the circle with center at 0 of radius (big enough) R, C_r is the circle of radius (small enough) r with center at z (see Figure 12).

Before we will proceed further we need to fix which branch of $h(w)$ we are considering in $\mathcal{D}$ and check that this branch defines holomorphic function in $\mathcal{D}$. Since

$$\begin{aligned} h(w) &= \left(\frac{1}{z-w}\right) \frac{w}{\sqrt{(w+3)(w+1)(w-1)(w-3)}} \\ &= \frac{1}{(z-w)} \frac{w}{\sqrt{w+3}\cdot\sqrt{w+1}\cdot\sqrt{w-1}\cdot\sqrt{w-3}} \end{aligned}$$

we need to choose branches for each square root in the denominator. We indicate the branch for $\sqrt{w}$ in the complex plane with the cut along the negative part of the real part: if $w = re^{i\theta}$ where θ is the angle from positive real axis to w, $r = |w|$, then $\sqrt{w} = \sqrt{r}\, e^{\frac{i\theta}{2}}$. Similarly we choose branches for each square root with obvious modification. For such identified branches of $h(w)$ we can check that $h(w)$ is a holomorphic function on $\mathcal{D}$. For example, let us assume that we choose the path τ that goes only along γ_1 but not γ_2, starting with the point P on the real axis.

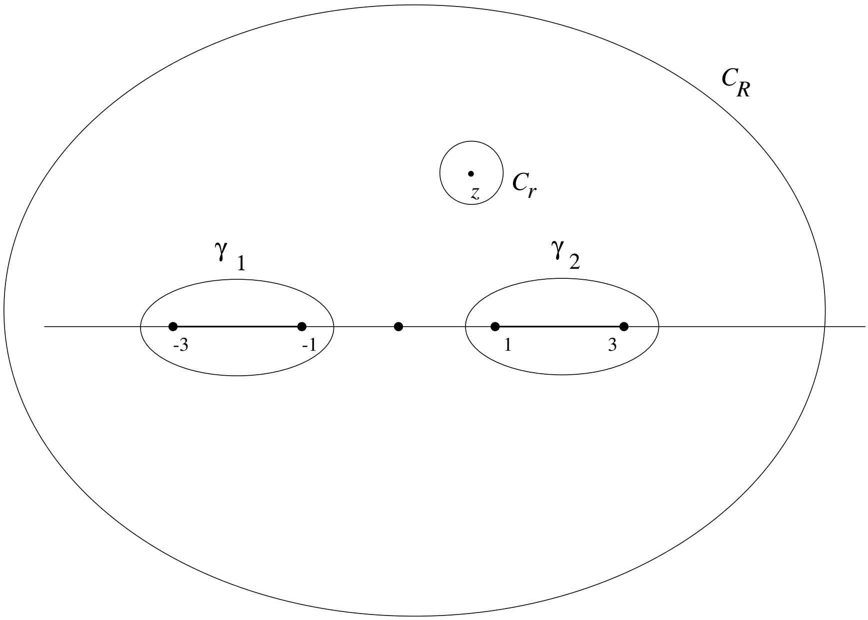


Figure 13.

$\sqrt{w-1}$ and $\sqrt{w-3}$ are holomorphic functions on τ, but both $\sqrt{w+1}$ and $\sqrt{w+3}$ are changing sign when we make their continuation along τ. Since $h(w)$ contains their product $\sqrt{w+3}\cdot\sqrt{w+1}$, it will not change. The same is true for a path that goes along γ_2, since in this case $\sqrt{w+3}$ and $\sqrt{w+1}$ will not change, only both $\sqrt{w-1}$ and $\sqrt{w-3}$ will change. Again, since they participate in $h(w)$ as a product, the function $h(w)$ will not change. Hence, the Monodromy Theorem guarantees that $h(w)$ with this choice of branches for the square root is holomorphic in $\mathcal{D}$.

