Algebra and Discrete Mathematics RESEARCH ARTICLE
Volume 13 **(2012)**. Number 2. pp. 209 – 219

One branch curve singularities with at most 2-parameter families of ideals

Yuriy A. Drozd, Ruslan V. Skuratovskii

Abstract. A criterion is given in order that the ideals of a one branch curve singularity form at most 2-parameter families. Namely, we present a list of plane curve singularities from the Arnold's classification which are the smallest among all one branch singularities having at most 2-parameter families of ideals.

Introduction

Ideals of commutative rings have been studied at least since the works of Dedekind on the ideals of algebraic numbers. The Dedekind domains, i.e. integrally closed noetherian domains of Krull dimension 1, are just domains such that all their ideals are invertible. If a domain A is not integrally closed, the theory of ideals becomes rather complicated. As it was noticed by Bass [2] and, independently, by Borevich and Faddeev [4], if A is of Krull dimension 1 and its integral closure R has 2 generators as A-module, every ideal is invertible over its multiplication ring (and vice versa). Moreover, in this case all finitely generated torsion free A-modules are direct sums of ideals. Jacobinski [13] and, independently, Drozd and Roiter [9] gave criteria for a commutative ring of Krull dimension 1 to have finitely many nonisomorphic torsion free modules. It so happens that it is also the case when it has finitely many ideal classes. As Greuel and Knörrer showed, in the local case these are just the rings dominating the *simple plane curve singularities A-D-E* of Arnold [1]. Schappert [14],

2010 MSC: Primary 13C05, Secondary 14H20.
Key words and phrases: Curve singularity, ideal, family of ideals, sandwich technique.

Drozd and Greuel [8] proved that a local ring of plane curve singularity only has 1-parameter families of ideals if and only if this singularity dominates a *strictly unimodal* plane curve singularity (in [1] they are called *unimodal* and *bimodal*). This time it is no more the case that torsion free modules behave in the same manner. Among strictly unimodal plane curve singularities only those of type T_{pq} are *tame*, i.e. only have 1-parameter families of indecomposable torsion free modules [7]. All others are *wild*, so have n-parameter families of nonisomorphic indecomposable torsion free modules for arbitrary n.

In this paper we find a criterion for a one branch curve singularity to have at most 2-parameter families of ideals. It so happens that such singularities can also be characterized using the Arnold lists from [1, Section 15.1]. Namely, they are just those dominating one of the singularities of type E_{30}, E_{32}, W_{24}, $W^{\sharp}_{2,*}$, W_{30}, N_{20}, N_{24} or N_{28} (see Theorem 1). To prove this result we use the "sandwich" technique, just as in the papers cited above. Certainly, the "one branch" condition is rather restrictive and one would like to get rid of it, but even in this case the calculations are cumbersome, so we had to restrict our ambition.

1. Main Theorem

We fix an algebraically closed field $\mathbf{k}$.

Definition 1. A *one branch curve singularity* is a complete local noetherian $\mathbf{k}$-algebra S of Krull dimension 1 without zero divisors and such that $S/\mathfrak{m} = \mathbf{k}$, where $\mathfrak{m}$ is the maximal ideal of S. It is called *plane* if $\mathfrak{m}$ is generated by 2 elements.

Such an algebra is indeed isomorphic to the completion of a local ring of a (singular) point p of an algebraic curve X over $\mathbf{k}$; this curve can be chosen plane if so is the singularity. Moreover, the curve X is irreducible in the formal neighbourhood of the point p, or, the same, p belongs to a unique branch (place, formal component) of X in the sense of [12] or [16].

It is known that the *normalization* of a one branch curve singularity S is isomorphic to the algebra $R = \mathbf{k}[[t]]$ of formal power series and R is finitely generated as S-module. So we always suppose that $t^r R \subset S \subset R$ for some r. For every element $x \in R$ let $v(x)$ be its *valuation*, i.e. $x = t^{v(x)}u$, where u is invertible in R. If S is plane and $\mathfrak{m} = (x, y)$, one can always suppose that $v(x) < v(y)$ and $v(x) \nmid v(y)$. Then we call the pair $\mathbf{v} = (v_1, v_2)$ the *valuation vector* of S. Obviously, it does not depend on the choice of such generators. Note that every plane curve singularity is Gorenstein [3].

