Algebra and Discrete Mathematics SURVEY ARTICLE
Volume 13 **(2012)**. Number 2. pp. 169 – 208


# S. N. Chernikov and the development of infinite group theory

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*On the 100$^{th}$ anniversary of the birth of Sergei Nikolaevich Chernikov*

Abstract. In this survey, the authors want to show the development and continuation of some studies, in which S.N. Chernikov stood as the main originator and to demonstrate clearly the extent of influence exerted by the ideas and results of S.N. Chernikov on the modern theory of infinite groups.

S.N. Chernikov, an extremely productive and creative researcher in the theory of infinite groups, was one of the founders of that subject. His creative work and his biography is an integral part of the history of the development of this important branch of modern group theory. His contributions to the field of algebra are profound, deep and important. However his ideas and his influence on the development of infinite groups, as exhibited in the work of his numerous students and followers, was also of tremendous importance. His Ph.D. students in Ural were Yu.N. Nefjedov, H.H. Mukhammedzhan, V.M. Glushkov, V.S. Charin, I.I. Eremin, M.I. Kargapolov, J.M. Gorchakov, N.V. Baeva (Chernikova), G.A. Malanyina, G.S. Shevtsov, G.F. Bachurin, Ya.D. Polovitskii, V.P. Shunkov, M.I. Sergeev, I.N. Abramovsky. The list of his Ukrainian students is even more impressive. These students were D.I. Zaitsev, F.N. Lyman, A.M. Andrukhov, L.M. Klyatckaya, O.N. Zub, A.F. Barannik, E.S. Alekseyeva, S.S. Levischenko, P.P. Baryshovets,

**2010 MSC:** 20-00, 20B07, 20C07, 20E15, 20E07, 20E25, 20E26, 20E28, 20E34, 20E36, 20E45, 20F14, 20F16, 20F18, 20F19, 20F22, 20F24, 20F50.

**Key words and phrases:** infinite groups, finiteness conditions, generalized soluble groups, generalized nilpotent groups, Chernikov groups, FC – groups, CC – groups.

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The influence of S.N. Chernikov was not restricted to his students however. His results and ideas influenced algebraists far beyond Chernikov's famous scientific school. The effect of this was not always immediate. Researchers often continued and naturally developed the ideas initiated by S.N. Chernikov's investigations, sometimes in unexpected ways. It is much like a small creek, which separates from a river, and flows for a while through a narrow channel, and then begins to merge with other streams and turns into a big river. S.N. Chernikov introduced many effective concepts that are continually used by many algebraists. Unfortunately, as sometimes happens, the originator of a concept is sometimes forgotten, as is the case with locally graded groups, first defined by S.N. Chernikov.

In this survey, we want to illustrate the development and continuation of some studies, in which S.N. Chernikov was the main originator. The complete picture cannot be presented here–such an endeavour would require much more time and space. For example, to collect and analyze all papers in which Chernikov groups are involved would be a major task. Therefore the content of this survey mainly reflects the mathematical interests of its authors. Nevertheless, the current survey demonstrates quite clearly the influence exerted by the ideas and results of S.N. Chernikov on the theory of infinite groups.

S.N. Chernikov began studying algebra independently by reading the books of D.A. Grave, N. G. Chebotarev, and O. Yu. Schmidt. As he himself said, by 1936 his research interests were finally decided, and were to be concerned with the theory of groups. These interests naturally led him to Moscow State University, and to A.G.Kurosh, whose graduate student he became in 1936. In the 1930's, Moscow State University was the leading mathematical center in the Soviet Union, during a period of very intense rapid development of group theory. Different areas of mathematics, especially geometry, topology, and the theory of automorphic functions, began dealing with problems in which various infinite groups played a significant role. Some of these problems required the study of the structure of infinite groups which clearly did not fit the results and methods already developed in the framework of the theory of finite groups.

The quest to develop the theory of infinite groups was therefore natural and vital, but the question of how this could be done and the direction in which to proceed remained. The theory of rings was a good model to follow since it was already a well-established algebraic theory in which the separation of finite and infinite rings was not apparent. The maximal and minimal conditions already played a key role in ring theory. It was therefore natural to use approaches that had proved to be productive and effective in the theory of rings and modules. It also should be noted that, in the 1920's and 1930's, contacts between German and Soviet mathematicians were quite close. In particular, O.Yu. Schmidt, the founder of the Russian, and later Soviet, group theory school, and P.S. Aleksandrov, the teacher of A.G. Kurosh, had extensive personal contacts with leading German mathematicians and often visited Germany. Some German mathematicians, in particular, one of the founders of modern algebra Emmy Noether, lectured at Moscow State University. This largely determined the fact that groups with finiteness conditions became one of the first areas to be studied in infinite groups. These investigations were strongly supported and encouraged by O.Yu. Schmidt who conducted his famous seminar at Moscow State University. The second leader of the seminar, A.G. Kurosh, then worked in topological groups. Being the main research center in algebra, this seminar attracted many young researchers. The program of this seminar was completely in harmony with the interests of Sergei Chernikov. In particular, the idea of considering groups with the minimal condition arose from this seminar.

In [KAG1932] A.G. Kurosh described abelian groups with the minimal condition on subgroups, but it is a much more difficult task to determine the non-abelian groups with this condition. Indeed, in general, the structure of such groups has not been obtained, and this situation is not likely to change for the foreseeable future. However the question arose as to what the limitations of such a study are. In the article [CSN1971-1], S.N. Chernikov discussed this point in some detail. The question of transferring some of the results of the theory of finite groups to infinite groups was studied by A.I. Uzkov who examined a group $G$ satisfying the following conditions:

(i) $G$ satisfies the minimal condition for all subgroups;

(ii) the set $\Lambda_n(G) = \{g \in G \mid |g| = n\}$ is finite for every positive integer $n$;

(iii) the center $\zeta(G)$ is finite.

Assuming that such groups are infinite, A.I. Uzkov obtained a classical theorem of Frobenius. S.N. Chernikov considered groups satisfying the first two conditions and showed that such a group has infinite center, so that the groups studied by A.I. Uzkov are finite. Chernikov exhibited this result at the seminar of O. Yu. Schmidt, and the intense discussion it generated turned into a general discussion on ways in which infinite group theory could be developed. One important outcome of this discussion was the formulation by O.Yu. Schmidt of his famous problem concerning infinite groups all of whose proper subgroups are finite. It should be noted that the result of A.I. Uzkov played a major role in the mathematical works of S.N. Chernikov, providing the first important step in his research into the study of groups with the minimal condition.

The problem of transferring the basic properties of finite nilpotent groups to infinite groups lead S.N. Chernikov to study groups which were later called groups with the normalizer condition ($N$- groups). Recall that a group $G$ satisfies *the normalizer condition* if $H \lneqq N_G(H)$ for every proper subgroup $H$ of $G$. These groups can be viewed as the groups all of whose subgroups are ascendant. Since a finite group is nilpotent if and only if it possesses this $N$-property, S.N. Chernikov in his paper [CSN1939] considered the conditions under which this property characterizes infinite nilpotent groups. More precisely, he described the $N$-groups satisfying the minimal condition for all subgroups [CSN1939]. Thus the groups which we now call Chernikov groups were introduced. However, it also became clear that the normalizer condition does not imply nilpotency of a group $G$ even when $G$ satisfies the minimal condition for all subgroups. An ascending central series terminating in the group itself may be infinite for such groups. Thus, in this way, another class of generalized nilpotent groups–the hypercentral groups ($ZA$ - groups)–appeared. The study of groups satisfying the minimal condition was continued in the papers [CSN1940-1, CSN1940-2].

Another key problem arose, that of constructing natural classes of infinite groups. This led S.N. Chernikov to the idea of "locality". As the chosen term indicates, this idea probably came from topology. Interestingly, the teacher of S.N. Chernikov, A.G. Kurosh, was very actively engaged in the fruitful study of topological groups, and Kurosh's teacher, P.S. Alexandrov, was one of the leading experts in topology. This idea was one of the most productive in the theory of infinite groups. In the works [CSN1939, CSN1940-1, CSN1940-2, CSN1940-3], new classes of infinite groups, arising from the imposition of certain properties on the system of finitely generated subgroups, were considered. Thus in the article [CSN1939], the key notions of local solvability and local nilpotency (this last term replaced the term of local specialty that was used originally in the article) appeared. In the mentioned article, the author also found that some properties of finitely generated subgroups (in one form or another) can be extended to the entire group. However, as we noted above, it became clear that in general the local nilpotency of a group, coupled with the minimal condition on subgroups, only implies the existence of an (possibly infinite) ascending central series terminating in the group. Thus the problem of finding a generalized form of solvability (respectively, nilpotency) in which the solubility (respectively nilpotency) of all finitely generated subgroups implies the solubility (respectively nilpotency) of the entire group arose. The idea of such a desired generalized form came with the concept of a Sylow set that was introduced by S.N. Chernikov in [CSN1940-3].

*The Sylow family of a periodic group $G$* is a family $\mathfrak{S}$ of normal subgroups of $G$, linearly ordered by inclusion, satisfying the following conditions:

(i) $G \in \mathfrak{S}$;

(ii) the intersection of every subfamily of $\mathfrak{S}$ belongs to $\mathfrak{S}$;

(iii) the union of every subfamily of $\mathfrak{S}$ belongs to $\mathfrak{S}$;

(iv) the factors of $\mathfrak{S}$ (that is the factor groups defined by successive members of the family $\mathfrak{S}$) are $p$-groups for some prime $p$;

(v) there are no distinct factors of the family $\mathfrak{S}$ corresponding to the same prime number $p$.

The next step was to define the notions of soluble and central systems introduced by S.N. Chernikov in [CSN1943]. The final forms of these concepts were based on the joint paper with A.G. Kurosh [KC1947] which identified ways of developing the theory of infinite groups for many years. The classes of generalized soluble and generalized nilpotent groups introduced and highlighted in this article are called the Kurosh-Chernikov classes. This paper established the relationships between some of these classes. Furthermore, questions regarding other possible relationships and properties of the groups in these classes were also raised. The article [KC1947] largely determined and stimulated key research in the theory of generalized solvable and generalized nilpotent groups; the questions contained in the article attracted the attention of numerous specialists.

One of the important natural follow-ups was to elucidate the structure of groups of the Kurosh-Chernikov classes under certain finiteness conditions. Sufficiently complete and clear ideas about the development of these studies can be found in the well known surveys due to S.N. Chernikov [CSN1959] and B.I. Plotkin [PBI1958], the famous book of A.G. Kurosh [KAG1967], the books of D.J.S. Robinson [RD1972-1, RD1972-2], and S.N. Chernikov [CSN1980]. On the one hand, these works gave a clearer understanding as to the structure of these classes, but on the other hand, they indicated the scope and limitations within which the provided methodology and tools could work. Many of the problems that have arisen in the framework of this theory received their solutions outside this theory. More precisely, in the framework of this theory, they were resolved positively, while beyond it, using completely different methods, some unique counterexamples have been created. Thus A.Yu. Olshanskii (see for example, [OA1981, Ch. 28]) produced a series of constructions of periodic infinite groups whose proper subgroups are finite cyclic, thereby obtaining examples of the type required in the problems of Schmidt and Chernikov. These unique constructions gave a negative answer for a number of other well-known problems. Other fascinating examples of finitely generated infinite periodic groups with many unusual properties have been created by R.I. Grigorchuk (see, for example, the survey [GRI2005]). These brilliant ideas of A. Yu Olshanskii and R. Grigorchuk showed that the traditional methods, techniques and approaches of the theory of generalized solvable groups do not control this theory. But it would be a mistake to regard this as a shortcoming of the theory of generalized solvable groups. Many methods of finite group theory or the theory of finite linear groups also do not work outside of these theories. However, these theories are not regarded as faulty. On the contrary, as S.N. Chernikov noted in this connection, these examples “set off the completion of the obtained positive results” and lead to a clear understanding of the important fact that the theory of generalized solvable groups, like other traditional areas of group theory, such as the theory of finite groups, the theory of Abelian groups, the theory of linear groups, etc. is saturated with many interesting and deep results, and it is an important specific part of the general theory of groups, which has its own objects of research, its methodology and rich history.