We can apply the Cauchy Integral Formula for the region $\mathcal{D}$ and a holomorphic function $h(w)$ in $\mathcal{D}$:

$$\int_{C_R} h(w)dw = \int_{\gamma} h(w)\ dw + \int_{C_r} h(w)\ dw. \tag{6.3}$$

Now

$$\left|\int_{C_R} h(w)\ dw\right| \le \int_{C_R} \frac{|w|\ dw}{|(z-w)||\sqrt{(w+3)(w+1)(w-1)(w-3)}|}$$
$$\sim \frac{R}{R\cdot R^2}\cdot 2nR = \frac{2\pi}{R}$$

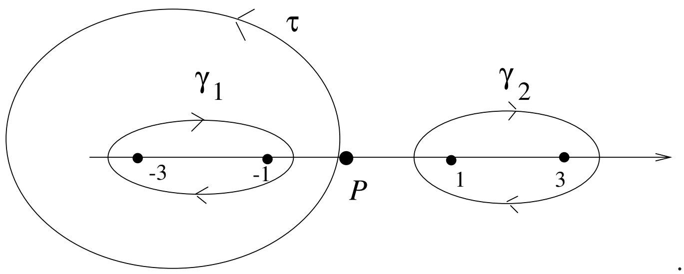


Figure 14.

and approaches 0 as $R \to +\infty$. From (6.3) it follows that

$$\int_{\gamma} h(w)\ dw = -\int_{C_r} h(w)\ dw. \tag{6.4}$$

Since the disc with boundary C_r contains only one singular point of $h(w)$, namely, $w = z$,

$$\begin{aligned}\int_{C_r} h(w)\ dw &= -2\pi i \cdot \ \text{Res}\ h(w)|_{w=z} \\ &= -2\pi i(-1)\frac{w}{\sqrt{(w+3)(w+1)(w-1)(w-3)}}\Big|_{w=z} \\ &= 2\pi i \frac{z}{\sqrt{(z+3)(z+1)(z-1)(z-3)}}.\end{aligned}$$

From (6.4) we have

$$\int_{\gamma} h(w)\ dw = -2\pi i \frac{z}{\sqrt{(z+3)(z+1)(z-1)(z-3)}}. \tag{6.5}$$

Letus investigate now the limit $\lim\limits_{\varepsilon \to 0} I(z)$

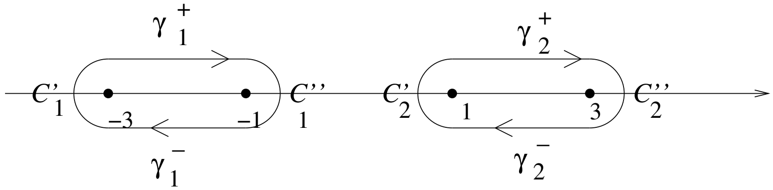


Figure 15.

Since, $I(z) = \int_{\gamma_1} h(w)\ dw + \int_{\gamma_2} h(w)\ dw$, we compute the limit for each $\int_{\gamma_i} h(w)\ dw$ $(i = 1, 2)$. We have

$$\int_{\gamma_1} h(w)\ dw = \int_{C_1'} h(w)\ dw + \int_{\gamma_1^+} h(w)\ dw + \int_{C_1''} h(w)\ dw + \int_{\gamma_1^-} h(w)\ dw. \quad (6.6)$$

First, we show that the integrals over C_1' and C_1'' converge to 0 as $\varepsilon \to 0$. We have for C_1':

$$\left| \int_{C_1'} h(w)\ dw \right| \leq \max_{C_1'} |h(w)| \text{ length } C_1' = \max_{C_1'} |h(w)| \cdot \pi\varepsilon.$$

