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RESEARCH ARTICLE

On c-normal and hypercentrally embeded subgroups of finite groups*

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Abstract. In this article, we investigate the structure of a finite group G under the assumption that some subgroups of G are c-normal in G. The main theorem is as follows:

Theorem A. *Let E be a normal finite group of G. If all subgroups of E_p with order d_p and $2d_p$ (if $p = 2$ and E_p is not an abelian nor quaternion free 2-group) are c-normal in G, then E is p-hypercyclically embedded in G.*

We give some applications of the theorem and generalize some known results.

1. Introduction

All groups considered in this paper are finite. We use conventional notions and notation, as in [3]. G always denotes a finite group, $|G|$ the order of G, $\pi(G)$ the set of all primes dividing $|G|$, G_p a Sylow p-subgroup of G for any prime $p \in \pi(G)$.

A well know result is that G is nilpotent if and only if every maximal subgroup of G is normal in G. In [11], Wang defined c-normality of a subgroup and prove that a finite group G is solvable if and only if every maximal subgroup of G is c-normal in G.

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Definition 1.1 ([11], Definition 1.1)**.** Let G be a group. We call a subgroup H is c-normal in G if there exit a normal subgroup N of G such that $HN = G$ and $H \cap N \leqslant H_G$.

The basic properties of c-normality are as follows.

Lemma 1.2 ([11], Lemma 2.1)**.** *Let G be a group. Then*

(1) *If H is normal in G , then H is c-normal in G.*

(2) *G is c-simple if and only if G is simple.*

(3) *If H is c-normal in G , $H \leqslant K \leqslant G$, then H is c-normal in K.*

(4) *Let $K \trianglelefteq G$ and $K \leqslant H$, Then H is c-normal in G if and only if H/K-normal in G/K.*

Several authors successfully use the c-normal property of some p-subgroups of G to determine the structure of G. (see [2],[5], [8-10]). Many results in previous papers have the following form: Suppose that G/E is supersolvable (or $G/E \in \mathcal{F}$, where $\mathcal{F}$ is a formation containing the class of all supersolvable groups), if some subgroups of E with prime power order are c-normal G , then G is supersolvable (or $G \in \mathcal{F}$). Actually, in a more general case, if we can get a criterion that E lies in the $\mathcal{F}$-hypercenter, then $G/E \in \mathcal{F}$ implies that $G \in \mathcal{F}$. In order to get good results, many authors have to impose the c-normal hypotheses on all the prime divisors or the minimal or maximal divisor p of $|G|$ rather than any prime divisor.

Let p be a fixed prime. In this paper, we mainly focus on how a normal subgroup E has the above property provided every p-subgroup of E with some fix order is c-normal in G. For this purpose, we introduce the concept of p-hypercentrally embeded:

Definition 1.3. A normal subgroup E is said to be p-hypercentrally embedded in G if every p-chief factor of G below E is cyclic.

It is of a lot interest to determine the structure of G with hypothesis that some p-subgroups are well suited in G. Many results on minimal p-subgroups and maximal subgroups of Sylow subgroups were obtained. Recently, people have more interest to get unified and general results ([8],[12]). That is, to consider the p-subgroups with the same order. For simplicity, we give the following notation of d_p .

Let E be a normal finite group of G. d_p is a prime power divisor of $|E_p|$ satisfying the following properties:If $|E_p| = p$ then $d_p = |E_p| = p$; if $|E_p| > p$ then $1 < d_p < |E_p|$.

In this paper, we will prove the following theorem:

Theorem A. *Let E be a normal finite group of G. If all subgroups of E_p with order d_p and $2d_p$ (if $p = 2$ and E_p is not an abelian nor quaternion free 2-group) are c-normal in G, then E is p-hypercyclically embedded in G.*

As an application of Theorem A, we have the following:

Theorem B. *Let E be a normal finite group of G such that both $N_G(E_p)$ and G/E are p-nilpotent. If either E_p is abelian or every subgroup of E_p with order d_p (d_p is a prime power divisor of $|E_p|$ and $1 < d_p < |E_p|$) and $2d_p$ (if $p = 2$ and E_p is not quaternion free) is c-normal in E, then G is p-nilpotent.*

2. Proof of the theorems

In this section, we will investigate how a normal subgroup E embedded in G if, for a fixed prime p, some subgroups of E_p are c-normal in G. First, we need some results about a normal subgroup with some subgroups being c-supplemented in G. Following [10], a group H is said to be c-supplemented in G if there exists a subgroup K of G such that $G = HK$ and $H \cap K \leqslant H_G$. It is clear from the definition that if a subgroup H is c-normal in G, then H is c-supplemented in G.

