Algebra in superextensions
of inverse semigroups

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ABSTRACT. We find necessary and sufficient conditions on an (inverse) semigroup X under which its semigroups of maximal linked systems $\lambda(X)$, filters $\varphi(X)$, linked upfamilies $N_2(X)$, and upfamilies $\upsilon(X)$ are inverse.

1. Introduction

In this paper we investigate the algebraic structure of various extensions of an inverse semigroup X and detect semigroups whose extensions $\lambda(X)$, $N_2(X)$, $\varphi(X)$, $\upsilon(X)$ are inverse semigroups.

The thorough study of extensions of semigroups was started in [9] and continued in [1]–[5]. The largest among these extensions is the semigroup $\upsilon(X)$ of all upfamilies on X.

A family $\mathcal{F}$ of subsets of a set X is called an *upfamily* if each set $F \in \mathcal{F}$ is not empty and for each set $F \in \mathcal{F}$ any subset $E \supset F$ of X belongs to $\mathcal{F}$. The space of all upfamilies on X is denoted by $\upsilon(X)$. It is a closed subspace of the double power-set $\mathcal{P}(\mathcal{P}(X))$ endowed with the compact Hausdorff topology of the Tychonoff product $\{0,1\}^{\mathcal{P}(X)}$. Identifying each point $x \in X$ with the upfamily $\langle x \rangle = \{A \subset X : x \in A\}$, we can identify

X with a subspace of $\upsilon(X)$. Because of that we call $\upsilon(X)$ an extension of X.

The compact Hausdorff space $\upsilon(X)$ contains many other important extensions of X as closed subspaces. In particular, it contains the spaces $N_2(X)$ of linked upfamilies, $\lambda(X)$ of maximal linked upfamilies, $\varphi(X)$ of filters, and $\beta(X)$ of ultrafilters on X; see [8]. Let us recall that an upfamily $\mathcal{F} \in \upsilon(X)$ is called

- *linked* if $A \cap B \neq \emptyset$ for any sets $A, B \in \mathcal{F}$;

- *maximal linked* if $\mathcal{F} = \mathcal{F}'$ for any linked upfamily $\mathcal{F}' \in \upsilon(X)$ that contains $\mathcal{F}$;

- a *filter* if $A \cap B \in \mathcal{F}$ for any $A, B \in \mathcal{F}$;

- an *ultrafilter* if $\mathcal{F} = \mathcal{F}'$ for any filter $\mathcal{F}' \in \upsilon(X)$ that contains $\mathcal{F}$.

The family $\beta(X)$ of all ultrafilters on X is called the *Stone-Čech extension* and the family $\lambda(X)$ of all maximal linked upfamilies is called the *superextension* of X, see [12] and [15]. The arrows in the following diagram denote the identity inclusions between various extensions of a set X.

$$
\begin{array}{ccc}
X \longrightarrow \beta(X) \longrightarrow & \lambda(X) & \\
\downarrow & \downarrow & \\
\varphi(X) \longrightarrow N_2(X) \longrightarrow & \upsilon(X). &
\end{array}
$$

Any map $f : X \to Y$ induces a continuous map

$$\upsilon f : \upsilon(X) \to \upsilon(Y), \quad \upsilon f : \mathcal{F} \mapsto \{A \subset Y : f^{-1}(A) \in \mathcal{F}\},$$

such that $\upsilon f(\beta(X)) \subset \beta(Y)$, $\upsilon f(\lambda(X)) \subset \lambda(Y)$, $\upsilon f(\varphi(X)) \subset \varphi(Y)$, and $\upsilon f(N_2(X)) \subset N_2(Y)$. If the map f is injective, then υf is a topological embedding, which allows us to identify the extensions $\beta(X)$, $\lambda(X)$, $\varphi(X)$, $N_2(X)$, $\upsilon(X)$ with corresponding closed subspaces in $\beta(Y)$, $\lambda(Y)$, $\varphi(Y)$, $N_2(Y)$, and $\upsilon(Y)$, respectively.

In [9] it was observed that any (associative) binary operation $* : X \times X \to X$ can be extended to an (associative) binary operation $* : \upsilon(X) \times \upsilon(X) \to \upsilon(X)$ defined by the formula:

$$\mathcal{A} * \mathcal{B} = \Big\langle \bigcup_{a \in A} a * B_a : A \in \mathcal{A}, \ \{B_a\}_{a \in A} \subset \mathcal{B} \Big\rangle$$

for upfamilies $\mathcal{A}, \mathcal{B} \in \upsilon(X)$. Here for a family $\mathcal{C}$ of non-empty subsets of X by

$$\langle \mathcal{C} \rangle = \{A \subset X : \exists C \in \mathcal{C} \text{ with } C \subset A\}$$

we denote the upfamily generated by the family $\mathcal{C}$.

According to [8], for each semigroup X, $\upsilon(X)$ is a compact Hausdorff right-topological semigroup containing the subspaces $\beta(X)$, $\lambda(X)$, $\varphi(X)$, $N_2(X)$ as closed subsemigroups. Algebraic and topological properties of these semigroups have been studied in [9], [1]–[5]. In particular, in [3] we studied properties of extensions of groups while [4] was devoted to extensions of semilattices. There are two important classes of semigroups that include all groups and all semilattices. Those are the classes of inverse and Clifford semigroups.

Let us recall that a semigroup S is *inverse* if for any element $x \in S$ there is a unique element x^{-1} (called *the inverse of* x) such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. A semigroup S is *regular* if each element $x \in S$ is *regular* in the sense that $x \in xSx$. It is known [13, II.1.2] that a semigroup S is inverse if and only if S is regular and idempotents of S commute. For a semigroup S by $E = \{x \in S : xx = x\}$ we denote the set of idempotents of S. It follows that for an inverse semigroup, E is a commutative subsemigroup of S and hence E is a *maximal semilattice* in S. Let us recall that a *semilattice* is a set endowed with an associative commutative idempotent operation. A semigroup S is called *linear* if $xy \in \{x, y\}$ for any points $x, y \in X$. Each linear commutative semigroup is a semilattice.