We recall the definition of the *parameter number of ideals* $\operatorname{par}(S) = \operatorname{par}(1, S)$ from [6, 8]. Remark first that every ideal of S is isomorphic to an S-submodule $M \subseteq R$ containing S [5]. Let $\mathrm{B}(d)$ be the closed subset of the Grassmannian $\mathrm{Gr}(d, R/S)$ consisting of those spaces which are S-submodules. Every point $b \in \mathrm{B}(d)$ gives rise to an S-submodule $M(b)$ of R which contains S. We set

$$O(b) = \{\, b' \in \mathrm{B}(d) \mid M(b') \simeq M(b) \,\},$$
$$\mathrm{B}(d, k) = \{\, b \in \mathrm{B}(d) \mid \dim O(b) = k \,\}$$

and

$$\operatorname{par}(S) = \max_{d,k} \{\, \dim \mathrm{B}(d, k) - k \,\}.$$

Note that both $O(b)$ and $\mathrm{B}(d, k)$ are locally closed subsets in $\mathrm{B}(d)$. Intuitively, $\operatorname{par}(S)$ is the biggest possible number of independent parameters that define isomorphism classes of S-ideals.

Definition 2. Let S be a one branch plane curve singularity, $\mathbf{v}$ be its valuation vector. We say that S is

- *of type* E_{6k} if $\mathbf{v} = (3, 3k + 1)$,
- *of type* E_{6k+2} if $\mathbf{v} = (3, 3k + 2)$,
- *of type* W_{6k} if $\mathbf{v} = (4, 2k + 1)$,
- *of type* $W^{\sharp}_{k,*}$ if $\mathbf{v} = (4, 4k + 2)$,
- *of type* N_{4k} if $\mathbf{v} = (5, k + 1)$.

Remark. In [8] it is shown that, if $\operatorname{char} \mathbf{k} = 0$, our definitions of singularities of types E and W are equivalent to those given in [1, § 15] in terms of the normal forms of equations. We do not precise the equations for singularities of type N, since they are complicated and we do not use them.

They say that a singularity S' *dominates* the singularity S, or is an *over-ring* of S, if $S \subseteq S' \subseteq R$.

Theorem 1. *Let S be a one branch curve singularity. The following conditions are equivalent:*

1) $\operatorname{par}(S) \leq 2$.

2) *Either* $\operatorname{char}\mathbf{k} \neq 2$ *and* S *dominates one of the following singularities:*

$$E_{30},\ E_{32},\ W_{24},\ W^{\sharp}_{2,*},\ W_{30},\ N_{20},\ N_{24},\ N_{28},$$

or $\operatorname{char}\mathbf{k} = 2$ *and* S *dominates one of the following singularities:*

$$E_{30},\ E_{32},\ W_{18},\ W^{\sharp}_{1,*},\ N_{20},\ N_{24}.$$

Proof. We suppose that $\operatorname{char}\mathbf{k} \neq 2$. If $\operatorname{char}\mathbf{k} = 2$, the calculations are quite similar (even easier, since less cases must be considered). As usually, we denote by $\langle\, a_1, a_2, \ldots, a_m \,\rangle$ the vector space (over $\mathbf{k}$) generated by $a_1, a_2, \ldots, a_m$.

Let $m = \dim R/\mathfrak{m}R$. It is known that also $\dim I/\mathfrak{m}I \leq m$ for all ideals I of S [5]. If $m > 5$, then the same observations as in [8, § 2.2] show that $\operatorname{par}(S) \geq 3$. If $m = 2$, S is a Bass ring [5], so has finitely many ideals up to isomorphism. For $m = 3$ the result follows from [10, Theorem 4.1]. For $m = 4$ it was proven in [15]. Hence, we only have to consider the case $m = 5$. Then S contains an element x with $v(x) = 5$, so we deal with singularities of type N. In section 2 we will calculate the ideals of the rings of types N_{4k} ($k \leq 7$) and show that there are at most 2-parameter families in these cases. Therefore, we must show now that $\operatorname{par}(1, S) \geq 3$ if $m = 5$ and S does not contain any element y with $6 \leq v(y) \leq 8$. If it is the case, $S \subseteq S_0$, where $S_0 = \mathbf{k} + \mathbf{k}x + t^9R$. The maximal ideal of S_0 is $\mathfrak{m}_0 = \mathbf{k}x + t^9R$. One easily checks that the S-ideals

$$I(\alpha, \beta, \gamma) = \langle\, 1, t + \alpha t^3 + \beta t^4 + \gamma t^8 \,\rangle + \mathfrak{m}_0, \text{ where } \alpha, \beta, \gamma \in \mathbf{k},$$

are pairwise non-isomorphic. It implies that $\operatorname{par}(1, S_0) \geq 3$, hence, $\operatorname{par}(1, S) \geq 3$ for every $S \subseteq S_0$. □