The research, conducted by S.N. Chernikov and his followers in the 40’s and 50’s, caused great interest among other famous algebraists such as O.Yu. Schmidt, A.I. Mal’tsev, R. Baer, K. Hirsch, B.H. Neumann, and P. Hall. This stimulated new research, which, in turn, provided both a direct and an indirect impact on the members of S.N. Chernikov’s School. This natural course of development led to the design and creation of a great and substantial new branch of the theory of infinite groups, known as “Groups with finiteness conditions”. In essence, S.N. Chernikov and the above mentioned mathematicians were its founders. A special position here belongs to S. N. Chernikov whose initiative and creative contributions, to a great extent, determined the direction of research in the field. By a finiteness condition we mean a property possessed by every finite group and by at least one infinite group. Imposing these finiteness conditions allows us to select such infinite groups that preserve certain properties of finite groups. The finiteness conditions identify many natural classes of groups at the interface of finite and infinite groups. Of course, the first finiteness conditions were the classical minimal and maximal conditions on subgroups of different families. These conditions provide very strong restrictions, and have a highly developed theory, rich in many deep and interesting results. Exploring groups with the minimal condition for normal subgroups and the groups with the minimal condition for abelian subgroups, S.N. Chernikov came to the very fruitful idea of exploring the effect that some major sub-systems of subgroups have on the group structure. Imposing some natural restriction on this concrete sub-system, S.N. Chernikov investigated important general properties of a group, and even described the structure of a group with the condition. It also became important to determine the influence of important natural systems of subgroups, such as the abelian and non-abelian subgroups, normal subgroups and their various generalizations, primary subgroups, and so on, on the entire group. The results obtained for groups with various finiteness conditions and groups with restrictions on important sub-systems have found a very good coverage in the books of A.G. Kurosh [KAG1967], D.J.S. Robinson [RD1972-1, RD1972-2], S.N. Chernikov [CSN1980], and in the survey articles [CSN1959, PBI1958, CSN1967, CSN1969, CSN1971-2, CSN1980-2, CZ1988, ZKC1972, AK2003, DS2009].

The influence of S.N. Chernikov on the development of group theory is not only evident from his results; there were areas which he helped create, but which were then developed by other algebraists. His influence on the formation of modern group theory, being the head of a quite large group-theoretical school, was extensive and our goal is to reflect some of this influence.

As we have already indicated, one of the first finiteness conditions considered by S.N. Chernikov was the finiteness of layers. A group $G$ is called layer-finite if it is periodic and the set $\mathbf{\Lambda}_n(G) = \{g \in G \mid |g| = n\}$ is finite for every positive integer $n$. In the paper [CSN1948-1], these groups were described in detail. In the article [CSN1957-1], S.N. Chernikov continued the study of these groups and obtained the following characterization.

*Let $G$ be a group. Then $G$ is layer-finite if and only if $G$ satisfies the following conditions:*

(i) *$G$ is periodic;*

(ii) *for each element $x \in G$ the conjugacy class $x^G = \{g^{-1}xg \mid g \in G\}$ is finite;*

(iii) *for each prime $p$ the Sylow $p$-subgroups of $G$ are Chernikov.*

This result shows that locally finite groups whose Sylow $p$-subgroups are Chernikov have a very special place among locally finite groups. In the papers [CSN1959, CSN1960], the locally finite groups, whose Sylow $p$-subgroups are finite for every prime $p$ were considered. The main result of these papers is:

*Let $G$ be a locally finite group. Suppose that the Sylow $p$-subgroups of $G$ are finite for all primes $p$. Then $G$ satisfies the following conditions:*

(i) *$O_{p'}(G)$ has finite index for each prime $p$;*

(ii) *$G$ can be embedded in the Cartesian product $Cr_{p\in\Pi(G)}G/O_{p'}(G)$ (so that $G$ is residually finite).*

Let $\pi$ be a set of primes. Recall that $\mathbf{O}_\pi(G)$ is the largest normal $\pi$-subgroup of $G$ and that if $p$ is a prime, then $p'$ denotes the set of all primes $q$ such that $q \neq p$. Thus $\mathbf{O}_{p'}(G)$ is the largest normal subgroup of $G$, which contains no $p$-elements.

The next important step in the study of locally finite groups whose Sylow $p$-subgroups are Chernikov was completed by a student of Chernikov, M.I. Kargapolov, in his work [KMI1961]. We note his results here.

*Let $G$ be a locally finite group. Suppose that the Sylow $p$-subgroups of $G$ are Chernikov for all primes $p$. Then $G/\mathbf{O}_{p'}(G)$ is Chernikov if and only if every simple section of $G$ containing elements of order $p$ is finite. In particular, if $G$ is locally soluble, then $G/\mathbf{O}_{p'}(G)$ is a Chernikov group.*

*Let $G$ be a periodic locally soluble group. Suppose that the Sylow $p$-subgroups of $G$ are Chernikov for all primes $p$. Then $G$ contains a normal divisible abelian subgroup $R$ such that $G/R$ is residually finite and the Sylow $p$-subgroups of $G/R$ are finite for each prime $p$.*

As a corollary the following statement can be deduced.

*Let $G$ be a periodic locally soluble group and suppose that the Sylow $p$-subgroups of $G$ are Chernikov for all primes $p$. Then*

(i) *the Sylow $p$-subgroups of $G$ are conjugate for each prime $p$;*

(ii) *$G$ can be embedded in the Cartesian product $\mathbf{Cr}_{p\in\Pi(G)}G/\mathbf{O}_{p'}(G)$ of Chernikov groups;*

(iii) *If $G$ is radical, then $G$ is countable.*

In connection with the last statement, the following question arose.

*Let $G$ be a locally soluble group whose Sylow $p$-subgroups are Chernikov for all primes $p$. Is $G$ countable?*

A negative answer to this problem was obtained by R. Baer in his paper [BR1969].

The conjugacy of the Sylow subgroups is a very powerful feature that made it possible to build a well-established theory of locally finite groups with Chernikov Sylow subgroups. Many theorems obtained for finite groups can be extended to infinite groups with this property. The results obtained in this area are well represented in the monograph of M.R. Dixon [DMR1994]. Here we note one more important result which was obtained by V.V. Belyaev in [BVV1981].

*Let $G$ be a locally finite group and suppose that the Sylow $p$-subgroups of $G$ are Chernikov for all primes $p$. Then $G$ contains a normal locally soluble subgroup of finite index.*

Groups with Chernikov Sylow subgroups were closely associated with certain natural numerical invariants defined on groups which originated from various properties and analogies with the concept of vector space dimension. In the transition to modules, the notion of dimension is naturally transformed into the concept of rank of a module. Every abelian group is a module over the ring $\mathbb{Z}$ of integers. Therefore, the concept of 0-rank (or $\mathbb{Z}$-rank) arises naturally in the theory of abelian groups, particularly for torsion-free groups. If $G$ is a torsion-free abelian group, then we can consider its divisible envelope $D$ as a vector space over the field $\mathbb{Q}$ and the 0-rank, $r_0(G)$, of $G$ is precisely the dimension of $D$ over $\mathbb{Q}$. If $G$ is an arbitrary abelian group, then we define $r_0(G) = r_0(G/\mathbf{Tor}(G))$. Thus,

$$r_0(G) = \mathbf{dim}_{\mathbb{Q}}(G \otimes_{\mathbb{Z}} \mathbb{Q}).$$

It follows that an abelian group $G$ has finite 0-rank $r$ if and only if $G/\mathbf{Tor}(G)$ is isomorphic to a subgroup of the additive group

$$\underbrace{\mathbb{Q} \otimes \cdots \otimes \mathbb{Q}}_{r}.$$

In particular, if an abelian group $G$ has a finite 0-rank, then $G/\mathbf{Tor}(G)$ has a finite series of subgroups, whose factors are isomorphic to subgroups of $\mathbb{Q}$.

The closest groups to infinite abelian ones are the nilpotent and hypercentral infinite groups. The first step in their study was the investigation of divisible hypercentral groups by S.N. Chernikov in his papers [CSN1946, CSN1948-2, CSN1948-3], CSN1949, CSN1950-1, CSN1950-2, CSN1950-3]. We recall that a group $G$ is called divisible (more precisely, divisible in the sense of Chernikov), if $G = G^n = \langle g^n \mid g \in G\rangle$ for every positive integer $n$. Divisible abelian groups possess an important property which we now describe. A group $G$ is called $\mathfrak{F}$-perfect, if $G$ contains no proper subgroup of finite index. Every divisible abelian group is $\mathfrak{F}$-perfect, and conversely, every abelian $\mathfrak{F}$-perfect group is divisible. The full description and main important properties of divisible hypercentral groups were obtained by S.N. Chernikov in the above-mentioned articles. We consider the more fundamental results here. For a description of divisible hypercentral groups S.N. Chernikov employed the construction of a generalized semidirect product. We recall this construction.

We say that a group $G$ is the *generalized semidirect product of a family* $\{K_\alpha \mid \alpha < \gamma\}$ of subgroups, where $\gamma$ is some ordinal, if the following conditions hold:

(i) the subgroup $L_\alpha = \langle K_\beta \mid \beta < \alpha\rangle$ is normal in $G$ for each $\alpha < \gamma$;

(ii) $L_\alpha \cap K_\alpha = \langle 1\rangle$ for each $\alpha < \gamma$;

(iii) $L_\gamma = G$.

*Let $G$ be a hypercentral group. Then the following assertions hold:*

(i) *$G$ is divisible if and only if $G$ is $\mathfrak{F}$-perfect.*

(ii) $\mathbf{Tor}(G)$ *is abelian.*

(iii) *$G$ has a family $\{K_\alpha \mid \alpha < \gamma\}$ of subgroups, satisfying the following conditions:*

  (a) $K_1 = \mathbf{Tor}(G)$, *$K_\alpha \cong \mathbb{Q}$ for all $1 < \alpha < \gamma$;*

(b) *$G$ is a generalized semidirect product of* $\{K_\alpha \mid \alpha < \gamma\}$;

(c) $\gamma = z(G)$ *is an invariant of the group* $G$.

In this regard, we note that in the article [MAI1949], A.I. Mal'tsev proved that every torsion-free locally nilpotent group $G$ can be embedded in a divisible torsion-free locally nilpotent group, and the minimal torsion-free divisible locally nilpotent group $D(G)$ containing $G$ (the divisible envelope of $G$) is uniquely determined up to isomorphism. If $G$ is torsion-free and hypercentral, then the divisible envelope of $G$ is likewise hypercentral, therefore the ordinal $\gamma = z(D(G))$ is an invariant of $G$. This invariant is called *the rational rank of the hypercentral group* $G$.

A group $G$ is called *polyrational*, if it has a subnormal series

$$\langle 1 \rangle = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G,$$

whose factors are isomorphic to subgroups of the additive group of rational numbers. If a hypercentral group $G$ has finite rational rank, then $G/\mathbf{Tor}(G)$ is polyrational. The study of polyrational groups, as well as links between the rational rank and other finiteness conditions, was continued by S.N. Chernikov's students, V.M. Glushkov and V.S. Charin in the works [GVM1950, GVM1951, GVM1952-1, GVM1952-2, CVS1949, CVS1951]. In particular, V.M. Glushkov considered the relationship between the rational rank and another important numerical invariant of a group, the special rank, which had been introduced by A.I. Mal'tsev in the paper [MAI1948]. A.I. Mal'tsev's definition also came from the notion of the dimension of a vector space.

It is well-known that if $A$ is a vector space of finite dimension $\mathbf{k}$ over a field $F$, and $B$ is a subspace of $A$, then $B$ is finite dimensional and the dimension of $B$ is at most $\mathbf{k}$. Accordingly, we say that a group $G$ *has finite special rank* $r(G) = r$, if every finitely generated subgroup of $G$ can be generated by $r$ elements and $r$ is the least positive integer with this property.

It is a well-known consequence of the structure theorem for finitely generated abelian groups that if $G$ is an abelian group with $k$ generators and $B$ is a subgroup of $G$, then $B$ is finitely generated and has at most $k$ generators. Thus a subgroup $H$ of a finitely generated abelian group $G$ is also finitely generated and the minimal number of generators of $H$ is at most the minimal number of generators of $G$. Thus finitely generated groups are examples of groups with finite rank. However, it is well-known that for non-abelian groups this sort of statement is not true. For example, the standard restricted wreath product $\mathbb{Z}\mathbf{wr}\mathbb{Z}$ of two infinite cyclic groups is a two-generator group and yet its base group is infinitely generated. Thus a subgroup of a finitely generated group need not even be finitely generated. Furthermore, it is well-known that a subgroup of a finitely generated nilpotent group is finitely generated. However, as the group $\mathbb{Z}_p\mathbf{wr}\mathbb{Z}_p$ for $p$ an odd prime shows, even in this case, it is possible to have a subgroup with more generators than the original group has. This group is two-generated but its base group is $p$-generated.

The special rank of a group $G$ is sometimes called *the Prüfer rank of* $G$, and is often called just the rank of $G$. We shall often follow this convention but also will use the full terminology to avoid possible confusion. H. Prüfer first defined groups of rank 1 in 1924 in his famous paper [PH1924]. The general concept of the special rank (and also the term *special rank*) were introduced by A.I. Mal'tsev [MAI1948]. Later, in the paper [BR1966], this rank was called the Prüfer rank.

In the paper [GVM1952-1, V.M. Glushkov discussed the relationship between the special and rational ranks in locally nilpotent groups. He proved the following result.

*Let $G$ be a locally nilpotent polyrational group. Then its rational rank coincides with its special rank.*

This result has been extended by D.I. Zaitsev [ZDI1971-1] to arbitrary polyrational groups.