A semigroup S is *Clifford* if it is a union of groups. An inverse semigroup S is Clifford if and only if $xx^{-1} = x^{-1}x$ for all $x \in X$. A semigroup S is called *sub-Clifford* if it is a union of cancellative semigroups. A semigroup S is sub-Clifford if and only if for any positive integer numbers $n < m$ and any $x \in S$ the equality $x^{n+1} = x^{m+1}$ implies $x^n = x^m$. Each subsemigroup of a Clifford semigroup is sub-Clifford and each finite sub-Clifford semigroup is Clifford. A commutative semigroup is Clifford if and only if it is inverse if and only if it is regular. A semigroup X is called *Boolean* if $x^3 = x$ for all $x \in X$. Each Boolean semigroup is Clifford.

It is well-known that the class of inverse (Clifford) semigroups includes all groups and all semilattices. Moreover, each inverse Clifford semigroup S decomposes into the union $S = \bigcup_{e \in E} H_e$ of maximal subgroups $H_e = \{x \in S : xx^{-1} = e = x^{-1}x\}$ indexed by the idempotents, which commute with all elements of S; see [13, II.2].

The algebraic structure of extensions of groups was studied in details in [3]. Extensions of semilattices were investigated in [4]. In particular,

in [4] it was shown that for a semigroup X the superextension $\lambda(X)$ is a semilattice if and only if the semigroup $\upsilon(X)$ is a semilattice if and only if X is a finite linear semilattice.

In Theorems 1.1–1.4 below we shall list all semigroups X whose extensions are (commutative) inverse semigroups.

For a natural number n by $C_n = \{z \in \mathbb{C} : z^n = 1\}$ we denote the cyclic group of order n and by L_n the linear semilattice $\{0, \ldots, n-1\}$ of order n, endowed with the operation of minimum. In particular, L_0 is an empty semigroup.

For two semigroups $(X, *)$ and $(Y, \star)$ by $X \sqcup Y$ we denote the disjoint union of these semigroups endowed with the semigroup operation

$$
x \circ y = \begin{cases}
x * y & \text{if } x, y \in X, \\
x & \text{if } x \in X \text{ and } y \in Y, \\
y & \text{if } x \in Y \text{ and } y \in X, \\
x \star y & \text{if } x, y \in Y.
\end{cases}
$$

The semigroup $X \sqcup Y$ will be called the *disjoint ordered union* of the semigroups X and Y. Observe that the operation of disjoint ordered union is associative in the sense that $(X \sqcup Y) \sqcup Z = X \sqcup (Y \sqcup Z)$ for any semigroups X, Y, Z.

We say that a subsemigroup X of a semigroup Y is *regular in Y* if each element $x \in X$ is regular in Y.

Theorem 1.1. *For a semigroup X and its superextension $\lambda(X)$ the following conditions are equivalent:*

(1) *$\lambda(X)$ is a commutative Clifford semigroup;*

(2) *$\lambda(X)$ is an inverse semigroup;*

(3) *the idempotents of the semigroup $\lambda(X)$ commute and $\lambda(X)$ is sub-Clifford or regular in $N_2(X)$;*

(4) *X is a finite commutative Clifford semigroup, isomorphic to one of the following semigroups: C_2, C_3, C_4, $C_2 \times C_2$, $L_2 \times C_2$, $L_1 \sqcup C_2$, L_n, or $C_2 \sqcup L_n$ for some $n \in \omega$.*

Theorem 1.2. *For a semigroup X and its semigroup of filters $\varphi(X)$ the following conditions are equivalent:*

(1) *$\varphi(X)$ is a commutative Clifford semigroup;*

(2) *$\varphi(X)$ is an inverse semigroup;*

(3) *the idempotents of the semigroup $\varphi(X)$ commute and $\varphi(X)$ is sub-Clifford or regular in $N_2(X)$;*

(4) *X is isomorphic to one of the semigroups: C_2, L_n or $L_n \sqcup C_2$ for some $n \in \omega$.*

Theorem 1.3. *For a semigroup X and its semigroup of linked upfamilies $N_2(X)$ the following conditions are equivalent:*

(1) *$N_2(X)$ is a commutative Clifford semigroup;*

(2) *$N_2(X)$ is an inverse semigroup;*

(3) *the idempotents of the semigroup $N_2(X)$ commute and $N_2(X)$ is sub-Clifford or regular;*

(4) *X is isomorphic to C_2 or L_n for some $n \in \omega$.*

Theorem 1.4. *For a semigroup X and its semigroup of upfamilies $\upsilon(X)$ the following conditions are equivalent:*

(1) *$\upsilon(X)$ is a finite semilattice;*

(2) *$\upsilon(X)$ is an inverse semigroup;*

(3) *the idempotents of the semigroup $\upsilon(X)$ commute and $\upsilon(X)$ is sub-Clifford or regular;*

(4) *X is a finite linear semilattice, isomorphic to L_n for some $n \in \omega$.*

Surprisingly, the following problem remains open.

Problem 1.5. *Characterize semigroups X whose Stone-Čech extension $\beta(X)$ is an inverse semigroup.* (Such semigroups have finite linear and finite cyclic subsemigroups; see Proposition 2.1.)

Theorems 1.1, 1.2, 1.3, and 1.4 will be proved in Sections 5, 6, 7, and 8, respectively.

2.   Commutativity in the Stone-Čech extension

In this section we establish some properties of semigroups whose Stone-Čech extension has commuting idempotents. Let us recall that a semigroup S is *cyclic* if $S = \{x^n : n \in \mathbb{N}\}$ for some element $x \in S$, called the *generator* of S.

Proposition 2.1. *If for a semigroup X all idempotents of the Stone-Čech extension $\beta(X)$ commute, then all cyclic subsemigroups and all linear subsemigroups of X are finite.*

Proof. First we show that each element $x \in X$ generates a finite cyclic subsemigroup $\{x^n\}_{n \in \mathbb{N}}$. If $\{x^n\}_{n \in \mathbb{N}}$ is infinite, then it is isomorphic to the semigroup $(\mathbb{N}, +)$. Then the Stone-Čech extension $\beta(X)$ contains a subsemigroup isomorphic to the Stone-Čech extension $\beta(\mathbb{N})$ of the semigroup $(\mathbb{N}, +)$. By Theorem 6.9 of [11], the semigroup $\beta(\mathbb{N})$ contains $2^{\mathfrak{c}}$ non-commuting idempotents and so does the semigroup $\beta(X)$ which is forbidden by our assumption. So, the cyclic subsemigroup $\{x^n\}_{n \in \mathbb{N}}$ is finite.