Lemma 2.1. *If N is a minimal abelian normal subgroup of G then all proper subgroups of N are not c-supplement in G.*

Proof. Suppose this Lemma is not true and let H be a proper subgroup of N which is c-supplemented in G. Obviously $H_G = 1$ since $H_G < N$ and N is a minimal normal subgroup of G. By the definition of c-supplement, there exit a proper subgroup M of G such that $G = HM$ with $H \cap M \leqslant H_G = 1$. Hence $NM \geqslant HM = G$. Since N is abelian, we know that $N \cap M \trianglelefteq G$. Hence $N \cap M = 1$. Therefore we have $|G| = |NM| = |N||M| > |H||M| = |HM|$, a contradiction to $G = HM$. □

For a saturated formation $\mathcal{F}$, the $\mathcal{F}$-hypercenter of a group G is denoted by $Z_{\mathcal{F}}(G)$ (see [3, p 389, Notation and Definitions 6.8(b)]). Let $\mathcal{U}$ denote the class of all supersolvable groups. In [2], Asaad gave the following result: Let p be a nontrivial normal p-subgroup, where p is an odd prime, if every minimal subgroup of P is c-supplemented in G, then $P \leqslant Z_{\mathcal{U}}(G)$. It is helpful to give a result for $p = 2$. In fact, we have the following property:

Property 2.2. *Let P be a normal 2-subgroup of G. If all minimal subgroups of P and all cyclic subgroups of P with order 4 (if P is neither abelian nor quaternion free) are c-supplemented in G, then $P \leqslant Z_\infty(G)$.*

Proof. Let Q be a Sylow q-subgroup of G $(q \neq p)$, we are going to show that PQ is 2-nilpotent. Suppose PQ is not 2-nilpotent, then PQ contains a minimal non 2-nilpotent subgroup H. By Ito's famous result, we know that $H = [H_2]H_q$, $exp(H_2) \leqslant 4$ and $H_2/\Phi(H_2)$ is a minimal normal subgroup of $H/\Phi(H_2)$. If $|H_2/\Phi(H_2)| = 2$, then we have $|H/\Phi(H_2) : H_q\Phi(H_2)/\Phi(H_2)| = |H_2/\Phi(H_2)| = 2$ and thus $H_q\Phi(H_2)/\Phi(H_2)$ is normal in $H/\Phi(H_2)$, which will lead to the nilpotent of H. Therefore $|H_2/\Phi(H_2)| > 2$. We distinguish the three cases:

Case 1. Every minimal subgroup of P and every cyclic subgroups with order 4 of P is c-supplemented in G. Let $\langle x\rangle$ be a subgroup of H_2 not contained In $\Phi(H_2)$, then $\langle x\rangle\Phi(H_2)/\Phi(H_2)$ is a nontrivial subgroup of $H/\Phi(H_2)$. Since $exp(H_2) \leqslant 4$, we know that $\langle x\rangle$ is c-supplemented in G and thus c-supplemented in H by [10, Lemma 2.1(1)]. By Lemma 2.1, we have $\langle x\rangle\Phi(H_2)/\Phi(H_2) = H_2/\Phi(H_2)$. But then $|H/\Phi(H_2) : H_q\Phi(H_2)/\Phi(H_2)| = |\langle x\rangle\Phi(H_2)/\Phi(H_2)| = 2$, a contradiction.