*Let $G$ be an arbitrary polyrational group. Then the rational rank of $G$ coincides with its special rank.*

In his paper [MAI1951], A. I. Mal'tsev introduced the following generalization of polyrational groups. Following A.I. Mal'tsev, we say that an abelian group $G$ is called an $A_1$-*group*, if $r_0(G)$ is finite. A group $G$ is said to be *a soluble* $A_1$-*group*, if $G$ has a finite subnormal series whose factors are abelian $A_1$-groups. Such groups have a finite subnormal series, the factors of which are periodic abelian or infinite cyclic. It was observed that the number of infinite cyclic factors is a numerical invariant of a group called the 0-rank or torsion-free rank of the group. The term torsion-free rank was first used by D.J.S. Robinson in [RD1972-3]. This concept has actively promoted by D.I. Zaitsev. In particular, in the papers [ZDI1975, ZDI1980-1, ZDI1980-2], he demonstrated the great effectiveness of this concept in solving a variety of group theory problems. In the paper [DKP2007] this concept was generalized as follows.

A group $G$ has 0-*rank* $r_0(G) = r$ if $G$ has an ascending series whose factors are either infinite cyclic or periodic and if the number of infinite cyclic factors is exactly $r$. If $G$ has an ascending series with periodic and infinite cyclic factors and the set of infinite cyclic factors is infinite, then we will say that the group $G$ *has infinite* 0-*rank*. Otherwise we will say that $G$ *has no* 0-*rank*.

In the paper [DKP2007], the groups having finite 0-rank were described in the broad class of generalized radical groups. A group $G$ is called *generalized radical*, if $G$ has an ascending series whose factors are locally nilpotent or locally finite.

We observe that every group $G$ always contains a unique largest normal periodic subgroup which we denote by $\mathbf{Tor}(G)$. We note that $\mathbf{Tor}(G)$ might be trivial. It is very easy to see that $\mathbf{Tor}(G)$ is generated by all the normal periodic subgroups of $G$. We note also that a product of normal locally finite subgroups is again locally finite, so that in every group $G$ the subgroup $Lf(G)$, generated by all normal locally finite subgroups, is the largest normal locally finite subgroup which we call *the locally finite radical.*

It follows easily from the definition that a generalized radical group $G$ has either an ascendant locally nilpotent subgroup or an ascendant locally finite subgroup. In the former case, the locally nilpotent radical of $G$ is non-trivial. In the latter case, $G$ contains a non-trivial normal locally finite subgroup, so the locally finite radical is non-trivial. Thus every generalized radical group has an ascending series of normal subgroups with locally nilpotent or locally finite factors. We observe also that a periodic locally generalized radical group is locally finite. As was shown in [DKP2007], the locally generalized radical groups of finite 0-rank have the following structure.

*Let $G$ be a locally generalized radical group of finite* 0-*rank. Then $G$ has normal subgroups $T \le L \le K \le S \le G$ such that*

(i) *$T$ is locally finite and $G/T$ is soluble-by-finite,*

(ii) *$L/T$ is torsion-free nilpotent,*

(iii) *$K/L$ is a finitely generated torsion-free abelian group,*

(iv) *$G/K$ is finite and $S/K$ is the soluble radical of $G/K$.*

*Moreover, if $r_0(G) = r$, then there are functions $f_1$ and $f_2$ such that $|G/K| \le f_1(r)$ and $dl(S/T) \le f_2(r)$.*

Here $dl(G)$ denotes the *derived length of the soluble group* $G$. The desire to find general properties common to both groups with Chernikov Sylow subgroups and groups of finite 0-rank gave rise to the following rank of a group $G$.

Let $p$ be a prime. We say that a group $G$ *has finite section p-rank* $r_p(G) = r$ if every elementary abelian $p$-section of $G$ is finite of order at most $p^r$ and there is an elementary abelian $p$-section $A/B$ of $G$ such that $|A/B| = p^r$.

We observe that a locally finite $p$-group $P$ has finite section $p$-rank if and only if $P$ is a hypercentral Chernikov group. Hence a locally finite group with Chernikov Sylow $p$-subgroups has finite section $p$-rank for every prime $p$. From the above description of a group $G$ of finite 0-rank we can see that $G/\mathbf{Tor}(G)$ has finite section $p$-rank for each prime $p$. The groups, having finite section $p$-rank for all primes $p$, were introduced by D.J.S. Robinson in [RD1968]. Taking into account the above-mentioned results concerning groups with Chernikov Sylow subgroups and groups of finite 0-rank, we can clarify the structure of the groups of finite section $p$-rank for all primes $p$.

*Let $G$ be a locally generalized radical group of finite section $p$-rank for all primes $p$. Then $G$ has finite 0-rank, moreover $r_0(G) \leq \frac{t(t+3)}{2}$ where $t = min\{r_p(G) | p \in \mathbf{P}\}$. Furthermore, $G$ has normal subgroups*

$$D \leq T \leq L \leq K \leq S \leq G$$

*satisfying the following conditions:*

(1) *$T$ is periodic and almost locally soluble;*

(2) *the Sylow $p$-subgroups of $G$ are Chernikov for all primes $p$;*

(3) *$D$ is a divisible abelian subgroup;*

(4) *the Sylow $p$-subgroups of $T/D$ are finite for all primes $p$ and $T/D$ is residually finite;*

(5) *$L/T$ is nilpotent and torsion-free;*

(6) *$K/L$ is abelian torsion-free and finitely generated;*

(7) *$G/K$ is finite and $|G/K| \leq f_3(t)$;*

(8) *$S/K$ is soluble and $dl(S/T) \leq f_4(t)$.*

We note some connections between the special and section ranks. Clearly, if a group $G$ has finite special rank, then the section $p$-rank of $G$

is finite for all primes $p$ and $r_p(G) \leq r(G)$ for all $p$. Conversely, it is clear that if $P$ is an abelian $p$-group, where $p$ is a prime, then $r_p(P) = r(P)$. If now $P$ is an arbitrary finite $p$-group then Yu.I. Merzlyakov proved in [MYuI1964] that if the special rank of every abelian subgroup of $P$ is at most $s$, then $r(P) \leq \frac{s(5s+1)}{2}$. In particular, if $r_p(P) = s$, then $r(P) \leq \frac{s(5s+1)}{2}$.

Suppose now that $G$ is a locally generalized radical group and $r_p(G) \leq b$ for some positive integer $b$. Let $T = \mathbf{Tor}(G)$. Then since the special rank and rational rank of a polyrational group coincide, we see that $G/T$ has special rank at most $\frac{b(b+3)}{2} + f_3(b)$. The subgroup $T$ is locally finite. Let $F$ be an arbitrary finite subgroup of $T$. Then, as noted above, every Sylow $p$-subgroup of $F$ has at most $\frac{b(5b+1)}{2}$ generators. It follows that $F$ has at most $\frac{b(5b+1)}{2} + 1$ generators [GR1989]. Hence $T$ has special rank at most $\frac{b(5b+1)}{2} + 1$, and it follows that $G$ has finite special rank at most

$$\frac{(b(b+3)}{2} + \frac{b(5b+1)}{2} + f_3(b) + 1 = b(3b+2) + f_3(b) + 1.$$

A vast array of articles has been devoted to different types of group of finite rank. Many important results on this topic were reflected in the survey [DKS2007], so we will not delve further into this subject.

Let us turn to another large and important area of group theory, having its origins in the work of S.N. Chernikov, namely the theory of $FC$-groups. S.N. Chernikov can be considered as one of the originators of this area. Some natural classes of $FC$-groups arose in the theory of groups many years ago. One of the first important results on $FC$-groups is a theorem which follows from the results of the paper [SCH1904] of I. Schur who studied the relationship between the factor group $G/\zeta(G)$ and the derived subgroup $[G, G]$. The results of I. Schur show that if a group $G$ has its center of finite index, then the derived subgroup $[G, G]$ is finite. Thus, in this paper, the essential classes of $FC$-groups, namely the classes of central-by-finite groups and finite-by-abelian groups were introduced. The next important work was an article by A.P. Ditsman [DAP1937], which argued that every finite $G$-invariant subset of a group $G$ whose elements have finite orders generates a finite normal subgroup.

In the paper of Baer [BR1940], the class of periodic $FC$-groups was introduced. From this work there followed the articles of P.A. Golberg [GPA1946] and S.N. Chernikov [CSN1955], and a series of other works, whose aim was to extend to periodic $FC$-groups the important theorems of the theory of finite groups. The term “$FC$-group” first appeared in the paper of R. Baer [BR1948]. This work, together with the works of S.N. Chernikov [CSN1948, CSN1957-1, CSN1957-2] and B.H. Neumann [NB1951, NB1954, NB1955] laid the foundations of the theory of $FC$-groups. In the paper [CSN1958], S.N. Chernikov first drew attention to the role of direct products of finite groups in the theory of periodic $FC$-groups. At the same time, his student M.I. Kargapolov published his article [KMI1958], which was also devoted to conditions for the embeddability of periodic $FC$-groups in a direct product of finite groups. A year later the important article of Ph. Hall [HP1959-2], also dedicated to this issue, appeared.

These papers laid the foundation of an important part of the theory of $FC$-groups, whose aim is to study conditions for the embeddability of a periodic $FC$-group in a direct products of finite groups, and to study the influence of the class $\mathbf{SD}\mathfrak{F}$ of subgroups of direct products of finite groups on the structure of $FC$-groups. Here, some interesting parallels with the theory of abelian groups arise. An important step in the development of the theory of $FC$-groups was done by another student of S.N. Chernikov - Yu.M. Gorchakov [GYM1961, GYM1962, GYM1965, GYM1971, GYM1974, GYM1976] and by M.J. Tomkinson [TM1975, TM1977, TM1978, TM1981]. We will not give consideration of these interesting and important results here because they are fully reflected in the monographs [GYM1978, TM1984] and the survey [TM1996]. We note some important results of the theory of $FC$-groups, which were obtained later and therefore were not reflected in the above-mentioned monographs and survey.

In the theory of $FC$-groups, the nilpotent groups of class at most 2 play a very specific role. Some important results indicate this. For example, Yu.M. Gorchakov showed that the factor group of an $FC$- group by the second hypercenter can be embedded in a direct product of finite groups (see [GYM1978, Corollary II.3.8]). Also, any $FC$-group contains an equipotent nilpotent subgroup of class 2 (T.Ya. Semenova; see [GYM1978, Theorem III.1.8]), but this is not true for abelian subgroups of $FC$-groups.

Following Yu.M. Gorchakov (see [GYM1978, Chapter I, Section 4]) a class $\mathfrak{D}$ is called a direct variety if it is closed under the taking of subgroups, factor groups, and direct products. If $\mathfrak{X}$ is a class of groups, then the direct variety generated by $\mathfrak{X}$ is equal to $\mathbf{QSD}\,\mathfrak{X}$ (see [GYM1978, Lemma I.4.2]). Nilpotent $FC$-groups of class 2 have also been considered in the paper [KLA1987]. We note the main results of this article.

Let $\mathfrak{N}_2$ denote the class of nilpotent groups having nilpotency class at most 2, and let $\mathfrak{F}\mathfrak{A}$ denote the class of finite-by-abelian groups.

*Let $\mathfrak{X}$ be the class of all nilpotent $FC$-groups having nilpotency class at most 2. Then $\mathfrak{X}$ is generated as a direct variety by all $FC$-groups having quasicyclic derived subgroups.*

*If $G \in \mathfrak{N}_2 \cap \mathbf{QSD}(\mathfrak{F}\mathfrak{A})$, then $G/\zeta(G)$ can be embedded in a direct product of finite groups.*

*Let $G$ be an $FC$-group and suppose that $G \in \mathfrak{N}_2$. If $[G, G]$ contains no element of order 2 and $G/\zeta(G) \in \mathbf{SD}\mathfrak{F}$, then $G \in \mathbf{QSD}(\mathfrak{F}\mathfrak{A})$.*

These results have the following important Corollaries. M.J. Tomkinson showed that the derived subgroup of a periodic $FC$-group belongs to the class $\mathbf{QSD}\mathfrak{F}$ [TM1984, Theorem 3.6]). In this connection, there appears the following problem: Does $\mathbf{QSD}(\mathfrak{F}\mathfrak{A})$ contains the class of all periodic $FC$-groups? [TM1984, Question 3.H]. Yu.M. Gorchakov has constructed an example of a $p$-group $G$ such that $[G, G] = \zeta(G)$ is a quasicyclic subgroup and $G/\zeta(G) \notin \mathbf{SD}\mathfrak{F}$ [GYM1978, Example II.2.11]. From the above result it follows that

*The class of periodic $FC$-groups is not contained in the direct variety generated by the groups with finite derived subgroups.*

Recall that a $p$-group $G$ is called extraspecial if $\zeta(G) = [G, G]$ is a subgroup of order $p$ and $G/\zeta(G)$ is an elementary abelian $p$-group, where $p$ is a prime. The paper [KLA1987] contains the following description of extraspecial groups from the class $\mathbf{QSD}\mathfrak{F}$:

*Let $G$ be an extraspecial $p$-group. If $G \in \mathbf{QSD}\mathfrak{F}$, then $G$ can be embedded in a central product of non-abelian subgroups of order $p^3$.*

This result gives an affirmative answer to Question 3G of [TM1984].