We recall the *sandwich procedure* used for calculation of ideals [5, 8]. Let S be a curve singularity, $\mathfrak{m} = \operatorname{rad} S$ and $S' = \operatorname{End}\mathfrak{m}$. We consider S' as an over-ring of S and set $\bar{S} = S'/\mathfrak{m}$. If I is an S-ideal, then $I' = S'I$ is an S'-ideal and $I' \supseteq I \supset \mathfrak{m}I = \mathfrak{m}I'$. So I is defined by the subspace $V = I/\mathfrak{m}I$ of the $\bar{S}$-module $W = I'/\mathfrak{m}I'$. This subspace is not arbitrary, but *generating* in the sense that $\bar{S}V = W$. Let $E = \operatorname{End} I'$, $E_0 = \{\, a \in E \mid aI' \subseteq \mathfrak{m}I' \,\}$ and $\bar{E} = E/E_0$. Then W is a $\bar{E}$-module As it was mentioned above, we can and always will suppose that $R \supseteq I \supseteq S$, thus $R \supseteq I' \supseteq S'$. Then $E \subseteq I'$, so $\bar{E} \subseteq W$, and two generating subspaces $V, V' \subseteq W$ define isomorphic ideals if and only if $V' = aV$ for an element $a \in \bar{E}$. Moreover, since we only consider subspaces containing the class of 1 (which we also

denote by 1), such element a belongs to V'. Let $\bar{W} = I'/\mathfrak{m}'I'$, where $\mathfrak{m}' = \operatorname{rad} S'$. Then the subspace $V \subseteq W$ is generating if and only if its image in $\bar{W}$ is the whole $\bar{W}$. Therefore, if $\dim \bar{W} = m = \dim R/\mathfrak{m}R$, then $\mathfrak{m}'I' = \mathfrak{m}I'$, hence the unique generating subspace of W is W itself, so the unique S-ideal I with $S'I = I'$ is I'. In the further calculations we will not consider such S'-ideals at all. The case $V = W$ will also be omitted, since then $I = I'$.

2. Description of ideals of singularities of type N

Since the calculations are quite analogous in all cases, we consider the "deepest" singularity of type N_{28}, when the valuation vector is $(5, 8)$. So, let $S \subset R = \mathbf{k}[[t]]$ be generated (as a complete local $\mathbf{k}$-algebra) by the elements x, y, where $v(x) = 5$, $v(y) = 8$. We may suppose that $t = y^2/x^3$. We also set $z = y/x$. Then $S \supset t^{28}R$. Moreover, since S is Gorenstein, every S-ideal is either principal or an S_0-ideal, where

$$S_0 = \operatorname{End} \mathfrak{m} = S + \langle\, t^{27} \,\rangle = \langle\, 1, x, y, x^2, xy, x^3, y^2, x^2y, x^4, xy^2 \,\rangle + t^{23}R$$

(see [3] or [5]). Consider the chain of rings $S_0 \subset S_1 \subset S_2 \subset S_3 \subset S_4$, where

$$\begin{aligned}
S_1 &= \operatorname{End} \mathfrak{m}_0 = \langle\, 1, x, y, x^2, xy, x^3, y^2 \,\rangle + t^{18}R,\\
S_2 &= \operatorname{End} \mathfrak{m}_1 = \langle\, 1, x, y, x^2, tx^2 \,\rangle + t^{13}R,\\
S_3 &= \operatorname{End} \mathfrak{m}_2 = \langle\, 1, z, x \,\rangle + t^8R,\\
S_4 &= \operatorname{End} \mathfrak{m}_3 = \langle\, 1, z \,\rangle + t^5R,
\end{aligned}$$

and $\mathfrak{m}_i = \operatorname{rad} S_i$. The S_3-ideals are known [13, 9, 10]; they are (except R, S_4 and S_3 itself):

$$\begin{aligned}
R_2 &= \langle\, 1 \,\rangle + t^2R,\\
R_3 &= \langle\, 1 \,\rangle + t^3R,\\
R_3^* &= \langle\, 1, t \,\rangle + t^3R,\\
S_4^* &= \langle\, 1, t^2, t^3 \,\rangle + t^5R.
\end{aligned}$$

The ideals R_3^* and S_4^* are indeed dual to R_3 and S_4 respectively, though we will not use this property. Note that it follows from [5] that every S_3-ideal is isomorphic either to an over-ring of S_3 or to the ideal dual to such an over-ring.