Now $|h(w)| = \frac{1}{|\sqrt{w+3}|} \cdot \left| \frac{w}{(z-w)\sqrt{(w+1)(w-1)(w-3)}} \right| \leq \frac{1}{\sqrt{\varepsilon}} M_1'(\varepsilon)$, where $M_1'(\varepsilon) = \max_{C_1'} \left| \frac{w}{(z-w)\sqrt{(w+1)(w-1)(w-3)}} \right|$. Since the function $\frac{w}{(z-w)\sqrt{(w+1)(w-1)(w-3)}}$ is holomorphic at $w = -3$ we have $M_1'(\varepsilon) \leq M_1'$ (some constant) for $0 < \varepsilon \leq \frac{1}{2}$. Hence $\left| \int_{C_1'} h(w)\ dw \right| \leq \frac{1}{\sqrt{\varepsilon}} \cdot M_1' \cdot \pi\varepsilon = \pi M_1' \sqrt{\varepsilon}$. Therefore, $\lim_{\varepsilon \to 0} \int_{C_1'} h(w)\ dw = 0$. Similarly, $\lim_{\varepsilon \to 0} \int_{C_1''} h(w)\ dw = 0$. Now, $\lim_{\varepsilon \to 0} \int_{\gamma_1^+} h(w)\ dw = \int_{-3}^{-1} [\lim_{\varepsilon \to 0} h(x + i\varepsilon)] dx$. Note that

$$\begin{aligned}
\lim_{\varepsilon \to 0} h(x + i\varepsilon) &= \frac{x}{(z-x)\sqrt{x+3} \cdot \sqrt{-(x+1)} \cdot i \cdot \sqrt{-(x-1)} \cdot i \cdot \sqrt{-(x-3)} \cdot i} \\
&= \frac{x}{(z-x) i^3 \sqrt{(x+3)(x+1)(x-1)(3-x)}} \\
&= (-i) \frac{-x}{(z-x)\sqrt{(x+3)(x+1)(x-1)(3-x)}} \\
&= (-i) \frac{|x|}{(z-x)\sqrt{(x+3)(x+1)(x-1)(3-x)}}.
\end{aligned}$$

On the other hand, for $x \in [-3, -1]$ $\lim_{\varepsilon \to 0} h(x - i\varepsilon) = -\lim_{\varepsilon \to 0} h(x + i\varepsilon)$, since 3 branches, namely $\frac{1}{\sqrt{w+1}}$, $\frac{1}{\sqrt{w-1}}$, and $\frac{1}{\sqrt{w-3}}$ change their signs. Because the direction of γ_1^- is the opposite to the direction of γ_1^+, we immediately obtain that

$$\lim_{\varepsilon \to 0} \int_{\gamma_1^-} h(w)\ dw = \lim_{\varepsilon \to 0} \int_{\gamma_1^+} h(w)\ dw,$$

and

$$\lim_{\varepsilon\to 0}\int_{\gamma_1} h(w)\ dw = -2i\int_{-3}^{-1}\frac{|x|\ dx}{(z-x)\sqrt{(x+3)(x+1)(x-1)(3-x)}}. \tag{6.7}$$

Similarly, we obtain

$$\lim_{\varepsilon\to 0}\int_{\gamma_2} h(w)\ dw = -2i\int_{1}^{3}\frac{|x|\ dx}{(z-x)\sqrt{(x+3)(x+1)(x-1)(3-x)}}, \tag{6.8}$$

and

$$\begin{aligned}\lim_{\varepsilon\to 0} h(x+i\varepsilon) &= \frac{x}{(z-x)\sqrt{x+3}\sqrt{x+1}\cdot\sqrt{x-1}\cdot i\sqrt{-(x-3)}}\\ &= -i\frac{x}{(z-x)\sqrt{(x+3)(x+1)(x-1)(3-x)}}\\ &= -i\frac{|x|}{(z-x)\sqrt{(x+3)(x+1)(x-1)(3-x)}}\end{aligned}$$