A group $G$ is called a $\mathfrak{Z}$-group if for each infinite cardinal $m$ and each subset $S$ of $G$ such that $|S| < m$ we have $|G : C_G(S)| < m$ [TM1984, Section 3]. This class of groups has appeared in a paper of P. Hall [HP1959]. The class of periodic $\mathfrak{Z}$-groups contains the direct variety generated by the class of finite groups [HP1959], but it was unknown if these two classes coincide [TM1984, Question 3F]). M.J. Tomkinson constructed an example of an extraspecial $p$-group that cannot be embedded in a central product of non-abelian groups of order $p^3$ [TM1984, Example 3.16]. By the above result this group cannot be contained in $\mathbf{QSD}\mathfrak{F}$, so we have the following assertion

*The class $\mathbf{QSD}\mathfrak{F}$ is not equal to the class of periodic $\mathfrak{Z}$-groups.*

The results of P. Hall, Yu.M. Gorchakov and M.J. Tomkinson show that the class $\mathfrak{V}$ of $FC$-groups, whose central factor groups can be embedded in a direct product of finite groups, plays an important role in the class of all $FC$-groups. In particular, it contains the class $\mathbf{QSD}\mathfrak{F}$ (see [GYM1978, Theorem I.4.6]). In this connection, the following problem naturally appears: is the class $\mathfrak{V}$ a direct variety? A negative answer to this question was obtained by L.A. Kurdachenko in the paper [KLA1992]. More precisely

*There exists an $FC$-group $G$ with the following properties:*

(i) *$G$ is a $p$-group where $p$ is a prime;*

(ii) $G/\zeta(G) \in \mathbf{SD}\mathfrak{F}$;

(iii) $G$ has an epimorphic image $B$ such that $B/\zeta(B) \notin \mathbf{SD}\mathfrak{F}$.

*In particular, the class $V$ is not a direct variety.*

If $G$ is a periodic $FC$-group, then $G/\zeta(G) \in \mathbf{SD}\mathfrak{F}$ (see [GYM1978, Corollary II.3.7]), and $[G, G] \in \mathbf{SD}\mathfrak{F}$ [TM1984, Corollary 2.27]). In this connection, the following question appears: If $G$ is an $FC$-group such that $[G, G] \in \mathbf{SD}\mathfrak{F}$, is it necessarily true that $G/\zeta(G) \in \mathbf{SD}\mathfrak{F}$ [TM1984, Question 2A]? Some results regarding this question can be found in the paper [KOT2000].

*Suppose that $G$ is an $FC$-group with $[G, G] \in \mathbf{SD}\mathfrak{F}$ and let $\pi = \Pi(\zeta(G) \cap [G, G])$, $A = G/[G, G]$.*

(i) *If $G/\zeta(G) \notin \mathbf{SD}\mathfrak{F}$, then there is a metabelian section $H = U/V$ of $G$ satisfying the following conditions:*

  (a) *$G/U$ is finite;*
  
  (b) *$H$ is a $p$-group for some $p \in \pi$;*
  
  (c) *$[H, H]$ is abelian and bounded ;*
  
  (d) *$H/\zeta(H) \notin \mathbf{SD}\mathfrak{F}$.*

(ii) *If the Sylow $p$-subgroups of $[G, G]$ are countable for every $p \in \pi$, then $G/\zeta(G) \in \mathbf{SD}\mathfrak{F}$.*

(iii) *If $A/A^p$ is countable for every $p \in \pi$, then $G/\zeta(G) \in \mathbf{SD}\mathfrak{F}$.*

Let $G$ be a residually finite group. We say that $G$ *is a strong residually finite group* if every factor group of $G$ is residually finite [KT2003]. The direct products of finite simple non-abelian groups, bounded abelian groups, and periodic $FC$-groups with finite Sylow subgroups are some examples of strong residually finite groups. If $G$ is a residually finite group, then $G$ can be endowed with a topology in which the family of all subgroups of finite index in $G$ forms a base of neighborhoods of the identity. In fact $G$ becomes a topological group and the topology is called the profinite topology on $G$. With this interpretation, a strong residually finite group $G$ is a residually finite group in which every normal subgroup is closed in the profinite topology on $G$. The strong residually finite $FC$-groups have been studied in the paper [KO2003]. Theorem 1 of this paper can be reformulated as follows.

*Let $G$ be an $FC$-group and let $T = \mathbf{Tor}(G)$. Then $G$ is strong residually finite if and only if the following conditions hold:*

(i) *$r_0(G/T)$ is finite;*

(ii) *$Sp(G/T)$ is empty;*

(iii) *$T$ is a strong residually finite group.*

Here $Sp(G/T)$ denotes the spectrum of $G/T$. The following result of the paper [KO2003] describes the locally nilpotent strong residually finite $FC$-groups.

*Let $G$ be a locally nilpotent $FC$-*group. *Then $G$ is a strong residually finite group if and only if $\zeta(G)$ contains a finitely generated torsion-free subgroup $V$ such that $G/V = \underset{p\in\Pi(G)}{\mathbf{Dr}}\, Z_p$, where $Z_p$ is a bounded central-by-finite group.*

Let $M$ be a minimal normal subgroup of a group $G$. Since $[M, G]$ is a $G$-invariant subgroup of $M$, either $[M, G] = \langle 1\rangle$ or $[M, G] = M$. In the first case $M \leq \zeta(G)$, that is, $M$ is *central in $G$*. In the second case, $C_G(M) \neq G$, and $M$ is *non-central in $G$*.

As usual, *the socle* of the group $G$, is the subgroup $\mathbf{Soc}(G)$ generated by all minimal normal subgroups of $G$. It is known that $\mathbf{Soc}(G) = \underset{\lambda\in\Lambda}{\mathbf{Dr}}\, M_\lambda$ is a direct product of some of these. Put

$$Z = \{\lambda \in \Lambda | M_\lambda \text{ is central in } G\} \text{ and } E = \{\lambda \in \Lambda | M_\lambda \text{ is not central in } G\}.$$

Then $\mathbf{Soc}(G) = S_1 \times S_2$, where $S_1 = \underset{\lambda\in Z}{\mathbf{Dr}}\, M_\lambda$, $S_2 = \underset{\lambda\in E}{\mathbf{Dr}}\, M_\lambda$. It is possible to prove that $S_1$ and $S_2$ are independent of the choice of the decomposition of $\mathbf{Soc}(G)$. The subgroup $S_1 = \mathbf{Socc}(G)$ is called *the central socle of the group $G$* and the subgroup $S_2 = \mathbf{Socnc}(G)$ is called the *non-central socle of the group $G$*.

Starting from the non-central socle we may construct *the upper non-central socle series of $G$*:

$$\langle 1\rangle = M_0 \leq M_1 \leq \dots M_\alpha \leq M_{\alpha+1} \leq \dots M_\gamma$$

where $M_1 = \mathbf{Socnc}(G)$, $M_{\alpha+1}/M_\alpha = \mathbf{Socnc}(G/M_\alpha)$ for all $\alpha < \gamma$ and $\mathbf{Socnc}(G/M_\alpha) = \langle 1 \rangle$.

The last term $M_\gamma$ of this series is called the *non-central hypersocle of the group* $G$ and will be denoted by $Z^*(G)$. Next result of the paper [KO2003] shows the role of the subgroup $Z^*(G)$.

*Let $G$ be an $FC$-group. Then $G$ is a strong residually finite group if and only if $G/Z^*(G)$ is a strong residually finite group.*

One of the main classes considered in the paper [KO2003] is the class of (hypercentral-by-hypercentral ) $FC$-groups.

*Let $G$ be a (hypercentral-by-hypercentral ) $FC$-group. Then $G$ is a strong residually finite group if and only if $G/Z^*(G) \leq T \times A$, where*

(i) *$A$ is a torsion-free abelian group of finite rank and $S_p(A)$ is empty.*

(ii) *$T$ contains a normal subgroup $L = \mathbf{Dr}_{p\in\Pi(L)} L_p$ where $L_p$ is a finite $p$-group.*

(iii) *$T/L = \mathbf{Dr}_{p\in\Pi(T/L)} Q_p$ where $Q_p$ is a bounded central-by-finite $p$-group.*

(iv) *The Sylow $p$-subgroups of $T$ are bounded central-by-finite groups.*

We next consider some other aspects of the theory of $FC$-groups and specifically some results that were obtained later. In the article [CSN1957-2], S. N. Chernikov gave a characterization of non-periodic $FC$-groups, which implies the following important result.

*A group $G$ is a non-periodic $FC$-group if and only if $G$ is a subgroup of a direct product $T \times A$ of a periodic $FC$-group $T$ and a torsion-free abelian group $A$.*

S.N. Chernikov drew attention to the fact that the case of non-periodic $FC$-groups has its own characteristics. If $G$ is an $FC$-group, then $\mathbf{Tor}(G)$ contains all elements of finite order. However it should be noted that not all properties of the periodic part $\mathbf{Tor}(G)$ can be transferred to the direct factor $T$. In particular, if $\mathbf{Tor}(G)$ is a subgroup of a direct product of finite groups, the conditions under which $G$ is embedded in a direct product of finite groups and abelian torsion-free groups are not clear. Also despite that we have here some interesting parallels with the theory of Abelian groups, the situation for FC-groups is much more complicated.

The first person to begin examining these questions about the structure of non-periodic $FC$-groups was L. A. Kurdachenko, one of S.N. Chernikov's students. His series of papers [KLA1977, KLA1979, KLA1981-1, KLA1981-2, KLA1983-1, KLA1983-2, KLA1984, KLA1986-1, KLA1986-2, KLA1987, KLA1988] was dedicated to this topic. We will not dwell here on these results, because they were sufficiently covered in detail in the survey [OS2009] of J. Otal and N.N. Semko.

We now consider another area which has its origins in the work of S.N. Chernikov, namely the consideration of groups with other restrictions on the conjugacy class of an element. This was the beginning of a large area of research that started with the *layer-Chernikov groups*, which were a natural extension of layer-finite groups (S.N. Chernikov called them *layer-extremal groups*). A group $G$ is called *layer-Chernikov* if the subgroup $\langle g \mid g \in G \text{ and } |g| = n\rangle$ is Chernikov for every positive integer $n$. S.N. Chernikov asked his student Ya.D.Polovitskii to study these groups and this he did in his work [PYa1962-1]. As a natural extension of the class of the layer-Chernikov groups, the class of periodic groups with Chernikov conjugacy classes ($CC$-groups) was introduced [PYa1962-2] (in this work, these groups were named *locally extremal groups*). The final form of the class of $CC$-groups was introduced in the paper [PYa1964]. We will not limit ourselves with the definition of Y.D. Polovitskii, but immediately look at the more general situation.

Let $G$ be a group and $S$ a $G$-invariant subset of $G$. Then the centralizer $C_G(S)$ is a normal subgroup of $G$. The corresponding factor group $\mathbf{Coc}_G(S) = G/C_G(S)$ is called the cocentralizer of the subset $S$ in the group $G$ ( L.A. Kurdachenko [KLA1993] ). The cocentralizers of many objects related to the group often influence the structure of the group and is a subject of study in many branches of group theory. For instance, in the theory of finite groups, cocentralizers of chief factors have played a significant role, one example being the case of formation theory, where local formations play an important role. Such local formations are defined using cocentralizers of chief factors.

Many types of infinite groups have been studied using cocentralizers of different conjugacy classes. Let $\mathfrak{X}$ be a class of groups. We say that a group $G$ has $\mathfrak{X}$*-conjugacy classes* (or $G$ is an $\mathfrak{X}C$-group) if $\mathbf{Coc}_G(g^G) \in \mathfrak{X}$ for each $g \in G$. For example, if $\mathfrak{X} = \mathfrak{I}$ is the class of all identity groups, then the class of $\mathfrak{I}C$-groups is exactly the class $\mathfrak{A}$ of all abelian groups. Therefore we may consider the class of $\mathfrak{X}C$-groups as a natural generalization of the class of abelian groups. When $\mathfrak{X} = \mathfrak{F}$ is the class of all finite groups, then the $\mathfrak{F}C$-groups are the groups with finite conjugacy classes or $FC$-groups, which we have briefly discussed already. This class is a suitable extension both of $\mathfrak{F}$ and $\mathfrak{A}$, and inherits many properties of these classes.

The class $\mathfrak{C}$ of Chernikov groups is a natural generalization of finite groups. Therefore the class of the groups with Chernikov conjugacy classes (the class of $CC$-groups) is the next natural and interesting generalization of abelian groups. As we mentioned above, Ya.D.Polovitskii [PYa1964] initiated the study of these groups. But the fundamental contribution to the development of the theory of $CC$-groups has been made by J. Otal. The most natural example of a $CC$-group is that of a direct product of Chernikov groups. Slightly more complicated examples can be constructed in the following way. Let $\{G_\lambda \mid \lambda \in \Lambda\}$ be a family of groups and consider their Cartesian product $C = \mathop{\mathbf{Cr}}_{\lambda\in\Lambda} G_\lambda$. The subgroup $\mathop{\mathbf{Zr}}_{\lambda\in\Lambda} G_\lambda = (\mathop{\mathbf{Dr}}_{\lambda\in\Lambda} G_\lambda)\zeta(\mathop{\mathbf{Cr}}_{\lambda\in\Lambda} G_\lambda)$ is called *the central product of the groups* $G_\lambda$, for $\lambda \in \Lambda$. Such central products of Chernikov groups are examples of $CC$-groups. Naturally one can ask what role central products of Chernikov groups play in the structure of $CC$-groups.