Next, assume that X contains an infinite linear subsemigroup L. Then $xy \in \{x, y\}$ for any elements $x, y \in L$. Choose any injective sequence $\{x_n\}_{n \in \omega}$ in L and define a 2-coloring $\chi : [\omega]^2 \to \{0, 1\}$ of the set $[\omega]^2 = \{(n, m) \in \omega^2 : n < m\}$ letting

$$\chi(n, m) = \begin{cases} 0 & \text{if } x_n x_m = x_n \\ 1 & \text{if } x_n x_m = x_m. \end{cases}$$

By Ramsey's Theorem [14] (see also [10, Theorem 5]), there is an infinite subset $\Omega \subset \omega$ and a color $k \in \{0, 1\}$ such that $\chi(n, m) = k$ for any pair $(n, m) \in [\omega]^2 \cap \Omega^2$. Consider the infinite linear subsemigroup $Z = \{x_n\}_{n \in \Omega}$ of X. By Theorem 1.1 of [4], each element of the semigroup $\beta(Z)$ is an idempotent. We claim that any two distinct free ultrafilters $\mathcal{U}, \mathcal{V} \in \beta(Z)$ do not commute (which is forbidden by our assumption). If the color $k = 0$, then $x_n x_m = x_n$ for any numbers $n < m$ in Ω, which implies that $\mathcal{U} * \mathcal{V} = \mathcal{U} \neq \mathcal{V} = \mathcal{V} * \mathcal{U}$. If $k = 1$, then $x_n x_m = x_m$ for any numbers $n < m$ in Ω and then $\mathcal{U} * \mathcal{V} = \mathcal{V} \neq \mathcal{U} = \mathcal{V} * \mathcal{U}$. $\qquad \square$

3.   The regularity of extensions of semigroups

In this section we shall prove some results related to the regularity of semigroups. Let us recall that an element $x \in S$ is *regular* in a semigroup S if $x \in xSx$.

Proposition 3.1. *Let X be a semigroup. An element $x \in X$ is regular in X if and only if the ultrafilter $\langle x \rangle$ is regular in the semigroup $\upsilon(X)$.*

Proof. The "if" part is trivial. To prove the "only if" part, assume that $\langle x \rangle$ is regular in $\upsilon(X)$ and find an upfamily $\mathcal{F} \in \upsilon(X)$ such that $\langle x \rangle = \langle x \rangle * \mathcal{F} * \langle x \rangle$. Then for some set $F \in \mathcal{F}$ we get $x \in xFx \subset xSx$, which means that x is regular in X. $\qquad \square$

Corollary 3.2. *A semigroup X is inverse if and only if X lies in some inverse semigroup $S \subset \upsilon(X)$.*

Proof. The "only if" part is trivial (just take $S = X$). To prove the "if" part, assume that a semigroup X lies in some inverse subsemigroup $S \subset \upsilon(X)$. The inverse semigroup S is regular and has commuting idempotents. Then the idempotents of the subsemigroup $X \subset S$ also commute. Each element $x \in X \subset S$ is regular in S and hence is regular in X by Proposition 3.1. Then the semigroup X is inverse, being a regular semigroup with commuting idempotents; see [13, II.1.2]. □

Let us recall that a non-empty subset I of a semigroup X is called an *ideal* in X if $XI \cup IX \subset I$.

Lemma 3.3. *Let X be a semigroup and $Z \subset X$ be a subsemigroup whose complement $X \setminus Z$ is an ideal in X. If for two upfamilies $\mathcal{A} \in \upsilon(Z) \subset \upsilon(X)$ and $\mathcal{B} \in \upsilon(X)$ we get $\mathcal{A} = \mathcal{A} * \mathcal{B} * \mathcal{A}$, then $\mathcal{A} = \mathcal{A} * \mathcal{B}_Z * \mathcal{A}$ for the upfamily $\mathcal{B}_Z = \{B \in \mathcal{B} : B \subset Z\} \in \upsilon(Z)$.*

Proof. It is clear that $\mathcal{A} * \mathcal{B}_Z * \mathcal{A} \subset \mathcal{A} * \mathcal{B} * \mathcal{A} = \mathcal{A}$. To prove the reverse inclusion, take any set $A \in \mathcal{A}$. It follows from $\mathcal{A} \in \upsilon(Z) \subset \upsilon(X)$ that $A \cap Z \in \mathcal{A} \subset (\mathcal{A} * \mathcal{B}) * \mathcal{A}$. So, we can find a set $C \in \mathcal{A} * \mathcal{B}$ and a family $\{A_c\}_{c \in C} \subset \mathcal{A}$ such that $\bigcup_{c \in C} c * A_c \subset A \cap Z$. For every $c \in C$ the inclusion $c * A_c \subset A \cap Z \subset Z$ implies $c \in Z$ (because $X \setminus Z$ is an ideal in X). So, $C \subset Z$. Since $C \in \mathcal{A} * \mathcal{B}$, there is a set $A \in \mathcal{A}$ and a family $\{B_a\}_{a \in A} \subset \mathcal{B}$ such that $\bigcup_{a \in A} a * B_a \subset C$. Since $X \setminus Z$ is an ideal in X, for every $a \in A$ the inclusion $a * B_a \subset C \subset Z$ implies $B_a \subset Z$ which means that $\{B_a\}_{a \in A} \subset \mathcal{B}_Z$ and hence $\mathcal{A} \subset \mathcal{A} * \mathcal{B}_Z * \mathcal{A}$. □

Corollary 3.4. *Let X be a semigroup and $S \in \{\beta(X), \lambda(X), \varphi(X), N_2(X), \upsilon(X)\}$ be one of its extensions. If the semigroup S is regular, then for any subsemigroup $Z \subset X$ whose complement $X \setminus Z$ is an ideal in X the semigroup $S \cap \upsilon(Z)$ is regular.*

Proof. Fix any upfamily $\mathcal{A} \in S \cap \upsilon(Z)$ and by the regularity of the semigroup S, find an upfamily $\mathcal{B} \in S$ such that $\mathcal{A} = \mathcal{A} * \mathcal{B} * \mathcal{A}$. By Lemma 3.3, $\mathcal{A} = \mathcal{A} * \mathcal{B}_Z * \mathcal{A}$ for the upfamily $\mathcal{B}_Z = \{B \in \mathcal{B} : B \subset Z\} \in \upsilon(Z)$. If $S \in \{\varphi(X), N_2(X), \upsilon(X)\}$, then $\mathcal{B}_Z \in S \cap \upsilon(Z)$ and hence $\mathcal{A}$ is regular in $S \cap \upsilon(Z)$.