Proposition 1. *Here is a complete list of representatives of the ideal classes of the ring S_2, which are not S_3-ideals, sorted by the induced S_3-ideals $I' = S_3 I$. We set $\tilde{I} = \mathfrak{m}_2 I'$.*

- $I' = S_3;\ \tilde{I} = \mathfrak{m}_2$*:*

  1) S_2,
  2) $F_1(\alpha,\beta) = \langle\, 1, z + \alpha z^3 + \beta z^4 \,\rangle + \mathfrak{m}_2$,
  3) $F_2(\alpha) = \langle\, 1, z^2 + \alpha z^3 \,\rangle + \mathfrak{m}_2$,
  4) $F_3(\alpha) = \langle\, 1, z^3 + \alpha z^4 \,\rangle + \mathfrak{m}_2$,
  5) $I_1 = \langle\, 1, z^4 \,\rangle + \mathfrak{m}_2$,
  6) $F_4(\alpha,\beta) = \langle\, 1, z + \alpha z^4, z^2 + \beta z^4 \,\rangle + \mathfrak{m}_2$,
  7) $F_5(\alpha) = \langle\, 1, z, z^3 + \alpha z^4 \,\rangle + \mathfrak{m}_2$,
  8) $F_6(\alpha) = \langle\, 1, z + \alpha z^3, z^4 \,\rangle + \mathfrak{m}_2$,
  9) $F_7(\alpha) = \langle\, 1, z^2, z^3 + \alpha z^4 \,\rangle + \mathfrak{m}_2$,
  10) $F_8(\alpha) = \langle\, 1, z^2 + \alpha z^3, z^4 \,\rangle + \mathfrak{m}_2$,
  11) $I_2 = \langle\, 1, z^3, z^4 \,\rangle + \mathfrak{m}_2$,
  12) $I_3 = \langle\, 1, z.z^2, z^3 \,\rangle + \mathfrak{m}_2$,
  13) $I_4 = \langle\, 1, z, z^2, z^4 \,\rangle + \mathfrak{m}_2$,
  14) $I_5 = \langle\, 1, z, z^3, z^4 \,\rangle + \mathfrak{m}_2$,
  15) $I_6 = \langle\, 1, z^2, z^3, z^4 \,\rangle + \mathfrak{m}_2$.

- $I' = S_4;\ \tilde{I} = \langle\, x, y \,\rangle + t^{10}R$*:*

  1) $F_9(\alpha,\beta) = \langle\, 1, z + \alpha t z^2 + \beta z^3 \,\rangle + \tilde{I}$, *where* $\alpha \neq 0$,
  2) $F_{10}(\alpha,\beta) = \langle\, 1, z^2 + \alpha t z^2 + \beta z^3 \,\rangle + \tilde{I}$, *where* $\alpha \neq 0$,
  3) $F_{11}(\alpha) = \langle\, 1, t z^2 + \alpha z^3 \,\rangle + \tilde{I}$,
  4) $F_{12}(\alpha,\beta) = \langle\, 1, z + \alpha t z^2, z^2 + \beta t z^2 \,\rangle + \tilde{I}$, *where* $\alpha \neq 0$ *or* $\beta \neq 0$,
  5) $F_{13}(\alpha,\beta) = \langle\, 1, z + \alpha z^3, t z^2 + \beta z^3 \,\rangle + \tilde{I}$,
  6) $F_{14}(\alpha) = \langle\, 1, z + \alpha t z^2, z^3 \,\rangle + \tilde{I}$, *where* $\alpha \neq 0$,
  7) $F_{15}(\alpha,\beta) = \langle\, 1, z^2 + \alpha z^3, t z^2 + \beta z^3 \,\rangle + \tilde{I}$,
  8) $F_{16}(\alpha) = \langle\, 1, z^2 + \alpha t z^2, z^3 \,\rangle + \tilde{I}$, *where* $\alpha \neq 0$,
  9) $I_7 = \langle\, 1, t z^2, z^3 \,\rangle + \tilde{I}$,
  10) $F_{17}(\alpha) = \langle\, 1, z, z^2, t z^2 + \alpha z^3 \,\rangle + \tilde{I}$,
  11) $F_{18}(\alpha,\beta) = \langle\, 1, z + \alpha t z^2, z^2 + \beta t z^2, z^3 \,\rangle + \tilde{I}$, *where* $\alpha \neq 0$ *or* $\beta \neq 0$,
  12) $I_8 = \langle\, 1, z, t z^2, z^3 \,\rangle + \tilde{I}$,

13) $I_9 = \langle\, 1, z^2, tz^2, z^3 \,\rangle + \tilde{I}$.