We observe that if $G$ is a $CC$-group, then $\mathbf{Tor}(G)$ contains all the elements of $G$ of finite order. In a similar fashion, if the subgroup $G_\lambda$ is periodic for each $\lambda \in \Lambda$, then

$$\mathop{\mathbf{Zrr}}_{\lambda\in\Lambda} G_\lambda = (\mathop{\mathbf{Dr}}_{\lambda\in\Lambda} G_\lambda)\mathbf{Tor}(\zeta(\mathop{\mathbf{Cr}}_{\lambda\in\Lambda} G_\lambda))$$

contains all the elements of finite order in $\mathop{\mathbf{Zr}}_{\lambda\in\Lambda} G_\lambda$. The subgroup $\mathop{\mathbf{Zrr}}_{\lambda\in\Lambda} G_\lambda$ is called *the central restricted product of the groups* $G_\lambda$, $\lambda \in \Lambda$.

Let $G$ be a group and $\mathfrak{X}$ be a class of groups. The family $\{H_\lambda \mid \lambda \in \Lambda\}$ is said to be an $\mathfrak{X}$-*residual family for the group* $G$ if it satisfies the conditions:

(i) $H_\lambda$ is a normal subgroup of $G$;

(ii) $\bigcap_{\lambda\in\Lambda} H_\lambda = \langle 1 \rangle$;

(iii) $H/H_\lambda \in \mathfrak{X}$ for all $\lambda \in \Lambda$.

If a group $G$ has an $\mathfrak{X}$-residual family, then we say that $G$ *is a residually* $\mathfrak{X}$-*group*, denoted by $G \in \mathbf{R}\mathfrak{X}$. For example, if $G$ is a $CC$-group, then $G/\zeta(G)$ is residually Chernikov.

As for abelian groups, the first step here is to consider countable groups. The first result here was obtained by Ya.D. Polovitskii [PYa1962-2].

*Let $G$ be a periodic countable $CC$-group. If $G$ is a residually Chernikov group, then $G$ is isomorphic to some subgroup of a direct product of Chernikov groups.*

A complete picture for the countable case was obtained by J. Otal and his student M. Gonzales. In their paper [GO1990], they obtained the following result.

*Let $G$ be a countable $CC$-group. Then $G$ is a section of a direct product of Chernikov-by-(finitely generated abelian) $CC$-groups. If $G$ is periodic, then $G$ is a section of a direct product of Chernikov groups.*

The next natural steps in the study of the structure of $CC$-groups consist of a consideration of $CC$-groups with countable central factor-group $G/\zeta(G)$ and residually Chernikov $CC$-groups.

If $\mathfrak{X}$ is a class of groups then let $\mathbf{SD}\mathfrak{X}$ denote the class of all subgroups of direct products of $\mathfrak{X}$-groups and let $\mathbf{QSD}\mathfrak{X}$ denote the class of all sections of direct products of $\mathfrak{X}$-groups. Also let $\mathfrak{A}^{\mathbf{tf}}$ denote the class of all torsion-free abelian groups. We next exhibit some important results from the paper [GO1995].

*Let $G$ be a $CC$-group.*

(i) *Suppose that $G$ is a residually Chernikov $CC$-group and that $G/\zeta(G)$ is countable. Then $G \in \mathbf{SD}(\mathfrak{C} \cup \mathfrak{A}^{\mathbf{tf}})$. Moreover, if $G$ is periodic, then $G \in \mathbf{SD}\mathfrak{C}$.*

(ii) *If $G$ is a residually Chernikov $CC$-group, then $G \in \mathbf{QSD}(\mathfrak{C} \cup \mathfrak{A}^{\mathbf{tf}})$. Moreover, if $G$ is periodic, then $G \in \mathbf{QSD}\mathfrak{C}$.*

These results allow us to obtain the following information about the structure of the central factor group and the derived subgroup of a $CC$-group.

*Let $G$ be a $CC$-group. Then $G/\zeta(G) \in \mathbf{QSD}(\mathfrak{C} \cup \mathfrak{A}^{\mathbf{tf}})$* and $[G, G] \in \mathbf{QSD}\mathfrak{C}$. *If $G$ is periodic, then $G/\zeta(G) \in \mathbf{QSD}\mathfrak{C}$. If $G$ is a residually Chernikov $CC$-group, then $G/\zeta(G) \in \mathbf{SD}\mathfrak{C}$* and $[G, G] \in \mathbf{SD}\mathfrak{C}$.

We say that a group $G$ is a *c-residually Chernikov group* if it has a Chernikov residual family $\{H_\lambda \mid \lambda \in \Lambda\}$ such that $\{\langle S\rangle^G \cap H_\lambda \mid \lambda \in \Lambda\}$ is countable for every finite subset $S$ of $G$. In this case $\{H_\lambda \mid \lambda \in \Lambda\}$ is said to be *a Chernikov c-family* of $G$. In [GO1996], J. Otal and M. Gonzales continued their investigation of the structure of $CC$-groups and they considered the role of the central and central restricted products of Chernikov groups in the general structure of a $CC$-group. Here are the main results of this article.

*Let $G$ be an arbitrary $CC$-group. Then $G/\zeta(G)$ is always a c-residually Chernikov group.*

Furthermore, the following conditions are equivalent:

(i) *$G$ is a $CC$-group and $G$ is c-residually Chernikov group;*

(ii) *$G$ is isomorphic to some subgroup of $\mathbf{Zr}_{\lambda\in\Lambda} G_\lambda$, where $G_\lambda$ is a Chernikov group for each $\lambda \in \Lambda$.*

*In addition,* if *$G$ is periodic then the following conditions are equivalent:*

(i) *$G$ is a $CC$-group and $G$ is c-residually Chernikov group; .*

(ii) *$G$ is isomorphic to some subgroup of $\mathbf{Zrr}_{\lambda\in\Lambda} G_\lambda$,* where *$G_\lambda$ is a Chernikov group for each $\lambda \in \Lambda$.*

From these results we can deduce the following facts about a $CC$-group $G$:

(i) *$G/\zeta(G)$ is isomorphic to a subgroup of $\mathbf{Zr}_{\lambda\in\Lambda} G_\lambda$, where $G_\lambda$ is a Chernikov group for each $\lambda \in \Lambda$.*

(ii) If $G/\zeta(G)$ is periodic, then $G/\zeta(G)$ *is isomorphic to a subgroup of $\mathbf{Zrr}_{\lambda\in\Lambda} G_\lambda$, where $G_\lambda$ is a Chernikov group for each $\lambda \in \Lambda$.*

(iii) *If the orders of the elements of $G/[G,G]\zeta(G)$ are bounded, then $G/\zeta(G)$ is isomorphic to a subgroup of a direct product of Chernikov groups.*

Assuming the *continuum hypothesis* holds then we can also deduce the following fact.

Let $G$ be residually Chernikov $CC$-group. Then $G$ is isomorphic to a subgroup of $\mathbf{Zr}_{\lambda\in\Lambda} G_\lambda$ where $G_\lambda$ is a countable $CC$-group for each $\lambda \in \Lambda$. In particular, if $G$ is periodic, then $G$ is isomorphic to a subgroup of $\mathbf{Zrr}_{\lambda\in\Lambda} G_\lambda$ where $G_\lambda$ is a countable $CC$-group for each $\lambda \in \Lambda$.

The study of $\mathfrak{X}C$-groups for other important classes of groups $\mathfrak{X}$ is not as advanced. When $\mathfrak{X}$ is the class of polycyclic-by-finite groups this study was begun by S. Franciosi, F. de Giovanni and M. Tomkinson [FdeGT1990]. The class of minimax groups contains the classes of Chernikov groups and polycyclic-by-finite groups, where a group is called *minimax* if it has a finite subnormal series whose factors are either polycyclic-by-finite or Chernikov groups. A group $G$ is called an *$MC$-group* or a group with minimax conjugacy classes if $\mathbf{Coc}_G(g^G)$ is minimax for each $g \in G$, a class introduced by L.A. Kurdachenko in [KLA1993]. The following results were obtained in this paper.

*Let $G$ be a $MC$-group. Then*

(i) *for each element $g \in G$ the subgroup $\langle g \rangle^G$ is minimax;*

(ii) *for each element $g \in G$ the subgroup $[G, g]$ is minimax.*

*Let $G$ be a $MC$-group. Then the following assertions are equivalent:*

(i) *$G$ is a Baer-nilpotent group.*

(ii) *$G$ is a locally nilpotent group.*

(iii) *$G$ is a $\bar{Z}$-group.*

(iv) *$G$ is a hypercentral group.*

(v) *$G$ has an ascending central series of a length at most $2\omega$.*

(vi) *$G$ is a group with normalizer condition.*

(vii) *$G$ is an $\tilde{N}$-group.*

*Let $G$ be an $MC$-group. Then the following assertions are equivalent:*

(i) *$G$ is an $SN$-group.*

(ii) *$G$ is an $SN^*$ - group.*

(iii) *$G$ is an $\overline{SN}$-group.*

(iv) *$G$ is an $SI$-group.*

(v) *$G$ is a hyperabelian group.*

(vi) *$G$ is a hypoabelian group.*

(vii) *$G$ is an $\overline{SI}$-group.*

(viii) *$G$ is a locally soluble group.*

(ix) *$G$ is a radical group.*

(x) *$G$ has an ascending series of normal subgroups whose factors are abelian and the length is at most $\omega$.*

(xi) *$G$ has a descending series of normal subgroups whose factors are abelian and the length is at most $\omega + 1$.*

(xii) *$G$ is a group with the normalize condition.*

(xiii) *$G$ is an $\tilde{N}$-group.*

The study of $MC$-groups was continued in [KO2001] and in the papers [KS1996] and [KLA1996], other classes of $\mathfrak{X}C$-groups were considered.

A subgroup $H$ of a group $G$ is called *almost normal* if it is normal in a subgroup of finite index, which is equivalent to saying that $N_G(H)$ has finite index in $G$. A subgroup $H$ of $G$ is almost normal if and only if the set $\mathbf{cl}_G(H) = \{H^g \mid g \in G\}$ is finite. Clearly, normal subgroups are almost normal. In [NB1955], B. H. Neumann considered groups in which every subgroup is almost normal and he showed that such groups were central-by-finite. S.N. Chernikov [CSN1957-1] considered groups in which $C_G(A)$ has finite index for each abelian subgroup $A$, and obtained a similar result to that of Neumann. In view of this result, the question as to the structure of those groups $G$ in which $N_G(A)$ has finite index for each abelian subgroup $A$ (i.e. every abelian subgroup is almost normal) was naturally raised. The answer to it was obtained by I.I. Eremin, a student of S.N. Chernikov, who proved in [EI1959]that such groups are centre-by-finite. In the article [EI1960], I.I. Eremin began to study groups in which all infinite subgroups are almost normal. Locally almost solvable groups with this property were described later by N.N. Semko, S.S. Levischenko and L.A. Kurdachenko [SLK1983] and the results can be summarized in the following way.

If $H$ is a subgroup of the group $G$, then we denote the set of all conjugates of $H$ in $G$ by $\mathbf{Cl}_G(H) = \{H^g | g \in G\}$, the conjugacy class of $H$ in $G$. Certainly, $|\mathbf{Cl}_G(H)| = |G : N_G(H)|$. We call the subgroup $\bigcap_{g\in G} N_G(H^g) = \bigcap_{g\in G} N_G(H)^g$ *the normalizer of the conjugacy class of $H$ in $G$* and denote it by $N_G(\mathbf{Cl}_G(H))$. If $\mathfrak{X}$ is a class of groups then the group $G$ is said to have $\mathfrak{X}$-*classes of conjugate subgroups* if $G/N_G(\mathbf{Cl}_G(H)) \in \mathfrak{X}$ for every subgroup $H$ of $G$ [KOS2004].

When $\mathfrak{I}$ is the class of identity groups, then clearly $G$ has $\mathfrak{I}$-classes of conjugate subgroups if and only if every subgroup of $G$ is normal so $G$ is a Dedekind group. If $\mathfrak{X} = \mathfrak{F}$, then we obtain the groups with finite conjugacy classes of subgroups; these are exactly the groups considered by B.H. Neumann.

The next natural step is to consider classes of infinite groups close to finite groups that have been well-studied from different points of view. The first candidates are the classes $\mathfrak{C}$ of all Chernikov groups, and $\mathfrak{P}$ of all polycyclic-by-finite groups. In the paper [PYD1977], Ya.D. Polovitskii considered groups with Chernikov classes of conjugate subgroups and proved that a periodic group with Chernikov classes of conjugate subgroups is central-by-Chernikov. By contrast, in the paper [KO2005], an example was constructed showing that this result is not true in general. In the paper [KO2005], the following description of groups with Chernikov classes of conjugate subgroups was obtained.