(since $x>0$ in this case).) Therefore

$$\lim_{\varepsilon\to 0}\int_{\gamma} h(w)\ dw = -2i\pi\cdot g(z). \tag{6.9}$$

Comparing (6.5) and (6.9) we derive

$$f(z) = \frac{z}{\sqrt{(z+3)(z+1)(z-1)(z-3)}}, \tag{6.10}$$

as required. □

Lemma 6.1. *The spectral measure of the operator $4M-1$ and the initial vector δ_η, where M is the Markov operator on the one-ended Screier graph corresponding to the orbit of the point $\eta = 1^\infty$ is the continuous measure with the following density function:*

$$p_1(x) = \frac{1}{4\pi}\sqrt{\frac{(x+1)(x+3)}{(x-1)(3-x)}}, \qquad x\in[-3,-1]\cup[1,3].$$

Proof. The proof is similar to the proof of Lemma 5.1 with some modifications. We need to prove that the Stieltjes transform of the measure $p_1(x)\ dx$ is the function computed in section 4

$$m(z) = -\frac{1}{4}\left(1-\sqrt{\frac{(z+1)(z+3)}{(z-1)(z-3)}}\right) = \frac{2z}{z^2-4z+3+\sqrt{z^4-10z^2+9}}.$$

Consider

$$I_1(z) = \int_\gamma h_1(w)\ dx, \tag{6.11}$$

where

$$h_1(w) = \frac{1}{(z-w)}\sqrt{\frac{(w+1)(w+3)}{(w-1)(w-3)}}. \tag{6.12}$$

Then we follow exactly the same scheme as in the proof of Lemma 5.1. The only difference is that the function $h_1(w)$ is not regular any more at $w=\infty$, and that is why we need to calculate $\int\limits_{C_R} h_1(w)\ dw$ using the residue of $h_1(w)$ at $w=\infty$. Using substitution $w=\frac{1}{u}$ we have

$$\begin{aligned} h_1\left(\frac{1}{u}\right) &= \frac{1}{\left(z-\frac{1}{u}\right)}\sqrt{\frac{\left(\frac{1}{u}+1\right)\left(\frac{1}{u}+3\right)}{\left(\frac{1}{u}-1\right)\left(\frac{1}{u}-3\right)}} \\ &= \frac{1}{\left(z-\frac{1}{u}\right)}\sqrt{\frac{(1+u)(1+3u)}{(1-u)(1-3u)}} = \frac{u}{(uz-1)}\sqrt{\frac{(1+u)(1+3u)}{(1-u)(1-3u)}}. \end{aligned}$$

Now

$$\begin{aligned} h_1(w)\ dw &= h_1\left(\frac{1}{u}\right) d\left(\frac{1}{u}\right) = h_1\left(\frac{1}{u}\right)\left(-\frac{1}{u^2}\right) du \\ &= -\frac{1}{u(uz-1)}\sqrt{\frac{(1+u)(1+3u)}{(1-u)(1-3u)}}\ du \\ \operatorname{Res}\ h_1\left(\frac{1}{u}\right) d\left(\frac{1}{u}\right)\Big|_{u=0} &= -\left[\frac{1}{uz-1}\sqrt{\frac{(1+u)(1+3u)}{(1-u)(1-3u)}}\right]\Bigg|_{u=0} \\ &= 1. \end{aligned}$$

Hence, $\int\limits_{C_R} h_1(w)\ dw = +2\pi i$. Therefore,

$$\int_\gamma h_1(w)\ dw = \int_{C_R} h_1(w)\ dw - \int_{C_r} h_1(w)\ dw = +2\pi i - \int_{C_r} h_1(w)\ dw. \tag{6.13}$$

Now

$$\int_{C_r} h_1(w)\ dw = -2\pi i \cdot \operatorname{res}\ h_1(w)|_{w=z}$$

$$= -2\pi i \cdot (-1)\sqrt{\frac{(w+1)(w+3)}{(w-1)(w-3)}}\Big|_{w=z}$$

$$= 2\pi\sqrt{\frac{(z+1)(z+3)}{(z-1)(z-3)}}.$$

Hence,

$$I_1(z) = \int_\gamma h_1(w)\ dw = +2\pi i - 2\pi i\sqrt{\frac{(z+1)(z+3)}{(z-1)(z-3)}}. \tag{6.14}$$