- $I' = S_4^*$; $\tilde{I} = \langle\, x, tz^2, y \,\rangle + t^{10}R$:

1) $F_{19}(\alpha, \beta) = \langle\, 1, t^2 + \alpha z + \beta z^2 \,\rangle + \tilde{I}$,
2) $F_{20}(\alpha, \beta) = \langle\, 1, t^2 + \alpha z^2, z + \beta z^3 \,\rangle + \tilde{I}$,
3) $F_{21}(\alpha, \beta) = \langle\, 1, t^2 + \alpha z, z^2 + \beta z^3 \,\rangle + \tilde{I}$,
4) $F_{22}(\alpha, \beta) = \langle\, 1, t^2 + \alpha z + \beta z^2, z^3 \,\rangle + \tilde{I}$,
5) $F_{23}(\alpha) = \langle\, 1, t^2, z + \alpha z^3, z^2 \,\rangle + \tilde{I}$,
6) $F_{24}(\alpha) = \langle\, 1, t^2 + \alpha z, z^2, z^3 \,\rangle + \tilde{I}$,
7) $F_{25}(\alpha) = \langle\, 1, t^2 + \alpha z^2, z, z^3 \,\rangle + \tilde{I}$.

- $I' = R_3$; $\tilde{I} = \langle\, x \,\rangle + t^8 R$:

1) $F_{26}(\alpha, \beta) = \langle\, 1, z + \alpha tz + \beta tz^2 \,\rangle + \tilde{I}$, *where* $\alpha \neq 0$,
2) $F_{27}(\alpha, \beta) = \langle\, 1, tz + \alpha z^2 + \beta tz^2 \,\rangle + \tilde{I}$,
3) $F_{28}(\alpha, \beta) = \langle\, 1, z, tz + \alpha z^2 + \beta tz^2 \,\rangle + \tilde{I}$,
4) $F_{29}(\alpha, \beta) = \langle\, 1, z + \alpha tz, z^2 + \beta tz^2 \,\rangle + \tilde{I}$, *where* $\alpha \neq 0$,
5) $F_{30}(\alpha) = \langle\, 1, z + \alpha tz, tz^2 \,\rangle + \tilde{I}$, *where* $\alpha \neq 0$,
6) $F_{31}(\alpha, \beta) = \langle\, 1, tz + \alpha tz^2, z^2 + \beta tz^2 \,\rangle + \tilde{I}$,
7) $F_{32}(\alpha) = \langle\, 1, tz + \alpha z^2, tz^2 \,\rangle + \tilde{I}$,
8) $F_{33}(\alpha) = \langle\, 1, z, tz, z^2 + \alpha tz^2 \,\rangle + \tilde{I}$, *where* $\alpha \neq 0$,
9) $I_{10} = \langle\, 1, z, tz, tz^2 \,\rangle + \tilde{I}$,
10) $F_{34}(\alpha) = \langle\, 1, z + \alpha tz, z^2, tz^2 \,\rangle + \tilde{I}$, *where* $\alpha \neq 0$,
11) $I_{11} = \langle\, 1, z, z^2, tz^2 + \alpha z^3 \,\rangle + \tilde{I}$.

- $I' = R_3^*$; $\tilde{I} = \langle\, x, z^2 \,\rangle + t^8 R$:

1) $F_{35}(\alpha, \beta) = \langle\, 1, t + \alpha z + \beta tz \,\rangle + \tilde{I}$, *where* $\alpha \neq 0$,
2) $F_{36}(\alpha, \beta) = \langle\, 1, t, z + \alpha tz + \beta tz^2 \,\rangle + \tilde{I}$,
3) $F_{37}(\alpha, \beta) = \langle\, 1, t + \alpha z, tz + \beta tz^2 \,\rangle + \tilde{I}$,
4) $F_{38}(\alpha, \beta) = \langle\, 1, t + \alpha z + \beta tz, tz^2 \,\rangle + \tilde{I}$,
5) $F_{39}(\alpha) = \langle\, 1, t, z + \alpha tz^2, tz \,\rangle + \tilde{I}$,
6) $F_{40}(\alpha) = \langle\, 1, t, z + \alpha tz, tz^2 \,\rangle + \tilde{I}$,
7) $F_{41}(\alpha) = \langle\, 1, t + \alpha z, tz, tz^2 \,\rangle + \tilde{I}$.

- $I' = R_2$; $\tilde{I} = \langle\, x \,\rangle + t^7 R$:

1) $F_{42}(\alpha, \beta) = \langle\, 1, t^2 + \alpha z^2, z + \beta tz \,\rangle + \tilde{I}$, *where* $\beta \neq 0$,

2) $F_{43}(\alpha,\beta) = \langle\, 1, t^2+\alpha z, tz \,\rangle + \tilde{I}$,
3) $I_{12}(\alpha,\beta) = \langle\, 1, t^2, z, tz \,\rangle + \tilde{I}$,
4) $F_{44}(\alpha,\beta) = \langle\, 1, t^2, z+\alpha tz, z^2 \,\rangle + \tilde{I}$, *where* $\alpha \neq 0$,
5) $F_{45}(\alpha) = \langle\, 1, t^2+\alpha z, tz, z^2 \,\rangle + \tilde{I}$.