Case 2. Every minimal subgroup of P is c-supplemented in G and P is an abelian 2-group. Let $\langle x\rangle$ be a subgroup of H_2 not contained In $\Phi(H_2)$. If $|x| = 2$, then we can get a contradiction by using exactly the same argument as we did in Case 1.Therefore we may assume that $\Omega_1(H_2) \leqslant \Phi(H_2)$, where $\Omega_1(H_2)$ is a subgroup generated by all minimal subgroup of H_2. Since H is a minimal non 2-nilpotent group and $\Phi(H_2)H_q < H$, $\Phi(H_2)H_q$ is a nilpotent group. As a result, H_q acts trivially on $\Omega_1(H_2)$. Note that H_2 is also an abelian 2-group, by [4, Theorem2. 4] H_q also acts trivially on H_2, a contradiction.

Case 3. Every minimal subgroup of P is c-supplemented in G and P is a non-abelian quaternion free 2-group. If H_2 is abelian, then we can get the same contradiction as Case 2. Hence we may assume that H_2 is also a non-abelian quaternion free 2-group. Applying [6, Theorem 2.7], H_q acts on $H_2/\Phi(H_2)$ with at least one fixed point. Bare in mind that $H_2/\Phi(H_2)$ is a minimal normal subgroup of $H/\Phi(H_2)$, we have $|H_2/\Phi(H_2)| = 2$, again a contradiction.

The above proof shows that PQ is 2-nilpotent and thus $Q \trianglelefteq PQ$. Note that P is a normal subgroup of G, we have $[P, Q] = 1$. Note that we can choose Q to be a Sylow q-subgroup of G for any $q \neq p$, we have $[P, O^2(G)] = 1$. Let H/K be a G-chief factor of P. The fact $[P, O^{2'}(G)] = 1$ yields that $G/C_G(H/K)$ is a 2-group. But by [3, A, Lemma 13.6], we have $O_2(G/C_G(H/K)) = 1$. Consequently $G/C_G(H/K) = 1$ for any G-chief factor of P, in other words, $P \leqslant Z_\infty(G)$. □

As an application of Property 2.2, we have:

Corollary 2.3. *If all minimal subgroups of G_2 and all cyclic subgroups of G_2 with order 4 (if G_2 is neither abelian nor quaternion free) are c-supplemented in G, then G is 2-nilpotent.*

Proof. Suppose this corollary is not true and let G be a counterexample with minimal order. Obviously the hypothesis is inhered by all subgroups of G, G is actually a minimal non 2-nilpotent group. Hence G_2 is a normal subgroup in G. Applying Property 2.2 to G_2, we get a contradiction. □

By combining [2, Theorem 1.1] and Property 2.2, we have:

Lemma 2.4. *Let P be a normal p-subgroup of G. If all cyclic subgroups of P with order p or 4 (if P is a non-abelian and not quaternion free 2-group) are c-supplement in G, then $P \leqslant Z_{\mathcal{U}}(G)$.*

Next, we will show that if that some class of p-subgroup is c-normal in G, then G is p-solvable.

Lemma 2.5. *If G_p is c-normal in G then G is p-solvable.*

Proof. Suppose this Lemma is not true and considered G to be a counterexample with minimal order. Clearly the hypothesis holds for any quotient group of G, the minimal choice of G implies that $O_p(G) = O_{p'}(G) = 1$. By the definition of c-normal, there exit a normal subgroup H of G such that $G = G_pH$ and $H \cap G_p \leqslant (G_p)_G$. But $(G_p)_G = O_p(G) = 1$, hence H is a p' normal subgroup of G. The fact $O_{p'}(G) = 1$ indicates that $H = 1$ and thus $G = G_p$, a contradiction. □

Lemma 2.6. *Let d_p be a prime power divisor of $|G_p|$ with $d_p > 1$. If every subgroup of $|G_p|$ with order d_p and $2d_p$ (If $p = 2$ and G_2 is neither abelian nor quaternion free)) is c-normal in G then G is p-solvable.*

Proof. Suppose this Lemma is not true and considered G to be a counterexample with minimal order. According to Lemma 2.5 we may assume that $1 < d_p < |G_p|$.

(1) $O_{p'}(G) = 1$.
Since the hypothesis holds for $G/O_{p'}(G)$, the minimal choice of G yields that $O_{p'}(G) = 1$.