If $S = \beta(X)$, then $\mathcal{B}_Z$ is a filter on Z and we can enlarge it to an ultrafilter $\tilde{\mathcal{B}}_Z \in \beta(Z)$. Then $\mathcal{A} = \mathcal{A} * \mathcal{B}_Z * \mathcal{A} \subset \mathcal{A} * \tilde{\mathcal{B}}_Z * \mathcal{A}$ implies that $\mathcal{A} = \mathcal{A} * \tilde{\mathcal{B}}_Z * \mathcal{A}$ by the maximality of the ultrafilter $\mathcal{A}$. So, $\mathcal{A}$ in regular in the semigroup $\beta(Z)$. By analogy we can consider the case $S = \lambda(X)$. □

4.   The extensions of the exceptional semigroups from Theorem 1.1

In this section we describe the structure of the extensions of the exceptional semigroups from Theorem 1.1(4).

We start with studying the superextensions of these semigroups. First note that for each set X of cardinality $1 \leq |X| \leq 2$ the superextension $\lambda(X)$ coincides with $\beta(X) = X$. If a set X has cardinality $|X| = 3$, then $\lambda(X) = X \cup \{\triangle\}$ where $\triangle = \{A \subset X : |A| \geq 2\}$. For a set X of cardinality $|X| = 4$ the superextension $\lambda(X) = \{x, \triangle_x, \square_x : x \in X\}$ consists of 12 elements, where

$$\triangle_x = \{A \subset X : |A \setminus \{x\}| \geq 2\} \ \text{ and}$$
$$\square_x = (X \setminus \{x\}) \cup \{A \subset X : x \in A, \ |A| \geq 2\} \ \text{ for } x \in X.$$

Given two semigroups X, Y we shall write $X \cong Y$ if these semigroups are isomorphic.

Proposition 4.1. *For finite exceptional semigroups we have the following isomorphisms:*

(1) $\lambda(C_2) = C_2$.

(2) $\lambda(C_3) \cong L_1 \sqcup C_3$.

(3) $\lambda(C_4) \cong (C_2 \sqcup L_1) \times C_4$.

(4) $\lambda(C_2 \times C_2) \cong (C_2 \sqcup L_1) \times C_2 \times C_2$.

(5) $\lambda(L_1 \sqcup C_2) \cong L_1 \sqcup L_1 \sqcup C_2$.

(6) $\lambda(L_2 \times C_2) \cong \big(L_1 \sqcup (L_2 \times L_2) \sqcup L_1\big) \times C_2$.

Proof. 1–4. The first four statements were proved in [5, §6].

5. For the semigroup $X = L_1 \sqcup C_2 = \{0, 1, -1\}$ the superextension $\lambda(X) = \{0, \triangle, 1, -1\}$ has the structure of the ordered union $\{0\} \sqcup \{\triangle\} \sqcup \{1, -1\}$, which is isomorphic to the semigroup $L_1 \sqcup L_1 \sqcup C_2$.

6. The semigroup $L_2 \times C_2 = \{0, 1\} \times \{-1, 1\}$ has two idempotents $e = (0, 1)$ and $f = (1, 1)$ and two elements $a = (0, -1)$ and $b = (1, -1)$ of order 2 such that $a^2 = e$ and $b^2 = f$. The superextension

$$\lambda(L_2 \times C_2) = \{x, \triangle_x, \square_x : x \in L_2 \times C_2\}$$

has the 6-element set of idempotents $E = \{e, \square_e, \triangle_a, \triangle_b, \square_f, f\}$, isomorphic to the semilattice $L_1 \sqcup (L_2 \times L_2) \sqcup L_1$. The semigroup $\lambda(L_2 \times C_2)$ is

isomorphic to the product $E \times C_2$ under the isomorphism $h : E \times C_2 \to \lambda(L_2 \times C_2)$ defined by

$$h : (x, g) \mapsto \begin{cases} x & \text{if } g = 1, \\ xb & \text{if } g = -1. \end{cases}$$

$\square$

The following proposition was proved in [4, 3.1].

Proposition 4.2. *For every $n \in \mathbb{N}$ the semigroup $\upsilon(L_n)$ is a finite semilattice. Consequently, the semigroups $\lambda(L_n)$, $\varphi(L_n)$, $N_2(L_n)$ also are finite semilattices.*

We recall that a semigroup X is called *Boolean* if $x = x^3$ for all $x \in X$. It is clear that each Boolean semigroup is Clifford and each commutative Boolean semigroup is inverse.

Proposition 4.3. *For every $n \in \mathbb{N}$ and the semigroup $X = C_2 \sqcup L_n$ the superextension $\lambda(X)$ is a finite commutative Boolean semigroup whose maximal semilattice $E(\lambda(X))$ coincides with the set $\lambda(X) \setminus \{a\}$ where a is the unique element generating the subgroup C_2 of $X = C_2 \sqcup L_n$. Moreover, $ea = a$ for any idempotent e of $\lambda(X)$.*

Proof. Observe that $X \setminus \{a\}$ is a linear semilattice such that $xa = a$ for all $x \in X \setminus \{a\}$. We identify the point a with the principal ultrafilter $\langle a \rangle$ generated by a. For the convenience of the reader we divide the proof of Proposition 4.3 into a series of claims.

Claim 4.4. *For each $\mathcal{F} \in \lambda(X) \setminus \{a\}$ we get $a * \mathcal{F} = \mathcal{F} * a = \langle a \rangle$.*

Proof. Since $\mathcal{F} \neq \langle a \rangle$, there is a set $F \in \mathcal{F}$ with $a \notin F$. Then $a * F = F * a = \{a\}$, which implies $a * \mathcal{F} = a * \mathcal{F} = \langle a \rangle$. $\square$

Claim 4.5. *Each element $\mathcal{F} \in \lambda(X) \setminus \{a\}$ is an idempotent.*

Proof. Since the upfamilies $\mathcal{F}$ and $\mathcal{F} * \mathcal{F}$ are maximal linked, it suffices to check that $\mathcal{F} \subset \mathcal{F} * \mathcal{F}$. Fix any set $F \in \mathcal{F}$ and consider two cases. If $a \notin F$, then $F = F * F \in \mathcal{F} * \mathcal{F}$. So, assume that $a \in F$. Since $\mathcal{F} \neq \langle a \rangle$, there is a non-empty set $F_a \in \mathcal{F}$ that does not contain the point a. Then $a * F_a = \{a\} \subset F$. For each $x \in F \setminus \{a\}$, let $F_x = F$ and observe that $x * F \subset \{x\} \cup F \subset F$. Then $\bigcup_{x \in F} x * F_x \subset \{a\} \cup F = F$ and hence $F \in \mathcal{F} * \mathcal{F}$. $\square$

Claim 4.6. $\mathcal{U} * \mathcal{V} = \mathcal{V} * \mathcal{U}$ *for any maximal linked systems* $\mathcal{U}, \mathcal{V} \in \lambda(X)$.