- $I' = R;\ \tilde{I} = t^5 R$:

1) $F_{46}(\alpha,\beta) = \langle\, 1, t+\alpha t^4, t^2+\beta t^4 \,\rangle + \tilde{I}$,
2) $I_{13} = \langle\, 1, t, t^2, t^3 \,\rangle + \tilde{I}$,
3) $I_{14} = \langle\, 1, t, t^2, t^4 \,\rangle + \tilde{I}$.

In all these formulae α *and* β *denote some elements from the field* $\mathbf{k}$. *Moreover, all quotient spaces* $W = I'/\mathfrak{m}_2 I'$ *are of dimension* 5, *so all* S*-ideals* I *such that* $S_2 I = I'$ *are actually* S_2*-ideals. Therefore, we need not consider them in the further calculations.*

Proof. We only consider the case when $I' = S_4$, since all other cases are quite similar (mostly easier). Then $W = \langle\, 1, z, z^2, tz^2, z^3 \,\rangle$, where we denote the class of an element by the same symbol as the element itself, and $\bar{W} = \langle\, 1, tz^2 \,\rangle$. Therefore, the dimension of a generating subspace V is at least 2. We also suppose that 1 is an element of a basis of V.

If $\dim V = 2$, there are the following possibilities:

1) $V = \langle\, 1, v \,\rangle$, where $v = z + \gamma z^2 + \alpha tz^2 + \eta z^3$ and $\alpha \neq 0$, since V must project onto $\bar{W}$. Set $\beta = \eta - \gamma^2$ and $a = 1 - tu$, where $u = z + \alpha tz^2 + \beta z^3$. Then $aV = \langle\, 1, u \,\rangle$ and the preimage of aV in S_4 is $F_9(\alpha,\beta)$ from the list. On the other hand, the image of $F_9(\alpha,\beta)$ in W is $V(\alpha,\beta) = \langle\, 1, z+\alpha tz^2+\beta z^3 \,\rangle$. If $V(\alpha',\beta') = aV(\alpha,\beta)$, then $a \in V(\alpha',\beta')$, so we may suppose that $a = 1 - \lambda(z + \alpha' tz^2 + \beta' z^3)$. Then the condition $a(z+\alpha tz^2+\beta z^3) \in V(\alpha',\beta')$ implies that $\alpha' = \alpha$ and $\beta' = \beta$. Therefore, the ideals $F_9(\alpha,\beta)$ are pairwise nonisomorphic. Further on we omit such verifications of nonisomorphy, since they are easy and straightforward.
2) $V = \langle\, 1, z^2 + \alpha tz^2 + \beta z^3 \,\rangle$ gives rise to $F_{10}(\alpha,\beta)$. Again one easily checks that all these ideals are nonisomorphic.
3) $V = \langle\, 1, tz^2 + \alpha z^3 \,\rangle$ gives rise to $F_{11}(\alpha)$.

If $\dim V = 3$, there are the following possibilities:

4) $V = \langle\, 1, u, v \,\rangle$, where $u = z + \alpha tz^2 + \gamma z^3$, $v = z^2 + \beta tz^2 + \eta z^3$ with $\alpha \neq 0$ or $\beta \neq 0$. Let $a = 1 - \eta u' - \gamma v'$, where $u' = z + \alpha tz^2$, $v' = z^2 + \beta tz^2$; then $aV = \langle\, 1, u', v' \,\rangle$, so gives rise to $F_{12}(\alpha,\beta)$.

5) $V = \langle 1, u, v \rangle$, where $u = z+\gamma z^2+\eta z^3$, $v = tz^2+\beta z^3$. Set $\alpha = \eta-\gamma^2$ and $u' = z + \alpha z^3$. Then $(1 - \gamma u')V = \langle 1, u', v \rangle$, so it gives rise to $F_{13}(\alpha, \beta)$.

6) In the same way $V = \langle 1, z + \beta z^2 + \alpha tz^2, z^3 \rangle$ is reduced to $\langle 1, z + \alpha tz^2, z^3 \rangle$ and gives rise to $F_{14}(\alpha)$.

7) $V = \langle 1, z^2 + \alpha z^3, tz^2 + \beta z^3 \rangle$ gives rise to $F_{15}(\alpha, \beta)$.

8) $V = \langle z^2 + \alpha tz^2, z^3 \rangle$ gives rise to $F_{16}(\alpha)$.

9) $V = \langle z^2, tz^2, z^3 \rangle$ gives rise to I_7.

Finally, if $\dim V = 4$, there are the following possibilities:

10) $V = \langle 1, u, v, w \rangle$, where $u = z + \beta z^3$, $v = z^2 + \gamma z^3$, $w = tz^2 + \alpha z^3$. Then $(1-\gamma z-\beta z^2)V = \langle 1, z, z^2, tz^2+\alpha z^3 \rangle$ and gives rise to $F_{17}(\alpha)$.