(2) *Every subgroup with order d_p and $2d_p$ (if $p = 2$ and G_2 is neither abelian nor quaternion free) is normal in G. In particular, $O_p(G) > 1$.*
Suppose there exit a subgroup K with order d_p or $2d_p$ (if $p = 2$) that is not normal in G. Then there exit a proper normal subgroup

L such that $G = KL, K \cap L \leqslant K_G$. Since G/L is a p-group, we can find a normal subgroup M containing L such that $|G/M| = p$. But $d_p < |G_p|$ so M still satisfies the hypothesis of this Lemma, thus M is p-solvable by the minimal choice of G and so is G.

(3) *Let N is a minimal normal subgroup contained in $O_p(G)$, then $|N| = d_p$.*
If $d_p > |N|$, then G/N satisfies the hypotheses of this Lemma and thus is p-solvable by the minimal choice of G. Since N is a p-group we can get that G is p-solvable, a contradiction.

(4) $d_p = p$.
Suppose $d_p > p$. From (3) we know that $|N| = d_p > p$ and thus N is not cyclic. Let H be a subgroup of G_p containing N such that $|H : N| = p$. Let M_1 and M_2 be two different maximal subgroup of H. By (2), both M_1 and M_2 are normal in G. Consequently $H/N = M_1M_2/N$ is also normal in G/N. Hence every subgroup of G/N with order p is normal in G/N. If $p = 2$ and and G_2 is neither abelian nor quaternion free, then by using a similar argument we know that every subgroup of G/N with order 4 is also normal in G/N. As a result, we see that G/N satisfies the hypothesis of this Lemma and the choice of G implies that G/N is p-solvable, thus G is p-solvable, a contradiction.

(5) *Final contradiction.*
If $p = 2$, then from (4) and Corollary 2.3, G is 2-nilpotent. So we may assume p is an odd prime. By (2) and (4) we know that every subgroup with order p is normal. Take a subgroup $\langle x\rangle$ with order p , it's easy to see that $G_p \leqslant C_G\langle x\rangle$. If $C_G\langle x\rangle < G$ then from the choice of G we know that $C_G\langle x\rangle < G$ is p-solvable. But $G/C_G\langle x\rangle < G$ is cyclic and thus G is p-solvable, contradict to the choice of G. Therefore we have $C_G\langle x\rangle = G$, that is, every minimal subgroup of order p is contained $Z(G)$. From Ito's theorem G is p-nilpotent, a contradiction. □

Now, we will study the properties of p-hypercyclically embedding. In [7, p. 217], a normal subgroup E is said to be hypercyclically embedded in G if every chief factor of G below E is cyclic. If a normal subgroup E is hypercyclically (p-hypercyclically) embedded in G, then E is solvable (p-solvable) and every normal subgroup of G contained in E is also hypercyclically (p-hypercyclically) embedded in G. The following lemma shows that for a p-solvable normal subgroup E, we can deduce that E

is hypercyclically (p-hypercyclically) embedded in G from the maximal p-nilpotent normal subgroup of E $F_p(E)$.

Lemma 2.7. *A p-solvable normal subgroup E is hypercyclically (p-hypercyclically) embedded in G if and only if $F_p(E)$ is hypercyclically (p-hypercyclically) embedded in G. In particular, if E is a p-solvable normal subgroup with $O_{p'}(E) = 1$, then E is hypercyclically embedded in G if and only if $O_p(E)$ is hypercyclically embedded in G.*

Proof. We only need to prove the sufficiency. Suppose the assertion is false and let (G, E) be a counterexample with $|G||E|$ minimal. We claim that $O_{p'}(E) = 1$. Indeed, since $F_p(E/O_{p'}(E)) = F_p(E)/O_{p'}(E)$, it's easy to verify that the hypothesis still holds for $(G/O_{p'}(E), E/O_{p'}(E))$. If $O_{p'}(E) \neq 1$, then the the minimal choice of (G, E) implies that $E/O_{p'}(E)$ hypercyclically (or p-hypercyclically) embedded in $G/O_{p'}(E)$. Since we have that $O_{p'}(E)$ is a normal subgroup of G contained in E, $O_{p'}(E)$ is hypercyclically (or p-hypercyclically) embedded in G . Therefore we have E hypercyclically (or p-hypercyclically) embedded in G, a contradiction.