On the other hand,

$$\begin{aligned}\lim_{\varepsilon\to 0} I_1(z) &= 2\int_{-3}^{-1} \frac{1}{(z-x)} \frac{i\sqrt{-(x+1)}\cdot\sqrt{x+3}}{i\sqrt{-(x-1)}\cdot i\sqrt{-(x-3)}}\ dx \\ &\quad + 2\int_{1}^{3} \frac{1}{(z-x)} \frac{\sqrt{x+1}\cdot\sqrt{x+3}}{\sqrt{x-1}\cdot i\sqrt{-(x-3)}}\ dx \\ &= -2i\left[\int_{-3}^{-1} \frac{1}{(z-x)}\sqrt{\left(\frac{x+1}{x-1}\right)\left(\frac{x+3}{3-x}\right)}\ dx\right. \\ &\quad \left. + \int_{1}^{3} \frac{1}{(z-x)}\sqrt{\left(\frac{x+1}{x-1}\right)\left(\frac{x+3}{3-x}\right)}\ dx\right] \\ &= -2i\cdot 4\pi S_1(z) = -8\pi i m(z),\end{aligned}$$

where $m(z)$ is the Stieltjes transform of the measure μ_1. Comparing the last equation with (6.14) we obtain

$$+2\pi i - 2\pi i\sqrt{\frac{(z+1)(z+3)}{(z-1)(z-3)}} = -8\pi i m(z),$$

and hence,

$$m(z) = -\frac{1}{4} + \frac{1}{4}\sqrt{\frac{(z+1)(z+3)}{(z-1)(z-3)}} = -\frac{1}{4}\left[1 - \sqrt{\frac{(z+1)(z+3)}{(z-1)(z-3)}}\right] \tag{6.15}$$

as required. □

References

[Akh65] N. I. Akhiezer, *The classical moment problem and some related questions in analysis*, Translated by N. Kemmer, Hafner Publishing Co., New York, 1965. MR MR0184042 (32 #1518)

[Ale72] S. V. Alešin, *Finite automata and the Burnside problem for periodic groups*, Mat. Zametki **11** (1972), 319–328. MR MR0301107 (46 #265)

[Aus09] T. Austin, *Rational group ring elements with kernels having irrational dimension*, (available at *http://arxiv.org/abs/math.GR/0909.2360*), 2009.

[BG00a] L. Bartholdi and R. I. Grigorchuk, *On the spectrum of Hecke type operators related to some fractal groups*, Tr. Mat. Inst. Steklova **231** (2000), no. Din. Sist., Avtom. i Beskon. Gruppy, 5–45. MR MR1841750 (2002d:37017)

[BG00b] Laurent Bartholdi and Rostislav I. Grigorchuk, *Spectra of non-commutative dynamical systems and graphs related to fractal groups*, C. R. Acad. Sci. Paris Sér. I Math. **331** (2000), no. 6, 429–434. MR MR1792481 (2001i:37012)

[BGK+07] Ye. V. Bondarenko, R. I. Grigorchuk, R.V. Kravchenko, Y. V. Muntyan, V. V. Nekrashevich, D. M. Savchuk, and Z. Sunik, *On classification of the groups, which are generated by automatons with three states on alphabet with two letters, and on some questions concerned with such groups*, Naukovy Visnyk Chernivetskogo Universytetu: Zbirnyk Naukovyh Prats **336-337** (2007).