11) $V = \langle 1, z + \alpha tz^2, z^2 + \beta tz^2, z^3 \rangle$ gives rise to $F_{18}(\alpha, \beta)$.

12) $V = \langle 1, z+\alpha z^2, tz^2, z^3 \rangle$. Then $(1-\alpha z)V = \langle 1, z, tz^2, z^3 \rangle$ and gives rise to I_8.

13) $V = \langle 1, z^2, tz^2, z^3 \rangle$ gives rise to I_9. □

Now we have to find, for every S_2-ideal I from this list, all S_1-ideals I^1 such that $S_2I^1 = I$, then to find, for every I^1, all S_0-ideals I^0 such that $S_1I^0 = I^1$ or $S_1I^0 = I$. Since the calculations are quite similar for all ideals I and very much alike the calculations from the preceding proof (even easier), we only present several "typical" cases.

Case 1. (This case is the most complicated.)
$I = S_2$, $\mathfrak{m}_1S_2 = \mathfrak{m}_0S_2 = \mathfrak{m}_1$.
It gives new S_1-ideals:

1) S_1,

2) $I_1^1(\alpha, \beta) = \langle 1, u(\alpha, \beta) \rangle + \mathfrak{m}_1$, where $u(\alpha, \beta) = z^2x(1 + \alpha z + \beta z^2)$,

3) $I_2^1(\alpha) = \langle 1, z^3x + \alpha z^4x \rangle + \mathfrak{m}_1$,

4) $I_3^1 = \langle 1, z^4x \rangle + \mathfrak{m}_1$,

5) $I_4^1(\alpha, \beta) = \langle 1, z^2x + \alpha z^4x, z^3x + \beta z^4x \rangle + \mathfrak{m}_1$,

6) $I_5^1(\alpha) = \langle 1, z^2x + \alpha z^3x, z^4x \rangle + \mathfrak{m}_1$,

7) $I_6^1 = \langle 1, z^3x, z^4x \rangle + \mathfrak{m}_1$.

$S_1/\mathfrak{m}_0 = \langle\, 1, t^{19}, t^{22} \,\rangle$. It gives new S_0-ideals:
$S_0,\ \langle\, 1, t^{19} + \alpha t^{22} \,\rangle,\ \langle\, 1, t^{22} \,\rangle$.
$\mathfrak{m}_0 I_1^1(\alpha,\beta) = \mathfrak{m}_1 I_1^1(\alpha,\beta)$ if $\beta \neq \alpha^2$. If $\beta = \alpha^2$, then
$I_1^1(\alpha,\alpha^2)/\mathfrak{m}_0 I_1^1(\alpha,\alpha^2) = \langle\, 1, u, t^{19} \,\rangle$, which gives new S_0-ideals
$\langle\, 1, u(\alpha,\alpha^2) + \gamma t^{19}, z^2xy + \alpha z^3 xy \,\rangle + \mathfrak{m}_0$.

Case 2. $I = F_1(\alpha,\beta)$; $\mathfrak{m}_1 I = \mathfrak{m}_0 I = \langle\, \alpha yz^2 + \beta yz^3, yz + \alpha yz^3 \,\rangle + \mathfrak{m}_1$.

a. $\alpha \neq 0$. Then the only new possibilities are
$I^1 = \langle\, 1, z + \alpha z^3 + \beta z^4 \,\rangle + \mathfrak{m}_1 I$,
$I^0 = \langle\, 1, z + \alpha z^3 + \beta z^4 \,\rangle + \mathfrak{m}_0 I^1$, where
$\mathfrak{m}_0 I^1 = \langle\, \alpha yz^2 + \beta yz^3, yz + \alpha yz^3 + \beta yz^4 \,\rangle + \mathfrak{m}_0$.

b. $\alpha = 0$, $\beta \neq 0$. Then the only new possibilities are
$I^1 = \langle\, 1, z + \beta z^4 \,\rangle + \mathfrak{m}_1 I$.
Since $\mathfrak{m}_0 I^1 = \mathfrak{m}_1 I^1$, no new I^0 occur.

c. $\alpha = \beta = 0$. Then the only new possibilities are
$I^1 = V + \mathfrak{m}_1 I$, where V is one of the following subspaces:
$\langle\, 1, z + \gamma yz^3 \,\rangle$, or $\langle\, 1, z + \gamma yz^3, yz^2 + \gamma' yz^3 \,\rangle$, or $\langle\, 1, z, yz^3 \,\rangle$.
In all cases $\mathfrak{m}_0 I^1 = \mathfrak{m}_1 I^1$, so no new I^0 occur.