Let N be a minimal normal subgroup of G contained in E. N is an abelian normal p-subgroup since E is p-solvable and $O_{p'}(E) = 1$. Consider the group $C_E(N)/N$. Let $L/N = O_{p'}(C_E(N)/N)$ and K be the Hall p' subgroup of L. Then $L = KN$. Since $K \leqslant L \leqslant C_E(N)$, we have $K = O_{p'}(L) \leqslant O_{p'}(G) = 1$. Consequently $O_{p'}(C_E(N)/N) = 1$ and we have $F_p(C_E(N)/N) = O_p(C_E(N)/N) \leqslant O_p(E)/N = F_p(E)/N$. As a result, we know that the hypothesis holds for $(G/N, C_E(N)/N)$ and the minimal choice of (G, E) yields that $C_E(N)/N$ is hypercyclically (or respectively p-hypercyclically) embedded in G/N. But $N \leqslant F_p(G)$ and thus N is also hypercyclically (or p-hypercyclically) embedded in G. Thus $C_E(N)$ is hypercyclically (or p-hypercyclically) embedded in G.

Since N is a normal p-subgroup which is hypercyclically (or respectively p-hypercyclically) embedded in G, we have that $|N| = p$. It yields $G/C_G(N)$ is a cyclic group. As a result, $EC_G(N)/C_G(N)$ is hypercyclically embedded in $G/C_G(N)$. Note that $E/C_E(N) = E/E \cap C_G(N)$ is G-isomorphic with $EC_G(N)/C_G(N)$, therefore $E/C_E(N)$ is hypercyclically embedded in $G/C_E(N)$. But $C_E(N)$ is also hypercyclically (or p-hypercyclically) embedded in G hypercyclically (or p-hypercyclically) embedded in G and thus E is hypercyclically (or p-hypercyclically) embedded in G, a final contradiction. □

Denote $\mathcal{A}(p-1)$ as the formation of all abelian groups of exponent divisible by $p-1$. The following proposition is well known:

Lemma 2.8 ([12], Theorem 1.4). *Let H/K be a chief factor of G, p is a prime divisor of $|H/K|$, then $|H/K| = p$ if and only if $G/C_G(H/K) \in \mathcal{A}(p-1)$.*

Let f be a formation function, and N be a normal subgroup of G. We say that G acts f-centrally on E if $G/C_G(H/K) \in f(p)$ for every chief factor H/K of G below E and every prime p dividing $|H/K|$ ([3], p. 387, Definitions 6.2). Fixing a prime p, define a formation function g_p as follows:

$$g_p(q) = \begin{cases} \mathcal{A}(p-1) & (\text{if } q = p) \\ \text{all finite group} & (\text{if } q \neq p) \end{cases}$$

From Lemma 2.8, we can see that E is p-hypercyclically embedded in G if and only if G acts g_p-centrally on E. By applying [3, p. 388, Theorem 6. 7], we get the following useful results:

Lemma 2.9. *A normal subgroup E of G is p-hypercyclically embedded in G if and only if $E/\Phi(E)$ is p-hypercyclically embedded in $G/\Phi(E)$.*

Lemma 2.10. *Let K and L be two normal subgroup of G contained in E. If E/K is p-hypercyclically embedded in G/K and E/L is p-hypercyclically embedded in G/L, then $E/L\cap K$ is p-hypercyclically embedded in $G/L\cap K$.*

The following proposition indicates that when $d_p = p$, the conclusion of Theorem A holds.

Proposition 2.11. *Let E be a normal subgroup of G. If all cyclic subgroups of E_p with order p and 4 (if $p = 2$ and E_p is not an abelian nor quaternion free 2-group) are c-normal in G, then E is p-hypercyclically embedded in G.*

Proof. Suppose this Theorem is not true and let (G, E) be a counterexample such that $|G| + |P|$ is minimal. Suppose $O_{p'}(E) \neq 1$, it's easy to verifies that $(G/O'_p(E), E/O'_p(E))$ satisfies the hypothesis of this Theorem and thus $E/O'_p(E)$ is p-hypercyclically embedded in $G/O_{p'}(E)$ by the minimal choice of (G, E). But then E is p-hypercyclically embedded in G. This contradiction implies that $O_{p'}(E) = 1$.