[BGN03] Laurent Bartholdi, Rostislav Grigorchuk, and Volodymyr Nekrashevych, *From fractal groups to fractal sets*, Fractals in Graz 2001, Trends Math., Birkhäuser, Basel, 2003, pp. 25–118. MR MR2091700

[Bri53] L. Brillouin, *Wave propagation in periodic structures. Electric filters and crystal lattices*, Dover Publications Inc., New York, N. Y., 1953, 2d ed. MR MR0052978 (14,704a)

[Gau84] Walter Gautschi, *On some orthogonal polynomials of interest in theoretical chemistry*, BIT **24** (1984), no. 4, 473–483. MR MR764820 (86d:65030)

[GL02] Rostislav I. Grigorchuk and Igor Lysenok, *Burnside problem*, The Consice handbook of algebra, A.V. Mikhalev and Gunter F. Pilz (eds.), Kluwer Academic Publishers, 2002, pp. 111–115.

[GLSŻ00] Rostislav I. Grigorchuk, Peter Linnell, Thomas Schick, and Andrzej Żuk, *On a question of Atiyah*, C. R. Acad. Sci. Paris Sér. I Math. **331** (2000), no. 9, 663–668. MR MR1797748 (2001m:57050)

[GN07] Rostislav Grigorchuk and Volodymyr Nekrashevych, *Self-similar groups, operator algebras and Schur complement*, J. Mod. Dyn. **1** (2007), no. 3, 323–370. MR MR2318495

[GNS00a] R. I. Grigorchuk, V. V. Nekrashevich, and V. I. Sushchanskiĭ, *Automata, dynamical systems, and groups*, Tr. Mat. Inst. Steklova **231** (2000), no. Din. Sist., Avtom. i Beskon. Gruppy, 134–214. MR MR1841755 (2002m:37016)

[GNS00b] ———, *Automata, dynamical systems, and infinite groups*, Tr. Mat. Inst. Steklova **231** (2000), no. Din. Sist., Avtom. i Beskon. Gruppy, 134–214. MR MR1841755 (2002m:37016)

[Gol64] E. S. Golod, *On nil-algebras and finitely approximable p-groups*, Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 273–276. MR MR0161878 (28 #5082)

[Gra09] I. Grabowski, *On the atiyah problem for the lamplighter groups*, (available at *http://arxiv.org/abs/math.GR/1009.0229*), 2009.

[Gri80] R. I. Grigorčuk, *On Burnside's problem on periodic groups*, Funktsional. Anal. i Prilozhen. **14** (1980), no. 1, 53–54. MR MR565099 (81m:20045)

[Gri83] R. I. Grigorchuk, *On the Milnor problem of group growth*, Dokl. Akad. Nauk SSSR **271** (1983), no. 1, 30–33. MR MR712546 (85g:20042)

[Gri84] ———, *Degrees of growth of finitely generated groups and the theory of invariant means*, Izv. Akad. Nauk SSSR Ser. Mat. **48** (1984), no. 5, 939–985. MR MR764305 (86h:20041)

[Gri88] ———, *Semigroups with cancellations of degree growth*, Mat. Zametki **43** (1988), no. 3, 305–319, 428. MR MR941053 (89f:20065)

[Gri00a] ———, *Branch groups*, Mat. Zametki **67** (2000), no. 6, 852–858. MR MR1820639 (2001i:20057)

[Gri00b] ———, *Just infinite branch groups*, New horizons in pro-p groups, Progr. Math., vol. 184, Birkhäuser Boston, Boston, MA, 2000, pp. 121–179. MR MR1765119 (2002f:20044)

[Gri05] Rostislav Grigorchuk, *Solved and unsolved problems around one group*, Infinite groups: geometric, combinatorial and dynamical aspects, Progr. Math., vol. 248, Birkhäuser, Basel, 2005, pp. 117–218. MR MR2195454

[Gri11] R. I. Grigorchuk, *Some topics of group actions on rooted trees*, The Proceedings of the Steklov Institute of Math. **273** (2011), 64–175.