Case 3. $I = I_1$; $\mathfrak{m}_1 I = \mathfrak{m}_0 I = \langle\, x, y, x^2, xy \,\rangle + t^{15}R$.
The only new possibility is $I^1 = \langle\, 1, \alpha tx^2 + z^4 \,\rangle + \mathfrak{m}_1 I$.
Since $\mathfrak{m}_0 I^1 = \mathfrak{m}_1 I^1$, so no new I^0 occur.

Case 4. $I = F_{11}(\alpha)$; $\mathfrak{m}_1 I = \mathfrak{m}_0 I = \langle\, x, y, x^2 \,\rangle + t^{13}R$.
The only new possibility is $I^1 = \langle\, 1, tz^2 + \alpha z^3 + \beta tx^2 \,\rangle + \mathfrak{m}_1 I$.
Since $\mathfrak{m}_0 I^1 = \mathfrak{m}_1 I^1$, so no new I^0 occur.

Case 5. $I = F_{20}(\alpha,\beta)$. If $\beta \neq 0$, then $\mathfrak{m}_0 I = \mathfrak{m}_1 I = \mathfrak{m}_2 I$, so no new S_1- and S_0-ideals occur. If $\beta = 0$, then $\mathfrak{m}_0 I = \mathfrak{m}_1 I = \langle\, t^2 x, zy, z^4 \,\rangle + \mathfrak{m}_1$ and we get new S_1-ideals
$I^1(\alpha) = \langle\, 1, t^2 + \alpha z^2, z \,\rangle + \mathfrak{m}_0 I$.
Again, $\mathfrak{m}_0 I^1(\alpha) = \mathfrak{m}_1 I_1(\alpha)$, so no new S_0-ideals occur.

References

[1] Arnold V.I., Varchenko A.N. and Gusein-Zade S.M. Singularities of Differentiable Maps. Nauka, Moscow, 1982.

[2] Bass H. Torsion free and projective modules. Trans. Amer. Math. Soc. 162 (1962), 319-327.

[3] Bass H. On the ubiquity of Gorenstein rings. Math. Z. 182 (1963), 8 –28.

[4] Borevich Z.I. and Faddeev D.K. Representations of orders with cyclic index. Trudy Matem. In-ta Steklova. 80 (1965), 51–65.

[5] Drozd Y.A., Ideals of commutative rings. Matem. Sb. 101 (1976), 334–348.

[6] Drozd Y.A. and Greuel G.-M. Semi-continuity for Cohen-Macaulay modules. Math. Ann. 306 (1996), 371–389.

[7] Drozd Y.A. and Greuel G.-M. Cohen–Macaulay module type. Composition Math. 89 (1993), 315–338.

[8] Drozd Y.A. and Greuel G.-M. On Schappert's characterization of strictly unimodal plane curve singularities. Singularities: The Brieskorn Anniversary Volume. Birkhäuser, 1998, p. 3–26.

[9] Drozd Y.A. and Roiter A.V. Commutative rings with a finite number of integral indecomposable representations. Izvestia AN SSSR. Ser. Matem. 31 (1967), 3–798.

[10] Drozd Y.A. and Skuratovskii R.V. Cubic rings and their ideals. Ukrainian Math. J. 62 (2010), 464–470.

[11] Greuel G.-M. and Knörrer H. Einfache Kurvensingularitäten und torsionsfreie Moduln. Math. Ann. 270 (1985), 417–425.

[12] Griffiths P.A. Introduction to Algebraic Curves. AMS, 1989.

[13] Jacobinski H. Sur les ordres commutatifs avec un nombre fini de réseaux indecomposables. Acta Math. 118 (1967), 1–31.

[14] Schappert A. A characterization of strictly unimodal plane curve singularities Singularities, Representations of Algebras and Vector Bundles. Lecture Notes in Math. Vol. 1273. Springer–Verlag, 1987, p. 168–177.

[15] Skuratovskii R.V. Ideals of one branch curve singularities of type W. Ukrainian Math. J. 61 (2009), 1257–1266.

[16] Walker R.J. Algebraic Curves. Springer–Verlag, 1978.

Contact information

Yu. A. Drozd Institute of Mathematics, National Academy of Sciences of Ukraine, Tereschenkivska 3, 01601 Kyiv, Ukraine
E-Mail: y.a.drozd@gmail.com, drozd@imath.kiev.ua
URL: www.imath.kiev.ua/∼drozd

R. V. Skuratovskii Kyiv National Taras Shevchenko University, Department of Mathematics, Volodymyrska 64, 01033 Kyiv, Ukraine
E-Mail: ruslcomp@mail.ru

Received by the editors: 31.01.2012
and in final form 31.01.2012.