From Lemma 2.6 and Lemma 1.2(3) we know that E is p-solvable and from Corollary F we know that $O_p(E) \leqslant Z_{\mathcal{U}}(G)$, thus $E \leqslant Z_{\mathcal{U}}(G)$ by Lemma 2.7, a contradiction. □

With the aid of all the preceding results, we can now prove the main theorem of this section.

Proof of Theorem A. Suppose this is not true and let (G, E) be a counterexample such that $|G| + |E|$ is minimal. If $|E_p| = p$, then E_p itself is c-normal in G and by Lemma 1.2, E_p is also c-normal in E. By Lemma 2.5 we know that E is p-solvable and consequently E is p-hypercyclically embedded in G since $|E_p| = p$. Therefore we may assume that $|E_p| > p$ and $1 < d_p < |E_p|$. By Proposition 2.11, we may further assume that $d_p > p$. Similar to step (1) in the proof of Lemma 2.6, we have $O_{p'}(E) = 1$. By Lemma 2.6, E is p-solvable. Let N be a minimal normal subgroup of G contained in E, then obviously $N \leqslant O_p(E)$.

(1) $|N| > p$.
Suppose $|N| = p$, then $d_p > |N|$ by our assumption that $d_p > p$. Hence $(G/N, E/N)$ also satisfies the hypothesis of this Theorem and therefore E/N is p-hypercyclically embedded in G/N by the choice of (G, E). If $|N| = p$, then E is p-hypercyclically embedded in G, a contradiction.

(2) $d_p > |N|$.
By Lemma 2.1 we have $d_p \geqslant |N|$. Suppose that $d_p = |N|$. Since $d_p < |E_p|$ by our assumption, let H be a subgroup of E_p such that N is a maximal subgroup of H . By (1), N is not cyclic and so is H. Hence we can choose a maximal subgroup K of H other than N . Obviously we have $H = NK$. If $N \cap K = 1$, then $|N| = |H|/|K| = p$, contradict to (1). Thus $N \cap K \neq 1$ and $|K : K \cap N| = |KN : N| = |H : N| = p$. Since $K_G \cap N \leqslant K \cap N < N$, we have $K_G \cap N = 1$. If $K_G \neq 1$, then $H = NK_G$ and $K = K \cap K_G N = (K \cap N)K_G$. As a result, $|K_G| = |K|/|K \cap N| = p$. But this contradicts to (1) because now we find a normal subgroup of G contained in $O_p(G)$ with order p. Therefore we have $K_G = 1$. Since $|K| = |N| = d_p$, K is c-normal in G by the hypothesis of this theorem. So there exists a proper normal subgroup L of G such that $G = KL$ and $K \cap L \leqslant K_G = 1$. Since $K \cap N \neq 1$ and $K \cap L = 1$, we have $N \neq L$ and thus $N \cap L = 1$. Consequently $|NL| = |N||L| = |K||L| = |KL| = |G|$ and thus $G = NL$. Let M an maximal subgroup of G containing L , then $|G : M| = p$ since G/L is a p-group. Obviously $G = NM$ and $N \cap M = 1$. But then $|N| = |G : M| = p$, a contradiction to (1).

(3) *N is the unique minimal normal subgroup of G contained in E and $N \nleqslant \Phi(E)$.*
Since $d_p > |N|$ by (2), it's easy to verify that $(G/N, E/N)$ still satisfies the hypothesis of this theorem. The minimal choice of (G, E) implies E/N is p-hypercyclically embedded in G/N. From

Lemma 2.10, N must be the unique minimal normal subgroup of G contained in E. From Lemma 2.9, we have $N \nleq \Phi(E)$.