*Let $G$ be a group with Chernikov classes of conjugate subgroups. Then the following assertions hold.*

(i) *$G$ contains an abelian normal subgroup $A$ such that $G/A$ is Chernikov.*

(ii) *$G/C_G(\mathbf{Tor}(A))$ is finite.*

(iii) *$[G, G]$ is Chernikov.*

By contrast, and perhaps surprisingly, groups with polycyclic-by-finite classes of conjugate subgroups behave in a fashion similar to that considered by B.H. Neumann, as the main result of the paper [KOSO2004] shows.

*A group $G$ has polycyclic-by-finite classes of conjugate subgroups if and only if $G/\zeta(G)$ is polycyclic-by-finite.*

In the article [CSN1947], S.N. Chernikov considered locally finite $p$-groups with the minimal condition for normal subgroups and these turned out to be Chernikov. Then, V.S. Carin [CVS1949-1] constructed an example of a metabelian group that is not Chernikov, but satisfies the minimum condition for normal subgroups. This work showed that the study of groups with restrictions on the normal subgroups requires the use of other techniques, namely techniques based on the theory of modules.

Let $G$ be a group, and $A$ a normal abelian subgroup of $G$. We set $H = G/A$, and define the natural action of $H$ on $A$ by the rule: $a^h = a^g$, whenever $h = gA \in H$, $g \in G$. This action extends naturally to an action of the group ring $\mathbb{Z}H$ on $A$ and this allows us to study this situation using the methods and techniques of group rings and their modules. This technique has been effectively implemented in the theory of finite groups and in the classical works of P. Hall [HP1955, HP1959-1], where he demonstrated the effectiveness of the theory of Noetherian modules in the study of infinite soluble groups. However, in situations where artinian modules appear (which happens in the study of soluble groups with the minimal condition for normal subgroups), the corresponding group ring can have a very complicated structure. Therefore, other approaches also need to be used. These other approaches include criteria for complementation of submodules, semisimplicity criteria, the search for generalizations of the classical theorems of Maschke and Fitting, the search for some canonical direct decompositions of Artinian modules, and so on. In this area, D.I. Zaitsev, the best Kiev student of S.N. Chernikov, has made significant contributions. We shall not discuss here the results pertaining to this broad topic, as they were described in detail in the survey of L.S. Kazarin and L.A. Kurdachenko [KK1992], and later in the book of L.A. Kurdachenko, J. Otal, and I.Ya. Subbotin [KOS2007].

Module theoretic methods also play an important role in the theory of factorizable groups, a topic which grew naturally out of the theory of groups with various systems of complemented subgroups created by S.N. Chernikov and developed by his students. The results of these studies are reflected in detail in the surveys of S.N. Chernikov [[CSN1969, CSN1971-2, CSN1978], and in his book [CSN1980-1]. New approaches, giving the possibility of significant progress in the theory of factorizable groups, were developed by D.I. Zaitsev (see the survey [KK1992]). Significant contributions to the theory of factorizable groups were made by N.S. Chernikov, a son of S.N. Chernikov, and Y.P. Sysak.

In order to illustrate S.N. Chernikov's great vision, we would like briefly to discuss his research ideas related to products of groups that initially did not seem significant, but later generated a good deal of research by many prominent algebraists. We are here talking about the uniform product of groups, or, as it is now called, mutually permutable products of subgroups.

Uniform products was a topic that S.N. Chernikov suggested to his Ph.D student V.P. Shunkov for his dissertation. V. P. Shunkov considered groups that decomposed into a *uniform product of Sylow p-subgroups* [SV1964]. Following S.N. Chernikov, we say that a group $G$ *is decomposed into a uniform product of subgroups* $H_\lambda, \lambda \in \Lambda$, if $\langle x\rangle \langle y\rangle = \langle y\rangle \langle x\rangle$ for every element $x \in H_\lambda, y \in H_\mu$, and for each pair of indices $\lambda, \mu \in \Lambda$. V.P. Shunkov [SV1964] described periodic groups that decompose into the uniform product of its Sylow $p$-subgroups.

*Let $G$ be a group. Then $G$ can be decomposed into the uniform product of its Sylow $p-$subgroups for all prime $p$ if and only if $G = A \leftthreetimes B$ where $A$ is a normal abelian subgroup, $A$ is quasicentral in $G$, $B = \mathop{\mathbf{Dr}}_{p\in\Pi(B)} B_p$, $B_p$ is a Sylow $p$- subgroup of $B$, $p \in \pi(B)$, $\Pi(A) \cap \Pi(B) = \varnothing$.*

Further investigations of uniform products of different kinds of subgroups were done in the articles written by V. G. Vasil'ev [VVG1977, VVG1978], and by S. N. Chernikov's Kiev students A. M. Andruhov [AAM1971], D. I. Zaitsev [ZD1982], V. V. Tsybulenko, and S. G. Kolesnik

[TVKS1993]. The following concept of a quasicentral product of groups is a natural generalization of uniform products (see [SI1982], [SI1983]). We call a group $G$ *a quasicentral product of a subgroup* $A$ *by a subgroup* $B$ if $G = AB$ and every element of $B$ normalizes each subgroup of $A$ [S1984]. As examples of quasicentral products we mention periodic groups decomposed into uniform product of its Sylow $p-$subgroups [SV1964] and soluble $\bar{T}$-groups (groups in which normality is a transitive relation) [RD1964]. V.P. Shunkov's research was followed in 1989 by M.Asaad and others who dedicated their research to totally (the same as the uniform product) and mutually permutable products. Since that time there has developed a series of general methods for studying these kind of products and we refer the reader to the monograph [BBAERRMA2010] where this area was discussed in detail.

It is interesting to note that V.P. Shunkov admitted in a conversation with one of the authors of the current survey that he did not consider his pioneer research on the uniform product as an important achievement. Nevertheless, S.N. Chernikov, who suggested this topic to V.P. Shunkov, clearly saw its potential at the beginning. Perhaps this is one of S.N. Chernikov's great qualities: he could recognize the potential of a research topic at the beginning. That is why, in particular, he grew such a distinguished scientific school.

We now touch on another interesting area of the theory of infinite groups. It is well-known that the property "to be a normal subgroup" is not transitive. Accordingly, the group $G$ is called a $T$*-group* if every subnormal subgroup of $G$ is normal. The group $G$ is called a $\bar{T}$-group if every subgroup of $G$ is a $T$-group. Finite $T$-groups have been studied by many authors since 1942. The investigation of infinite soluble $T$-group was begun by D. J. S. Robinson in [RD1964] where he obtained the main results of their structure. After this, $T$-groups and $\bar{T}$-groups were studied by many other algebraists, mmong whom were students of S.N. Chernikov, namely M.I. Kargapolov and I.N. Abramovsky [AI1966, AbK1958]. The Ph.D. thesis of I.N. Abramovsky was devoted to the elucidation of various (especially local) properties of these groups. Some natural generalizations of $T$-groups were studied by I.Ya. Subbotin, and N.F. Kuzennii.

An interesting property of the finite $\bar{T}$-groups is the fact that every subgroup is pronormal. So there is a natural question concerning groups all of whose subgroups are pronormal and the connection between such groups and $\bar{T}$-groups arises. Recall that a subgroup $H$ of $G$ is called *pronormal* (in $G$), if the subgroups $H$ and $H^g$ are conjugate in $\langle H, H^g\rangle$ for each element $g \in G$. Pronormal subgroups arise naturally in finite soluble groups when studying Sylow subgroups, Hall subgroups, system normalizers, Carter subgroups, and so on. The term "pronormal subgroup" is due to P. Hall. Unlike the Sylow subgroups of a finite group, pronormal subgroups carry no conditions associated with order and therefore are natural candidates for study in infinite groups with no further restrictions. N.F. Kuzennii and I.Ya. Subbotin in [KNS1987 -1] obtained a complete description of periodic locally graded groups with all subgroups pronormal, as well as non-periodic locally solvable groups of this kind. It turned out that, unlike with finite groups [PT1971], the class of groups with all subgroups pronormal and the class of $\bar{T}$-groups are different. As was shown by N.F. Kuzennii and I.Ya. Subbotin in a further paper [KNS1987-2], such a coincidence takes place only for groups with all cyclic subgroups pronormal. It should be noted that S.N. Chernikov immediately recognized the promise of the study of groups saturated with pronormal subgroups and greatly encouraged its research. It turned out that further research in this area makes it possible to establish new and interesting connections and generalizations. We say that a subgroup $H$ of $G$ is *transitively normal* if $H$ is normal in every subgroup $K \geq H$ in which $H$ is subnormal. Each pronormal subgroup is transitively normal, so it is natural to pose the question of when the transitively normal subgroups are pronormal. Some of the conditions for this have been found by L.A. Kurdachenko and I.Ya. Subbotin in the article [KS2006].

The following characterization of finite nilpotent groups is also related to pronormality: *a finite group is nilpotent if and only if each pronormal subgroup is normal.* In fact in every locally nilpotent group all pronormal subgroups are normal [KNS1988]. It appears to be unknown whether the converse holds, although an affirmative answer holds for some fairly large classes of groups, as obtained by L.A. Kurdachenko, J. Otal, I.Ya. Subbotin [KOS2002-2], L.A Kurdachenko, I.Ya. Subbotin [KS2003] and L.A. Kurdachenko, A. Russo, G. Vincenzi [KRV2006].

A subgroup $H$ of $G$ is called *weakly pronormal in* $G$, if for every pair of subgroups $K$ and $L$ such that $H \leq K \trianglelefteq L$ we have $L \leq N_G(H)K$ (in this case, we say that $H$ has the Frattini property). For finite solvable groups, T. Peng in [PT1971] proved that weak pronormality implies pronormality. This result was extended by L.A. Kurdachenko, J. Otal, I.Ya. Subbotin in [KOS2005] to groups having an ascending series of normal subgroups the factors of which are $N$-groups.

Pronormality is also not a transitive property. Therefore it is natural to investigate the groups in which the property of being a pronormal subgroup is transitive. A quite complete result here was obtained by

L.A. Kurdachenko and I.Ya. Subbotin in [KS2002]. It is interesting to note that the question concerning the transitivity of a particular case of pronormality, namely abnormality, is more complicated. Recall that a subgroup $H$ of $G$ is called abnormal (in $G$), if $g \in \langle H, H^g \rangle$ for each element $g$ of $G$. The best result concerned with abnormality being a transitive relation, as well as other important properties of abnormal subgroups, can be found in the article [KS2005] and the survey [KOS2007].

As we noted above, S.N. Chernikov encouraged the study of the influence of properties of important subgroup families on the group structure. The dual problem of the effect on the group structure due to important families of factor groups (i.e., of all proper factor groups, all finite factor groups, all factor groups by infinite normal subgroups, etc.) has also been studied. This interesting topic turned out to be useful and it is now quite well developed. Significant contributions to this theory have been made by L.A. Kurdachenko and I.Ya. Subbotin, Kiev students of S.N. Chernikov. We will not dwell on this because these issues have been detailed in the book, L.A. Kurdachenko, J. Otal, I.Ya. Subbotin [KOS2002-1].

S.N. Chernikov initiated the study of the influence of dense systems of different subgroups on the structure of a group. This approach was clearly articulated in the work [CSN1975]. This theme also proved to be very fruitful, not only for S.N. Chernikov and his students, but also many other well-known algebraists. This topic was covered in the monograph of N.N. Semko [SN1998], so we will not discuss it further here.

Some ideas and S.N. Chernikov's results have been developed in works of the Gomel algebraic school. In paper [ASMSh2005] S.N. Chernikov's result on finite groups with complemented primary cyclic subgroups has been included in the general theory of generalized the central elements. Developing S.N. Chernikov's idea on groups with a dense system of subnormal subgroups, L.N. Zakrevskaya, L.A. Shemetkov and A.E. Shmigirev investigated [ZL1984, ShA2003, ShSh2004, ShA2004] finite groups with a dense system of $\mathfrak{F}$-subnormal subgroups.

Of course it is impossible in even a large survey to briefly exhibit all the research and development of infinite groups that has grown from the visionary ideas of S.N. Chernikov. Hopefully this survey illustrates the great creativity and generosity of this prominent mathematician and caring teacher.

## References

[AAM1971] Andruhov, A. M. Uniform products of infinite cyclic groups. (Russian) Groups with systems of complemented subgroups (Russian), pp. 59–146, 225. Izdanie Inst. Mat. Akad. Nauk Ukrain. SSR, Kiev, 1971.

[AbK1958] Abramovsky I.N., Kargapolov M.I. Finite groups with transitivity for normal subgroups. Uspehi Math. Nauk – 1958, 13, number 3, 242–243.

[AK2003] Artemovich O.D., Kurdachenko L.A. Groups, which are rich on X – subgroups. Bulletin Lviv University, series mekhanik and math. – 2003, 61, 218–237

[BBAERRMA2010] Ballester-Bolinches A., Esteban-Romero R., Asaad M., Products of Finite Groups, De Gryuter, 2010.