[GŠ06] Rostislav Grigorchuk and Zoran Šunik̀, *Asymptotic aspects of Schreier graphs and Hanoi Towers groups*, C. R. Math. Acad. Sci. Paris **342** (2006), no. 8, 545–550. MR MR2217913 (2006k:20048)

[GŠ07] ———, *Self-similarity and branching in group theory*, Groups St. Andrews 2005, I, London Math. Soc. Lecture Note Ser., vol. 339, Cambridge Univ. Press, Cambridge, 2007, pp. 36–95.

[GSŠ07a] Rostislav Grigorchuk, Dmytro Savchuk, and Zoran Šunić, *The spectral problem, substitutions and iterated monodromy*, Probability and mathematical physics, CRM Proc. Lecture Notes, vol. 42, Amer. Math. Soc., Providence, RI, 2007, pp. 225–248. MR 2352271 (2008h:20040)

[GSŠ07b] Rostislav Grigorchuk, Dmytro Savchuk, and Zoran Šunik̀, *The spectral problem, substitutions and iterated monodromy*, Probability and Mathematical Physics, A Volume in Honor of Stanislav Molchanov, CRM Proceedings and Lecture Notes, vol. 42, American Mathematical Society, Providence, Rhode Island USA, 2007, pp. 225–248.

[GŻ99] Rostislav I. Grigorchuk and Andrzej Żuk, *On the asymptotic spectrum of random walks on infinite families of graphs*, Random walks and discrete potential theory (Cortona, 1997), Sympos. Math., XXXIX, Cambridge Univ. Press, Cambridge, 1999, pp. 188–204. MR MR1802431 (2002e:60118)

[GŻ01] ———, *The lamplighter group as a group generated by a 2-state automaton, and its spectrum*, Geom. Dedicata **87** (2001), no. 1-3, 209–244. MR MR1866850 (2002j:60009)

[GŻ04] ______, *The Ihara zeta function of infinite graphs, the KNS spectral measure and integrable maps*, Random walks and geometry, Walter de Gruyter GmbH & Co. KG, Berlin, 2004, pp. 141–180. MR MR2087782

[Hoř63] J. Hořeiš, *Transformations defined by finite automata (in Russian)*, Problemy Kibernetiki **9** (1963), 23–26.

[Kes59] Harry Kesten, *Symmetric random walks on groups*, Trans. Amer. Math. Soc. **92** (1959), 336–354. MR MR0109367 (22 #253)

[LW10] F. Lehner and S. Wagner, *Free lamplighter groups and a question of atiyah*, (available at *http://arxiv.org/abs/math.GR/1005.2347*), 2010.

[Nek05] Volodymyr Nekrashevych, *Self-similar groups*, Mathematical Surveys and Monographs, vol. 117, American Mathematical Society, Providence, RI, 2005. MR MR2162164 (2006e:20047)

[PSZ10] M. Pichot, T. Schick, and A. Zuk, *Closed manifolds with transcendental l2-betti numbers*, (available at *http://arxiv.org/abs/math.GR/1005.1147*), 2010.

[Ser97] Jean-Pierre Serre, *Répartition asymptotique des valeurs propres de l'opérateur de Hecke T_p*, J. Amer. Math. Soc. **10** (1997), no. 1, 75–102. MR MR1396897 (97h:11048)

[vN29] John von Neumann, *Zur allgemeinen Theorie des Masses*, Fund. Math. **13** (1929), 73–116 and 333, = *Collected works*, vol. I, pages 599–643.

[Whe84] John C. Wheeler, *Modified moments and continued fraction coefficients for the diatomic linear chain*, The Journal of Chemical Physics **80** (1984), no. 1, 472–476.

Contact information

R. I. Grigorchuk Department of Mathematics, Mailstop 3368 Texas A&M University College Station, TX 77843-3368, USA
E-Mail: grigorch@math.tamu.edu

Ya. S. Krylyuk Mathematics Department, De Anza College, 21250 Stevens Creek Blvd, Cupertino, CA 95014, USA

Received by the editors: 03.04.2012
and in final form 03.04.2012.