(4) *Final contradiction.*
By (3), there exit a maximal subgroup M of E such that $E = NM$. $E_p = E_p \cap NM = N(E_p \cap M)$. Clearly $E_p \cap M < E_p$ since N is not contained in M, so we can choose a maximal subgroup K of E_p such that $E_p \cap M \leqslant K$. Note that now $E_p = NK$, if $N \cap K = 1$, then by simple calculation we know that $|N| = p$, contradict to (1). Hence $1 < N \cap K < N$. Clearly $|N| < d_p \leqslant |K|$, so we can choose a subgroup H with order d_p such that $1 < N \cap K < H \leqslant K$. Because $N \neq H$ and N is the unique minimal normal subgroup of G contained in E, we have $H_G = 1$. By the hypothesis of this Theorem, H is c-normal in E and hence there exit a normal subgroup L of G such that $G = HL$ and $H \cap L \leqslant H_G = 1$. Therefore $E = E \cap HL = H(E \cap L)$ and $E \cap L$ is a non trivial normal subgroup of G contained in E. But since $H \cap (E \cap L) \leqslant H \cap L = 1$ and $H \cap N \neq 1$, we have $N \nleq E \cap L$, contradicts to N being the unique minimal normal subgroup of G contained in E. □

Remark. The conclusion of Theorem A does not hold if we replace "c-normal" with "c-supplemented" in the hypothesis. One can take A_5 for a example. Obviously every subgroup of A_5 with order 5 is c-supplemented in A_5, but A_5 is not 5-hypercyclically embedded in itself.

Corollary 2.12. *Let $d_p(G)$ be a prime power divisor of $|G_p|$ satisfying the following properties: If $|G_p| = p$ then $d_p(G) = |G_p| = p$; if $|G_p| > p$ then $1 < d_p(G) < |G_p|$. Suppose that all of the subgroups of G_p with order $d_p(G)$ and $2d_p(G)$ (if $p = 2$ and G_p is not an abelian nor a quaternion free 2-group) are c-normal in G. Then G is p-supersolvable.*

Corollary 2.13. *Let E be a normal finite group of G and suppose that G/E is p-supersolvable. Suppose that all of the subgroups of E_p with order d_p and $2d_p$ (if $p = 2$ and E_p is not an abelian nor quaternion free 2-group) are c-normal in G. Then G is p-supersolvable.*

It is clear that G is p-nilpotent implies G is p-supersolvable but the converse is not true. However, The following lemma reveals a connection between p-nilpotent and p-supersolvable through the p-nilpotency of $N_G(G_p)$.

Lemma 2.14. *G is p-nilpotent if and only if G is p-supersolvable and $N_G(G_p)$ is p-nilpotent.*

Proof. Suppose this lemma is not true and let G be a minimal counterexample. Since $N_{G/O_{p'}(G)}(G_pO_{p'}(G)/O_{p'}(G)) = N_G(G_p)O_{p'}(G)/O_{p'}(G)$, we have that $O_{p'}(G) = 1$ by induction.

Let N be a minimal normal subgroup of G. Then $|N| = p$ since G is p-supersolvable and $O_{p'}(G) = 1$. It's easy to verify that G/N still satisfy the hypothesis of this lemma. Again from induction we know that N is the unique minimal normal subgroup of G, $\Phi(G) = 1$ and $N = C_G(N)$. But the fact $|N| = p$ implies that $G_p \leqslant C_G(N)$. Therefore we have $G_p = N$. It follows that $G = N_G(G_p)$ is p-nilpotent, a contradiction. □

Now we can prove Theorem B by using Theorem A and Lemma 2.14.

Proof of Theorem B. Suppose this is not true. Let (G, E) be a counterexample such that $|G| + |E|$ is minimal. We first claim that E is p-nilpotent. Since $N_G(E_p)$ is p-nilpotent, $N_E(E_p) = N_G(E_p) \cap E$ is also p-nilpotent. If E_p is abelian, then $N_E(E_p) = C_E(E_p)$ and hence E is p-nilpotent by Burnside's theorem. If E_p is not abelian, then every subgroup of E_p with order d_p (d_p is a prime power divisor of E_p and $1 < d_p < |E_p|$) and $2d_p$ (if $p = 2$ and E_p is not quaternion free) is c-normal in E by hypothesis and Lemma 1.2(3). We know from Corollary 2.12 that E is p-supersolvable. It follows from Lemma 2.14 that E is p-nilpotent.