[BR1940] Baer R. Sylow theorems of infinite groups. Duke Math. J. – 6 (1940), 598–614.

[BR1948] Baer R. Finiteness properties of groups. Duke Math.J.– 15(1948), 1021–1032.

[BR1966] Baer R. Local and global hypercentrality and supersolubility I, II. Indagationes mathematical – 1966, 28, 93–126

[BR1969] Baer R. Lokal endlich–auflösbare Gruppen mit endlichen Sylowuntergruppen. Journal für die reine und angewandte Mathematik – 239/240(1969), 109–144.

[BVV1981] Belyaev V.V. Locally finite groups with Chernikov Sylow p – subgroups. Algebra i Logika–20 (1981), 605–619. English transl. Algebra and Logic – 20(1981), 393–402.dachenko L.A. Groups, which are rich on X-subgroups. Bulletin Lviv University, series mekhanik and math. – 2003, 61, 218–237

[BR1940] Baer R. Sylow theorems of infinite groups. Duke Math. J. – 6 (1940), 598–614.

[BR1966] Baer R. Local and global hypercentrality and supersolubility I, II. Indagationes mathematical – 1966, 28, 93–126

[BR1969] Baer R. Lokal endlich – auflösbare Gruppen mit endlichen Sylowuntergruppen. Journal für die reine und angewandte Mathematik – 239/240(1969), 109–144.

[BVV1981] Belyaev V.V. Locally finite groups with Chernikov Sylow p – subgroups. Algebra i Logika – 20 (1981), 605–619 English transl. Algebra and Logic – 20(1981), 393–402.

[CSN1939] Chernikov S.N. Infinite special groups. Math. Sbornik – 1939, 6, 199–214

[CSN1940-1] Chernikov S.N. Infinite locally soluble groups. Math. Sbornik – 1940, 7, 35–61

[CSN1940-2] Chernikov S.N. To theory of infinite special groups. Math. Sbornik – 1940, 7, 539–548.

[CSN1940-3] Chernikov S.N. On groups with Sylow sets. Math. Sbornik – 1940, 8, 377–394.

[CSN1943] Chernikov S.N. To theory of locally soluble groups. Math. Sbornik – 1943, 13, 317–333.

[CSN1946] Chernikov S.N. Divisible groups possesses an ascending central series. Math. Sbornik – 1946, 18, 397–422.

[CSN1947] Chernikov S.N. To the theory of finite p – extensions of abelian p - groups. Doklady AN USSR–1947, 58, 1287–1289.

[CSN1948-1] Chernikov S.N. Infinite layer – finite groups. Math. Sbornik – 1948, 22, 101–133.

[CSN1948-2] Chernikov S.N. To the theory of divisible groups. Math. Sbornik – 1948, 22, 319–348.

[CSN1948-3] Chernikov S.N. A complement to the paper "To the theory of divisible groups". Math. Sbornik – 1948, 22, 455–456.

[CSN1949] Chernikov S.N. To the theory of torsion – free groups possesses an ascending central series. Uchenye zapiski Ural University – 1949, 7, 3–21.

[CSN1950-1] Chernikov S.N. On divisible groups with ascending central series. Doklady AN USSR – 1950, 70, 965–968.

[CSN1950-2] Chernikov S.N. On a centralizer of divisible abelian normal subgroups in infinite periodic groups. Doklady AN USSR – 1950, 72, 243–246.

[CSN1950-3] Chernikov S.N. Periodic ZA – extension of divisible groups. Math. Sbornik – 1950, 27, 117 – 128.

[CSN1955] Chernikov S.N. On complementability of Sylow $p$-subgroups in some classes of infinite groups. Math. Sbornik. – 37(1955), 557 – 566.

[CSN1957-1] Chernikov S.N. On groups with finite conjugacy classes. Doklady AN USSR – 1957, 114, 1177 – 1179

[CSN1957-2] Chernikov S.N. On a structure of groups with finite conjugate classes. Doklady AN SSSR – 115(1957), 60 – 63.

[CSN1958] Chernikov S.N. On layer – finite groups. Math. Sbornik – 1958, 45, 415–416.

[CSN1959] Chernikov S.N. Finiteness conditions in general group theory. Uspekhi Math. Nauk – 1959, 14, 45 – 96.

[CSN1960] Chernikov S.N. On infinite locally finite groups with finite Sylow subgroups. Math. Sbornik – 1960, 52, 647 – 652.

[CSN1967] Chernikov S.N. Groups with prescribed properties of a system of infinite subgroups. Ukrain. Math. Journal – 1967, 19, 111 – 131.

[CSN1969] Chernikov S.N. Investigations of groups with prescribed properties of subgroups. Ukrain. Math. Journal – 1969, 21, 193 – 209.

[CSN1971-1] Chernikov S.N. On a problem of Schmidt. Ukrain. Math. Journal – 1971, 23, 598 – 603

[CSN1971-2] Chernikov S.N. On groups with the restrictions for subgroups. "Groups with the restrictions for subgroups", NAUKOVA DUMKA: Kyiv – 1971, 17 – 39.

[CSN1975] Chernikov S.N. Groups with dense system of complement subgroups. "Some problems of group theory", MATH. INSTITUT: Kyiv – 1975, 5 – 29.

[CSN1980-1] Chernikov S.N. The groups with prescribed properties of systems of subgroups. NAUKA : Moskow – 1980.

[CSN1980-2] Chernikov S.N. Infinite groups, defined by the properties of system of infinite subgroups. "VI Simposium on group theory", NAUKOVA DUMKA: Kyiv – 1980, 5 – 22.

[KC1947] Kurosh A.G., Chernikov S.N. Soluble and nilpotent groups. Uspekhi Math. Nauk – 1947, 2, number 3, 18 – 59

[AI1966] Abramovsky I.N. Locally generalized Hamiltonian groups. Sibirian Math. J. – 1966, 7, number 3, 481 – 485.

[AbK1958] Abramovsky I.N., Kargapolov M.I. Finite groups with transitivity for normal subgroups. Uspehi Math. Nauk – 1958, 13, number 3, 242 – 243.

[AK2003] Artemovich O.D., KurAI1966. Abramovsky I.N. Locally generalized Hamiltonian groups. Sibirian Math. J. – 1966, 7, number 3, 481 – 485.

[CVS1949-1] Charin V.S. Remark on minimal condition for subgroups. Doklady AN USSR – 1949, 66, 575 – 576.

[CVS1949-2] Charin V.S. On divisible groups with root series of finite length. Doklady AN USSR – 1949, 66, 809 – 811.

[CVS1951] Charin V.S. To a theory of locally nilpotent groups. Mat. Sbornik – 29(1951), 433 – 454.

[CZ1988] Charin V.S., Zaitsev D.I. Groups with the finiteness conditions and other restrictions on subgroups. Ukrain. Math. J. – 1988, 40, 3, 277 – 287

[DAP1937] Ditsman A. P. On p – groups. Doklady Akad. Nauk SSSR – 15 (1937), 71–76.

[DMR1994] Dixon M.R. Sylow theory, formations and Fitting classes in locally finite groups. WORLD SCIENTIFIC : Singapore – 1994.

[DKP2007] Dixon M.R., Kurdachenko L.A., Polyakov N.V. On some ranks of infinite groups. Ricerche Mat. – 56(2007), no 1, 43 – 59

[DKS2007] Dixon M.R., Kurdachenko L.A., Subbotin I.Ya. On various rank conditions in infinite groups. Algebra and Discrete Mathematics, 2007, number 4, 23 – 43

[DS2009] Dixon M.R., Subbotin I.Ya. Groups with finiteness conditions on some subgroup systems: a contemporary stage. Algebra and Discrete Mathematics – 2009, Number 4, 29 – 54

[EI1959] Eremin I.I. Groups with finite conjugacy classes of abelian subgroups. Math. Sbornik – 1959, 47, 45 – 54.

[EI1960] Eremin I.I. Groups with finite conjugacy classes of infinite subgroups. Uch zap. Perm University – 1960, 17, 2, 13 – 14.

[FdeGT1990] Franciosi S., de Giovanni F. and Tomkinson M.J. Groups with polycyclic – by -finite conjugacy classes. Bolletino Unione Mat.Italiana – 1990, 4B, no 7, 35–55.

[GPA1946] Golberg P.A. Sylow $p$-subgroups of locally normal groups. Math. Sbornik. – 19(1946), 451 – 460.

[GO1990] Gonzales M., Otal J. P. Hall's covering group and embedding of countable CC – groups. Communications Algebra – 18(1990), 3405 - 3412.

[GO1995] Gonzales M., Otal J. Embedding theorems for residually Chernikov CC – groups. Proc. Amer. Math. Soc. – 123(1995), no 8, 2383 – 2332.

[GO1996] Gonzales M., Otal J. The extension of results due to Gorchakov and Tomkinson from FC – groups to CC – groups. Journal Algebra – 185(1996), 314 – 328.

[GVM1950] Glushkov V.M. To theory of ZA – groups. Doklady AN USSR – 1950, 74, 885 – 888.

[GVM1951] Glushkov V.M. On locally nilpotent torsion – free groups. Doklady AN USSR – 1951, 89, 157 – 160.

[GVM1952-1] Glushkov V.M. On some question of the theory of torsion – free nilpotent and locally nilpotent groups. Mat. Sbornik – 30(1952), no. 1, 79 – 104.

[GVM1952-2] Glushkov V.M. On central series of infinite groups. Mat. Sbornik – 31(1952), 491 – 496.

[GYM1961] Gorchakov Yu.M. On embedding of locally normal groups in direct products of finite groups. Doklady AN SSSR – 138 (1961), 26 – 28.

[GYM1962] Gorchakov Yu.M. On locally normal groups. Doklady AN SSSR – 147 (1962), 537 – 539.

[GYM1965] Gorchakov Yu.M. On locally normal groups. Math. Sbornik. – 67(1965), 244 – 254.

[GYM1971] Gorchakov Yu.M. Locally normal groups. Sib. Math. J.–12(1971), 1259–1272.

[GYM1974] Gorchakov Yu.M. Theorems of Prüfer – Kulikov type. Algebra and Logic – 13(1974), 655 – 661.

[GYM1976] Gorchakov Yu.M. Subgroups of direct products. Algebra and Logic – 15(1976), 622 – 627.

[GYM1978] Gorchakov Yu.M. The groups with finite classes of conjugacy elements. NAUKA : Moskow – 1978.

[GRI2005] Grigorchuk R.I. Solved and unsolved problems around one group. Geometric, Combinatorial and Dynamical Aspects. Progress in Mathematics Series, Vol. 248. Bartholdi, L.; Ceccherini – Silberstein, T.; Smirnova – Nagnibeda, T.; Zuk, A. (Eds.). 2005, 413 p., BIRKHÄUSER: pp. 117 – 218.

[GR1989] Guralnick R.M. On the number of generators of a finite group. Archiv Math. – 53 (1989), 521 – 523

[HP1954] Hall Ph. Finiteness conditions for soluble groups. Proc. London Math. Soc. – 4(1954), 419 – 436.

[HP1959-1] On the finiteness of certain soluble groups. Proc.London Math. Soc. – 9(1959), 595 – 632.

[HP1959] Hall Ph. Periodic FC – groups. Journal London Math. Soc. 34(1959), 289–304.

[KAG1932] Kurosh A.G. Zur Zerlegung unendlicher Gruppen. Math. Annalen – 1932, 106, 107 – 113

[KAG1967] Kurosh A.G. The theory of groups. NAUKA: Moskow – 1967.

[KK1992] Kazarin L.S., Kurdachenko L.A. The finiteness conditions and factorizations in infinite groups. Uspehi Math. Nauk – 1992, 47, 3, 75 – 114.

[KLA1977] Kurdachenko L.A. FC – groups whose periodic part can be embedded in a direct product of finite groups. Math. Notes – 21(1977), number 1, 6 – 12.

[KLA1979] Kurdachenko L.A. Structure of FC – groups whose periodic part can be embedded in a direct product of finite groups. Math. Notes – 25(1979), number 1, 10 – 15.

[KLA1981-1] Kurdachenko L.A. Embeddability of an FC – group in the direct product of certain finite groups and a torsion – free abelian group. Math. Notes – 29(1981), number 3, 186 – 193.

[KLA1981-2] Kurdachenko L.A. Some conditions for embeddability of an FC – group in a direct product of finite groups and a torsion – free abelian group. Mathematics of the USSR Sbornik – 42(1982), no 4, 499 – 514.

[KLA1983-1] Kurdachenko L.A. On some non – periodic FC – groups. "Groups and systems of their subgroups", Math. Institute: Kiev– 1983, 92 – 100.

[KLA1983-2] Kurdachenko L.A. FC – groups with bounded periodic part. Ukrain Math. J. – 35(1983), no 3, 324 – 327.

[KLA1984] Kurdachenko L.A. Central factorization of FC – groups. "The structure of groups and their subgroup characterization", Math. Institute: Kiev– 1984, 73 – 85.