Proof. The equality $\mathcal{U} * \mathcal{V} = \mathcal{V} * \mathcal{U}$ is trivial if $\mathcal{U}$ or $\mathcal{V}$ belongs to $\beta(X) = X$. So, we assume that the maximal linked systems $\mathcal{U}, \mathcal{V} \notin X$ are not ultrafilters.

First we prove that $\mathcal{U} * \mathcal{V} \subset \mathcal{V} * \mathcal{U}$. Fix any set $W \in \mathcal{U} * \mathcal{V}$. Without loss of generality, it is of the basic form $W = \bigcup_{u \in U} u * V_u$ for some set $U \in \mathcal{U}$ and a family $\{V_u\}_{u \in U} \subset \mathcal{V}$. Since $X \setminus \{a\}$ is a linear semilattice, (the proof of Theorem 2.5) [4] guarantees that:

$$(U \setminus \{a\}) * (V_u \setminus \{a\}) \subset W \quad \text{for some point} \quad u \in U \setminus \{a\}.$$

We consider three cases.

1) $a \notin V_u$ and $a \notin U$. Then $\mathcal{V} * \mathcal{U} \ni V_u * U = U * V_u = (U \setminus \{a\}) * (V_u \setminus \{a\}) \subset W$ and hence $W \in \mathcal{V} * \mathcal{U}$.

2) $a \notin V_u$ and $a \in U$. Then $a * V_u = \{a\}$. Since $\mathcal{V} \neq \langle a \rangle$, the set $V_a \setminus \{a\}$ is not empty and hence contains some idempotent. Then $aV_u = \{a\} \subset a * V_a \subset W$ and

$$\mathcal{V} * \mathcal{U} \ni V_u * U = U * V_u = (a * V_u) \cup (U \setminus \{a\}) * V_u =$$

$$\{a\} \cup (U \setminus \{a\}) * (V_u \setminus \{a\}) \subset \{a\} \cup W \subset W$$

and again $W \in \mathcal{V} * \mathcal{U}$.

3) $a \in V_u$. In this case $a \in u * V_u \subset W$. It follows from $\mathcal{U} \neq \langle a \rangle$ that $a \notin U_a$ for some set $U_a \in \mathcal{U}$. Let $U_v = U$ for all $v \in V_u \setminus \{a\}$ and observe that

$$\mathcal{V} * \mathcal{U} \ni \bigcup_{v \in V_u} v * U_v = a * U_a \cup ((V_u \setminus \{a\}) * U) =$$

$$\{a\} \cup ((V_u \setminus \{a\}) * (U \setminus \{a\})) \cup ((V_u \setminus \{a\}) * a) \subset \{a\} \cup W \cup \{a\} \subset W$$

and hence $W \in \mathcal{V} * \mathcal{U}$. Therefore, $\mathcal{U} * \mathcal{V} \subset \mathcal{V} * \mathcal{U}$.

The inclusion $\mathcal{V} * \mathcal{U} \subset \mathcal{U} * \mathcal{V}$ can be proved by analogy. □

□

Next, we study the structure of the space of filters $\varphi(X)$ of the finite exceptional groups from Theorem 1.2.

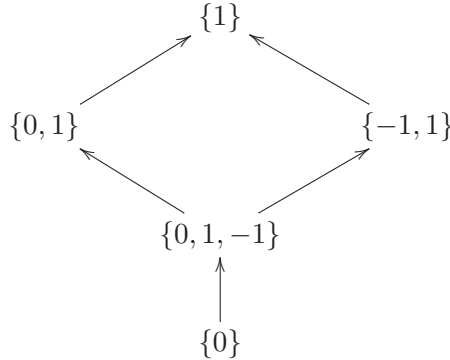**Proposition 4.7.**   (1) $\varphi(C_2) = N_2(C_2) \cong L_1 \sqcup C_2$;

(2) $\varphi(L_1 \sqcup C_2)$ *is a commutative Boolean semigroup isomorphic to the subsemigroup*

$$\{(e, x) \in (L_1 \sqcup (L_2 \times L_2)) \times C_2 : e \in L_1 \sqcup \{(0,0), (0,1)\} \Rightarrow (x = 1)\}$$

of the commutative Boolean semigroup $(L_1 \sqcup (L_2 \times L_2)) \times C_2$.

Proof. 1. The semigroup $\varphi(C_2)$ contains two ultrafilters and one filter $\mathcal{Z} = \langle C_2 \rangle$ generated by the set C_2. The filter $\mathcal{Z}$ is the zero of the semigroup $\varphi(C_2)$ and hence $\varphi(C_2)$ is isomorphic to $\{\mathcal{Z}\} \sqcup C_2$.

2. For the semigroup $X = L_1 \sqcup C_2 = \{0, 1, -1\}$ the semigroup $\varphi(X)$ contains 7 filters generated by all non-empty subsets of X. So, we can identify filters with their generating sets. Among these 7 filters there are 5 idempotents: $\{0\}, \{0, 1, -1\}, \{0, 1\}, \{1, -1\}$, and $\{1\}$ which form a semilattice E

$$\{1\}$$

$$\{0, 1\} \qquad \{-1, 1\}$$

$$\{0, 1, -1\}$$

$$\{0\}$$

isomorphic to $L_1 \sqcup (L_2 \times L_2)$. Two filters $\{-1\}$ and $\{0, -1\}$ generate 2-element subgroups with idempotents $\{1\}$ and $\{0, 1\}$, respectively. Since $\{-1\} * \{0, 1\} = \{0, -1\}$, the semigroup $\lambda(X)$ is isomorphic to the subsemigroup $\{(e, x) \in E \times C_2 : e \in \{\{0\}, \{0, 1, -1\}, \{1, -1\}\} \Rightarrow (x = 1)\}$ of the commutative Boolean semigroup $E \times C_2$. $\qquad \square$

Now we consider the the semigroups $\varphi(L_n)$ and $\varphi(L_n \sqcup C_2)$. We shall show that the latter semigroup has the structure of the reduced product of a semilattice and a group.