By induction, we have $O_{p'}(E) = 1$ and thus E must be a p-group. Therefore $G = N_G(E) = N_G(E_p)$ is p-nilpotent, a contradiction. □

Remark. In Theorem A we ask E_p to be c-normal in G provided that $|E_p| = p$. But we don't impose the c-normality on E_p in Theorem B under the same circumstance because E_p is abelian if $|E_p| = p$.

Corollary 2.15. *Suppose $N_G(G_p)$ is p-nilpotent. If either G_p is abelian or every subgroup of G_p with order d_p (d_p is a prime power divisor of G_p and $1 < d_p < |G_p|$) and $2d_p$ (if $p = 2$ and G_p is not quaternion free) is c-normal in G, then G is p-nilpotent.*

3. Applications

In this section, we give some applications to show that we can apply our results to generalize some known results.

Corollary 3.1 ([1, Theorem 3.4])**.** *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. If all minimal subgroups and all cyclic subgroups with order 4 of $G^{\mathcal{F}}$ are c-normal in G, then $G \in \mathcal{F}$.*

Proof. From Theorem A, we know that $G^{\mathcal{F}}$ is p-hypercentrally embedded in G for all $p \in \pi(G^{\mathcal{F}})$ and thus $G^{\mathcal{F}} \leqslant Z_{\mathcal{U}}(G)$. Since $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, we have that $Z_{\mathcal{U}}(G) \leqslant Z_{\mathcal{F}}(G)$. Consequently $G \in \mathcal{F}$ because $G/G^{\mathcal{F}} \in \mathcal{F}$ and $G^{\mathcal{F}} \leqslant Z_{\mathcal{U}}(G) \leqslant Z_{\mathcal{F}}(G)$. □

Corollary 3.2 ([8, Theorem 0.1]). *Let E be a normal subgroup of a group G of odd order such that G/E is supersolvable. Suppose that every non-cyclic Sylow subgroup P of E has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with order $|H| = |D|$ are c-normal in G. Then G is supersolvable.*

Proof. Let p be the minimal prime divisor of $|E|$. If E_p is cyclic, then E is p-nilpotent by [13, Lemma 2.8]. If E_p is not cyclic, then by Corollary B, E is p-supersolvable and thus p-nilpotent since now p is the minimal prime divisor of $|E|$. By repeating this argument we know that E has a Sylow-tower and therefore E is solvable. Let p be any prime divisor of $|E|$, If E_p is cyclic, then E is p-hypercentrally embedded in G since now E is p-solvable. If E_p is not cyclic, E is also p-hypercentrally embedded in G by Theorem A. As a result we have $E \leqslant Z_{\mathcal{U}}(G)$. It follows that G is supersolvable since G/E is supersolvable and $E \leqslant Z_{\mathcal{U}}(G)$. □

Corollary 3.3 ([5, Theorem 3.1]). *Let p be an odd prime dividing the order of a group G and P a Sylow-subgroup of G. If $N_G(P)$ is p-nilpotent and every maximal subgroup of P is c-normal in G, then G is p-nilpotent.*

By noting the fact that if p is a prime such that $(|G|, p-1) = 1$, then G is p-nilpotent if and only if G is p-supersolvable, we have the following two corollary:

Corollary 3.4 ([5, Theorem 3.4]). *Let p be the smallest prime number dividing the order of a group G and P a Sylow p-subgroup of G. If every maximal subgroup of P is c-normal in G, then G is p-nilpotent.*

Proof. If $|P| = p$, then G is p-nilpotent by [13, Lemma 2.8]. If $|P| > p$, then by Corollary 2.12, G is p-supersolvable. Hence G is p-nilpotent. □

Corollary 3.5 ([5, Theorem 3.6]). *Let p be the smallest prime number dividing the order of group G and P a Sylow p-subgroup of G. If every minimal subgroup of $P \cap G'$ is c-normal in G and when $p = 2$, either every cyclic subgroup of $P \cap G'$ with order 4 is also c-normal in or P is quaternion-free, then G is p-nilpotent.*

Corollary 3.6 ([5, Corollary 3.9]). *Let p be an odd prime number dividing the order of a group G and P a Sylow p-subgroup of G. If every minimal subgroup of $P \cap G'$ is c-normal in G, then G is p-supersolvable.*

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