[KLA1986-1] Kurdachenko L.A. Non – periodic FC – groups and related classes of locally normal groups and abelian groups without torsion. Sibir. Math. J. – 27(1986), no 2, 227 – 236.

[KLA1986-2] Kurdachenko L.A. FC – groups of countable torsion – free rank. Math. Notes – 40 (1986), number 3, 16 –30.

[KLA1987] Kurdachenko L.A. The nilpotent FC – groups of class two. Ukrain Math. J. – 39(1987), no 3, 255 – 259.

[KLA1988] Kurdachenko L.A. On some classes of FC – groups. "The researches of groups with the restrictions on subgroups", Math. Institute: Kiev– 1988, 34 – 41.

[KLA1992] Kurdachenko L.A. On some classes of periodic FC – groups. Contemporary Math., Proc. Intern. conf. on algebra, part 1 – 131(1992), 499 – 509

[KLA1993] Kurdachenko L.A. On groups with minimax conjugacy classes. "Infinite groups and adjoining algebraic structures", NAUKOVA DUMKA: Kiev – 1993, 160–177.

[KLA1996] Kurdachenko L.A. On normal closures of elements in generalized FC-groups. "Infinite groups 94 (Ravello 1994)", Walter de Gruyter: Berlin, 1996, 141–151.

[KMI1958] Kargapolov M.I. To theory of semi – simple locally normal groups. "Scientific reports of higher school. Fiz. – math. Sciences. 6 (1958), 3 – 7.

[KMI1961] Kargapolov M.I. Locally finite groups having normal systems with finite factors. Sibir. Math. J. – 2(1961), 853 – 873.

[KNS1987-1] Kuzennyj N.F., Subbotin I.Ya. Groups in which all subgroups are pronormal. Ukrain. Math. J. – 1987, 39, 3, 325 –329.

[KNS1987-2] Kuzennyj N.F., Subbotin I.Ya. Locally soluble groups in which all infinite subgroups are pronormal. Izvestiya VUZ, Math. – 1987, 11, 77 – 79.

[KNS1988] Kuzennyj N.F., Subbotin I.Ya. A new characterization of locally nilpotent IH – groups. Ukrain. Math. J. – 1988, 40, 322 – 326.

[KO2001] Kurdachenko L.A., Otal J. Frattini properties of groups with minimax conjugacy classes. Quaderni di Matematica – 8 (2001), 221 – 237

[KO2003] Kurdachenko L.A., Otal J. FC – groups all of whose factor groups are residually finite. Communications in Algebra – 31(2003), no 3, 1235 – 1251

[KO2005] Kurdachenko L.A., Otal J. Groups with Chernikov classes of conjugate subgroups. Journal Group Theory – 8(2005), no 1, 93 – 108

[KOSO2004] Kurdachenko L.A., Otal J., Soules P. Groups with polycyclic – by – finite conjugate classes of subgroups. Communications in Algebra – 32(2004), no 12, 4769–4784

[KOS2002-1] Kurdachenko L.A., Otal J., Subbotin I.Ya. Groups with prescribed quotient groups and associated module theory. WORLD SCENTIFIC (ISBN 981 – 02 – 4783 – 4) , New Jersey – 2002

[KOS2002-2] Kurdachenko L.A., Otal J., Subbotin I.Ya. On some criteria of nilpotency. Communications in Algebra – 2002, 30, no 2, 3755 – 3776.

[KOS2005] Kurdachenko L.A., Otal J., Subbotin I.Ya. Abnormal, Pronormal, Contranormal, and Carter Subgroups in Some Generalized Minimax Groups. Communications in Algebra – 2005, 33, no 12, 4595 – 4616

[KOS2007-1] Kurdachenko L.A., Otal J., Subbotin I.Ya. Artinian modules over group rings. Frontiers in Mathematics. BIRKHÄUSER: Basel – 2007.

[KOS2007-2] Kurdachenko L.A., Otal J., Subbotin I.Ya. On properties of abnormal and pronormal subgroups in some infinite groups. Groups St. Andrews 2005. Cambridge University Press, Series: London Mathematical Society Lecture Note Series (No.340) – 2007, 772 – 782.

[KOT2000] Kurdachenko L.A., Otal J., Tomkinson M.J. FC – groups whose central factor group can be embedded in a direct product of finite groups. Communications in Algebra – 28(2000), no 3, 1343 – 1350

[KRV2006] Kurdachenko L.A., Russo A., Vincenzi G. Groups without proper abnormal subgroups. Journal Group Theory – 2006, 9, 507 – 518

[KS1996] Kurdachenko L.A., Subbotin I.Ya. Groups with restrictions for cocentralizers of elements. Communications in Algebra – 24(1996), no 3, 1173 – 1187

[KS2002] Kurdachenko L.A., Subbotin I.Ya. On transitivity of pronormality. Comment. Matemat.Univ. Caroline – 2002, 43, no. 4, 583 – 594

[KS2003] Kurdachenko L.A., Subbotin I.Ya. Pronormality, contranormality and generalized nilpotency in infinite groups. Publicacions Matemàtiques – 2003, 47, number 2, 389 – 414

[KS2005] Kurdachenko L.A., Subbotin I.Ya. Abnormal subgroups and Carter subgroups in some infinite groups. Algebra and Discrete Mathematics – 2005, 1, 63 – 77.

[KS2006] Kurdachenko L.A., Subbotin I.Ya. Transitivity of normality and pronormal subgroups. AMS Special session on infinite groups. October 8 – 9, 2005, Bard College. "Combinatorial group Theory, discrete groups, and number theory. American Mathematical Society, Contemporary Mathematics, 2006, 421, 201 – 212.

[MAI1948] Mal'tsev A.I. On groups of finite rank. Mat. Sbornik – 22(1948), no. 2, 351–352.

[MAI1949] Mal'tsev A.I. Nilpotent torsion – free groups. Izvestiya AN USSR, series math. – 1949, 13, number 3, 201 – 212.

[MYuI1964] Merzlyakov Yu.I. On locally soluble groups of finite rank. Algebra i logika – 3(1964), no 2, 5 – 16.

[NB1951] Neumann B.H. Groups with finite classes of conjugate elements. Proc. London Math. Soc. 1(1951), 178 – 187.

[NB1954] Neumann B.H. Groups covered by permutable subsets. Journal London Math. Soc. 29(1954), 236 – 248.

[NB1955] Neumann B.H. Groups with finite classes of conjugate subgroups. Math. Z. 63(1955), 76 – 96.

[OA1981] Ol'shanskij A.Yu. Geometry of defining relations in groups. KLUWER ACAD. PUBL.: Dordrecht – 1991.

[OS2009] Otal J., Semko N.N. Groups with the small cocentralizers. Algebra and Discrete Mathematics – 2009, Number 4, 135 – 157.

[PT1971] Peng T.A. Pronormality in finite groups. J. London Math. Soc. – 1971, 3, no. 2, 301 – 306.

[PBI1958] Plotkin B.I. Generalized soluble and generalized nilpotent groups. Uspekhi Math. Nauk – 1958, 13, number 4, 89 – 172.

[PYa1962-1] Polovitsky Ya.D. Layer extremal groups. Math. Sbornik – 1962, 56, number 1, 95 – 106.

[PYa1962-2] Polovitsky Ya.D. On locally extremal and layer extremal groups. Math. Sbornik – 1962, 58, number 2, 685 – 694.

[PYa1964] Polovitsky Ya.D. Groups with extremal conjugacy classes. Sibirian Math. J. – 1964, 5, 891 – 895.

[PH1924] Prüfer H. Theorie der Abelschen Gruppen. Math. Z. – 1924, 20, 165 – 187.

[RD1964] Robinson D.J.S. Groups in which normality is a transitive relation, Proc. Cambridge Philos. Soc. – 1964, 60, 21 – 38.

[RD1968] Robinson D.J.S. Infinite soluble and nilpotent groups, QUEEN MARY COLLEGE, Mathematics Notes: London, 1968.

[RD1972-1] Robinson D.J.S. Finiteness conditions and generalized soluble groups, part 1. SPRINGER: Berlin – 1972.

[RD1972-2] Robinson D.J.S. Finiteness conditions and generalized soluble groups, part 2. SPRINGER: Berlin – 1972.

[RD1972-3] Intersections of primary powers of a group, Math. Z. 124 (1972), 119 – 132.

[SCH1904] Schur I. Über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen. J. Reine Angew. Math. – 127(1904), 20 – 50.

[SI1984] Subbotin, I.Ya. The quasicentral product of groups. *Structure of groups and their subgroups characterization*. Math. Inst. of Acad. of Sciences of Ukrainian SSR, Kiev, 1984, 126-139.

[SN1998] Semko N.N. Groups dense normal conditions for some systems of subgroups. MATH. INSTITUT: Kyiv – 1998.

[SLK1983] Semko N.N., Levischenko S.S., Kurdachenko L.A. On groups with infinite almost normal subgroups. Izvestiya VUZ, mathem., 1983, no 10, 57 – 63.

[SV1964] Schunkov, V.P.On groups decomposed into the uniform product of its $p$-subgroups. *DAN SSSR*, 154 (1964), 542-544.

[TM1975] Tomkinson M.J. Extraspecial sections of periodic FC – groups. Compositio Mathematica – 31(1975), number 3, 285 – 303.

[TM1977] Tomkinson M.J. Residually finite periodic FC – groups. Journal London Math. Soc. 16(1977), 221 – 228.

[TM1978] Tomkinson M.J. On the commutator subgroup of a periodic FC – group. Archiv Math. 31 (1978), 123 – 125.

[TM1981] Tomkinson M.J. A characterization of residually finite periodic FC – groups. Bull. London Math. Soc. 13(1981), 133 – 137.

[TM1984] Tomkinson M.J. FC – groups. PITMAN: Boston – 1984.

[TM1996] Tomkinson M.J. FC – groups: Recent Progress. Infinite groups 1994. Proceedings of the international conference, held in Ravello, Italy, May 23 – 27, 1994. WALTER DE GRUYTER: Berlin 1996, 271 – 285 (S. 166)

[TVKS1992] Tsybulenko, V. V.; Kolesnik, S. G. Groups that are decomposable into the uniform product of any full set of their Sylow p -subgroups. (Russian) Infinite groups and related algebraic structures (Russian), 354–366, Akad. Nauk Ukrainy, Inst. Mat., Kiev, 1993.

[VVG1977] Vasil'ev, V. G. Strong uniform products of cyclic p -groups. (Russian) Algebra i Logika 16 (1977), no. 5, 499–506, 623.

[VVG1978] Vasil'ev, V. G. Derived length of uniform products of cyclic groups. (Russian) Algebra i Logika 17 (1978), no. 3, 260–266, 357.

[ZDI1971-1] Zaitsev D.I. On soluble groups of finite rank. "The groups with the restrictions for subgroups", NAUKOVA DUMKA: Kiev – 1971, 115 – 130.

[ZDI1971-2] Zaitsev D.I. To the theory of minimax groups. Ukrain. Math. J. – 23(1971), no 5, 652 – 660.

[ZDI1975] Zaitsev D.I. The groups with the complemented normal subgroups, "The some problems of group theory", MATH. INST.: Kiev – 1975, 30 – 74.

[ZDI1980-1] Zaitsev D.I. The products of abelian groups. Algebra i logika – 19(1980), no 2, 94 – 106.

[ZD1980-2] Zaitsev D.I. The groups of operators of finite rank and their applications. Book «VI Symposium on group theory», NAUKOVA DUMKA: Kiev – 1980, 22–37.

[ZD1982] Zaĭtsev, D. I. Weakly uniform products of polycyclic groups. (Russian) Subgroup characterization of groups, pp. 13–26, 111, Akad. Nauk Ukrain. SSR, Inst. Mat., Kiev, 1982.

[ZKC1972] Zaitsev D.I., Kargapolov M.I., Charin V.S. Infinite groups with prescribed properties of subgroups. Ukrain. Math. J. – 1972, 24, 5, 618 – 633.

[ASMSh2005] Kh. A. Al-Sharo, E.A. Molokova and L.A. Shemetkov, Factorizable Groups and Formations, Acta Applicandae Mathematicae (2005), 85, 3-10.

[ZL1984] L.N. Zakrevskaya, Finite groups with s dense system of F -subnormal subgroups, In the book: Investigation of normal and subgroup structure of finite groups, Minsk: Nauka i technika, 1984, p. 71-88.

[ShA2003] A.E. Shmigirev, Solubility of finite groups with the condition of density for generalized subnormal subgroups, Proc. F. Scorina State Univ., No. 1, 2003, 118-120.

[ShSh2004] L.A. Shemenkov and A.E. Shmigirev, On finite groups with a dense system of subgroups, Dokl. NAN Belarus, No. 6, 2004, 29-31.

[ShA2004] A.E. Shmigirev, On finite groups with with the condition of density for generalized subnormal subgroups, Proc. F. Scorina State Univ., No. 6, 2004, 130-149.

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Received by the editors: 27.02.2012
and in final form 27.02.2012.