Let X, Y be two semigroups and I be an ideal in X. The *reduced product* $X \times_I Y$ is the set $I \cup ((X \setminus I) \times Y)$ endowed with the semigroup operation

$$a * b = \begin{cases} p_X(a) * p_X(b) & \text{if } p_X(a) * p_X(b) \in I, \\ (p_X(a) * p_X(b), p_Y(a) * p_Y(b)) & \text{if } p_X(a) * p_X(b) \notin I. \end{cases}$$

Here by $p_X : X \times_I Y \to X$ and $p_Y : (X \setminus I) \times Y \to Y$ we denote the natural projections. Let us recall that by Proposition 4.7(1), the semigroup $\varphi(C_2)$ is isomorphic to the commutative Boolean semigroup $L_1 \sqcup C_2$.

Proposition 4.8. *For every $n \in \mathbb{N}$ the semigroup*

(1) $\varphi(L_n)$ *is a finite semilattice, and*

(2) $\varphi(L_n \sqcup C_2)$ *is a commutative Boolean semigroup isomorphic to the reduced product* $\varphi(L_{n+1}) \times_{\varphi(L_n)} \varphi(C_2)$.

Proof. By Proposition 4.2, the semigroup $\varphi(L_n)$ is a finite semilattice.

Now consider the semigroup $X = L_n \sqcup C_2$. Since X is finite we can identify the semigroup $\varphi(X)$ with the commutative semigroup of all non-empty subsets of X. Let a be the generator of the cyclic group C_2 and $e = a^2$ be its idempotent. The idempotent semilattice $E = L_n \sqcup \{e\}$ of X is isomorphic to the linear semilattice L_{n+1}. So, we shall identify E with L_{n+1}. Observe that $\varphi(X) \setminus \varphi(L_n) = \{F \subset X : F \cap C_2 \neq \emptyset\}$ and the map $h : \varphi(L_{n+1}) \times_{\varphi(L_n)} \varphi(C_2) \to \varphi(X)$ defined by

$$
h(A) = \begin{cases} A & \text{if } A \subset \varphi(L_n) \\ (A \setminus C_2) \cup B & \text{if } (A, B) \in (\varphi(L_{n+1}) \setminus \varphi(L_n)) \times \varphi(C_2) \end{cases}
$$

is a required isomorphism between the semigroups $\varphi(X)$ and $\varphi(L_{n+1}) \times_{\varphi(L_n)} \varphi(C_2)$. $\qquad\square$

5.   Proof of Theorem 1.1

Given a semigroup X, we need to prove the equivalence of the following statements:

(1) $\lambda(X)$ is a commutative Clifford semigroup;

(2) $\lambda(X)$ is an inverse semigroup;

(3) the idempotents of $\lambda(X)$ commute and $\lambda(X)$ is sub-Clifford or regular in $N_2(X)$;

(4) X is a finite commutative inverse semigroup, isomorphic to one of the following semigroups: C_2, C_3, C_4, $C_2 \times C_2$, $L_2 \times C_2$, $L_1 \sqcup C_2$, L_n, or $C_2 \sqcup L_n$ for some $n \in \omega$.

We shall prove the implications $(4) \Rightarrow (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$.

The implication $(4) \Rightarrow (1)$ follows from Propositions 4.1—4.3 while $(1) \Rightarrow (2) \Rightarrow (3)$ are trivial or well-known; see [13, II.1.2].

To prove that $(3) \Rightarrow (4)$, assume that the idempotents of the semigroup $\lambda(X)$ commute and $\lambda(X)$ is sub-Clifford or regular in $N_2(X)$. Then the idempotents of the semigroup X also commute and hence the set $E = \{e \in X : ee = e\}$ of idempotents of X is a semilattice. For the convenience of the reader we divide the further proof into a series of claims.

Claim 5.1. *The semigroup $\lambda(X)$ is sub-Clifford or regular.*

Proof. If $\lambda(X)$ is not sub-Clifford, then it is regular in the semigroup $N_2(X)$ according to our assumption. We claim that $\lambda(X)$ is regular. Given any maximal linked system $\mathcal{A} \in \lambda(X)$, use the regularity of $\lambda(X)$ in $N_2(X)$ to find a linked upfamily $\mathcal{B} \in N_2(X)$ such that $\mathcal{A} = \mathcal{A} * \mathcal{B} * \mathcal{A}$. Enlarge $\mathcal{B}$ to a maximal linked upfamily $\tilde{\mathcal{B}} \in \lambda(X)$. Then $\mathcal{A} = \mathcal{A} * \mathcal{B} * \mathcal{A} \subset \mathcal{A} * \tilde{\mathcal{B}} * \mathcal{A}$ implies that $\mathcal{A} = \mathcal{A} * \tilde{\mathcal{B}} * \mathcal{A}$ by the maximality of the linked family $\mathcal{A}$. Therefore $\mathcal{A}$ is regular in $\lambda(X)$. □

Claim 5.2. *The semigroup X is inverse.*

Proof. Since the idempotents of the semigroup X commute, it suffices to check that X is regular. By our assumption, the semigroup X is regular or sub-Clifford. If $\lambda(X)$ is regular, then X is regular by Proposition 3.1. Now assume that $\lambda(X)$ is sub-Clifford. Then so is the semigroup X. Since the idempotents of the semigroup $\beta(X) \subset \lambda(X)$ commute, by Proposition 2.1, each cyclic subsemigroup $\{x^n\}_{n\in\mathbb{N}}$ of S is finite and hence is a group by the sub-Clifford property of X. Then the semigroup X is Clifford and hence regular. □

Claim 5.3. *The semilattice $E \subset X$ is linear and finite.*

Proof. Assuming that E is not linear, we can find two idempotents $x, y \in E$ such that $xy \notin \{x, y\}$. Now consider the maximal linked system $\mathcal{L} = \langle \{x, y\}, \{x, xy\}, \{y, xy\} \rangle$. It can be shown that $\mathcal{L} \neq \mathcal{L} * \mathcal{L} = \langle \{xy\} \rangle = \mathcal{L} * \mathcal{L} * \mathcal{L}$, which is not possible if the semigroup $\lambda(X)$ is sub-Clifford.

Next, we show that the element $\mathcal{L}$ is not regular in the semigroup $\upsilon(X)$, which is not possible if the semigroup $\lambda(X)$ is regular. Assuming that $\mathcal{L}$ is regular, find an upfamily $\mathcal{A} \in \upsilon(X)$ such that $\mathcal{L} * \mathcal{A} * \mathcal{L} = \mathcal{L}$. It follows from $\{x, y\} \in \mathcal{L} = \mathcal{L} * \mathcal{A} * \mathcal{L}$ that $\{x, y\} \supset \bigcup_{u\in L} u * B_u$ for some set $L \in \mathcal{L}$ and some family $\{B_u\}_{u\in L} \subset \mathcal{A} * \mathcal{L}$. The linked property of family $\mathcal{L}$ implies that the intersection $L \cap \{x, xy\}$ contains some point u. Now for the set $B_u \in \mathcal{A} * \mathcal{L}$ find a set $A \in \mathcal{A}$ and a family $\{L_a\}_{a\in A} \subset \mathcal{L}$ such that $B_u \supset \bigcup_{a\in A} a * L_a$. Fix any point $a \in A$ and a point $v \in L_a \cap \{y, xy\}$. Then $uav \in uaL_a \subset uB_u \subset \{x, y\}$. Since $u \in \{x, xy\}$ and $v \in \{y, xy\}$,

the element uav is equal to xby for some element $b \in \{a, ya, ax, yax\}$. So, $xby \in \{x, y\}$. If $xby = x$, then $xy = xbyy = xby = x \in \{x, y\}$. If $xby = y$, then $xy = xxby = xby = y \in \{x, y\}$. In both cases we obtain a contradiction with the choice of the idempotents x and y.

Since the idempotents of the semigroup $\beta(X) \subset \lambda(X)$ commute, the linear semilattice E is finite according to Proposition 2.1. $\qquad \square$

Since X is an inverse semigroup with finite linear semilattice E, we can apply Theorem 7.5 of [6] to derive our next claim.

Claim 5.4. *The semigroup X is inverse and Clifford.*

Since the semigroup X is inverse and Clifford, the idempotents of X commute with all elements of X; see Theorem II.2.6 in [13].

Claim 5.5. *Each subgroup H in X has cardinality $|H| \leq 4$.*

Proof. We lose no generality assuming that H coincides with the maximal group H_e containing the idempotent e of the group H. An upfamily $\mathcal{A} \in \upsilon(H)$ is called *left invariant* if $x\mathcal{A} = \mathcal{A}$ for any point $x \in H$. By $\overset{\leftrightarrow}{N}_2(H)$ denote the family of all left invariant linked systems on H and by $\overset{\leftrightarrow}{\lambda}(H) = \max \overset{\leftrightarrow}{N}_2(H)$ the family of all maximal elements of $\overset{\leftrightarrow}{N}_2(H)$. Elements of $\overset{\leftrightarrow}{\lambda}(H)$ are called *maximal invariant linked systems.* Zorn's Lemma guarantees that the set $\overset{\leftrightarrow}{\lambda}(H)$ is not empty.

We claim that $\overset{\leftrightarrow}{\lambda}(H)$ is a singleton. Assuming the opposite, fix two distinct maximal invariant linked systems $\mathcal{A}_1, \mathcal{A}_2 \in \overset{\leftrightarrow}{\lambda}(H)$. By Proposition 1 of [2], for every $i \in \{1, 2\}$ the set

$$\uparrow\mathcal{A}_i = \{\mathcal{L} \in \lambda(H) : \mathcal{L} \supset \mathcal{A}_i\}$$

is a left ideal in the compact right-topological semigroup $\lambda(H)$. By Ellis' Theorem [7] (see also [11, 2.5]), this left ideal contains an idempotent $\mathcal{E}_i \supset \mathcal{A}_i$. The idempotents $\mathcal{E}_1, \mathcal{E}_2$ do not commute because the products $\mathcal{E}_1 * \mathcal{E}_2$ and $\mathcal{E}_2 * \mathcal{E}_1$ belong to the disjoint left ideals $\uparrow\mathcal{A}_2$ and $\uparrow\mathcal{A}_1$, respectively (the left ideals $\uparrow\mathcal{A}_1$ and $\uparrow\mathcal{A}_2$ are disjoint by the maximality of the invariant linked systems $\mathcal{A}_1$ and $\mathcal{A}_2$).

Since $|\overset{\leftrightarrow}{\lambda}(H)| = 1$, we can apply Theorems 2.2 and 2.6 of [5] and conclude that the group H either has cardinality $|H| \leq 5$ or else H is isomorphic to the dihedral group D_6 or to the group $(C_2)^3$.

To finish the proof of Claim 5.5 it remains to show that H is not isomorphic to the groups C_5, D_6 or C_2^3.

C_5: If H is isomorphic to the 5-element cyclic group C_5, then the superextension $\lambda(X)$ contains an isomorphic copy of the semigroup $\lambda(C_5)$. By [5, §6.4], the semigroup $\lambda(H) \cong \lambda(C_5)$ contains two distinct elements $\mathcal{Z}, \Theta$ such that $\mathcal{L} * \Theta = \mathcal{Z}$ for any maximal linked system $\mathcal{L} \in \lambda(H)$. This implies that the element Θ is not regular in $\lambda(H)$. We claim that this element is not regular in $\lambda(X)$. Assuming the converse, find a maximal linked system $\mathcal{L} \in \lambda(X)$ such that $\Theta = \Theta * \mathcal{L} * \Theta$.

Let e be the idempotent of the maximal subgroup $H = H_e$ of X. Since X is inverse and Clifford, the idempotent e lies in the center of the semigroup X, so the shift $s_e : X \to eX$, $s_e : x \mapsto xe = ex$, is a well-defined homomorphism from the semigroup X onto its principal ideal $eX = Xe$. Since $e\Theta = \Theta$, for the maximal linked system $e\mathcal{L} \in \lambda(eX)$ we get $\Theta = \Theta * e\mathcal{L} * \Theta$, which means that Θ is regular in the semigroup $\lambda(eX)$. Since $eX \setminus H_e$ is an ideal in eX, Corollary 3.4 implies that the element Θ is regular in the semigroup $\lambda(H_e)$, which contradicts the choice of Θ. So, Θ is not regular in $\lambda(X)$ and the semigroup $\lambda(X)$ is not regular.

On the other hand, the property of Θ guarantees that $\Theta \neq \mathcal{Z} = \Theta * \Theta = \Theta * \Theta * \Theta$, which means that the semigroup $\lambda(X)$ is not sub-Clifford. In both cases we obtain a contradiction with Claim 5.1.

D_6: Next, assume that H is isomorphic to the dihedral group D_6. In this case H contains an element a of order 3 and element b of order 2 such that $ba = a^2 b$. Consider the maximal linked systems $\Delta = \langle \{e, a\}, \{e, a^2\}, \{a, a^2\} \rangle$ and $\Lambda = \langle \{e, b\}, \{e, ab\}, \{e, a, a^2\}, \{a, b, ab\}, \{a^2, b, ab\} \rangle$. It is easy to check that Δ and Λ are two non-commuting idempotents in $\lambda(X)$ (because $\{e, a, ab\} \in \Delta * \Lambda$ and $\{e, a, ab\} \notin \Lambda * \Delta$). So, the semigroups $\lambda(X) \supset \lambda(H)$ contains two non-commuting idempotents, which is a contradiction.

C_2^3: In the case $H \cong C_2^3$ fix three elements $a, b, c \in C_2^3$ generating the group C_2^3. Consider two maximal linked systems $\square_b = \langle \{e, a\}, \{e, b\}, \{e, ab\}, \{a, b, ab\} \rangle$ and $\square_c = \langle \{e, a\}, \{e, c\}, \{e, ac\}, \{a, c, ac\} \rangle$ and observe that they are non-commuting idempotents of $\lambda(H)$ (because $\{e, c, b, ab\} \in \square_b * \square_c$ and $\{e, c, b, ab\} \notin \square_c * \square_b$). □

Claim 5.6. *The semigroup $\lambda(X)$ is inverse.*

Proof. By Claim 5.1, the semigroup $\lambda(X)$ is regular or sub-Clifford. We claim that $\lambda(X)$ is regular. If not, then $\lambda(X)$ is sub-Clifford. Claims 5.3—5.5 imply that the semigroup X is finite and so is its superextension $\lambda(X)$. Being finite and sub-Clifford, the semigroup $\lambda(X)$ is Clifford and hence regular. Taking into account that the idempotents of $\lambda(X)$ commute, we conclude that the semigroup $\lambda(X)$ is inverse. □

Since the semilattice E is linear, we can write it as $E = \{e_1, \ldots, e_n\}$ where $e_i e_j = e_i \neq e_j$ for all $1 \leq i < j < n$. For every $i \leq n$ by $H_i = H_{e_i}$ denote the maximal subgroup of X that contains the idempotent e_i. By Claim 5.5, each subgroup H_i has cardinality $|H_i| \leq 4$ and hence is commutative. We claim that the semigroup X also is commutative. Indeed, given any points $x, y \in X$ we can find numbers $i, j \leq n$ such that $x \in H_i$ and $y \in H_j$. We lose no generality assuming that $i \leq j$. Then $xy = (xe_i)y = x(e_i y) = (ye_i)x = yx \in H_i$, so X is commutative.

Claim 5.7. *For any $1 < i < n$ the maximal subgroup H_i is trivial.*

Proof. Assume conversely that the subgroup H_i is not trivial and take any element $a \in H_i \setminus E$. Next, consider the maximal linked system $\Delta = \langle \{e_{i-1}, a\}, \{a, e_{i+1}\}, \{e_{i-1}, e_{i+1}\} \rangle$. We claim that Δ is not regular in $\lambda(X)$. Assume conversely that Δ is regular in $\lambda(X)$. Using Corollary 3.4, we can show that Δ is regular in the semigroup $\lambda(H_{i-1} \cup H_i \cup H_{i+1})$. Then we can find a maximal linked system $\mathcal{F} \in \lambda(H_{i-1} \cup H_i \cup H_{i+1})$ such that $\Delta = \Delta * \mathcal{F} * \Delta$. For the set $\{a, e_{i+1}\} \in \Delta$, find a set $A \in \Delta * \mathcal{F}$ with $A \subset H_{i-1} \cup H_i \cup H_{i+1}$ and a family $\{D_a\}_{a \in A} \subset \Delta$ such that $\{a, e_{i+1}\} \supset \bigcup_{a \in A} a * D_a$. Observe that such an inclusion is possible only if $D_a = \{a, e_{i+1}\}$ for all $a \in A$. But then $A * \{a, e_{i+1}\} \subset \{a, e_{i+1}\}$ implies $A = \{e_{i+1}\}$ and $\Delta * \mathcal{F} = \langle e_{i+1} \rangle$ which is not possible. $\square$

Claim 5.8. *If $n \geq 2$, then $|H_n| \leq 2$.*

Proof. Assume conversely that $|H_n| > 2$. Then two cases are possible.

1. The group H_n is cyclic. Fix a generator a of the cyclic group H_n and consider the maximal linked system $\Delta = \langle \{a, e_{n-1}\}, \{a, e_n\}, \{e_{n-1}, e_n\} \rangle$. We claim that the element Δ is not regular in the semigroup $\lambda(X)$. Assuming the opposite, we can find a maximal linked system $\mathcal{F} \in \lambda(X)$ with $\Delta * \mathcal{F} * \Delta = \Delta$. Then for the set $\{e_n, a\} \in \Delta$ we can find a set $A \in \Delta * \mathcal{F}$ and a family $\{D_a\}_{a \in A}$ of minimal subsets of Δ such that $\{e_n, a\} \supset \bigcup_{a \in A} a * D_a$. This inclusion is possible only if $\{a, e_n\} = D_a$ for all $a \in A$. The inclusion $A * \{a, e_n\} \subset \{a, e_n\}$ implies that $A = \{e_n\}$. Now for the set $A \in \Delta * \mathcal{F}$, find a minimal set $D \in \Delta$ and a family $\{F_d\}_{d \in D} \subset \mathcal{F}$ such that $\bigcup_{d \in D} d * F_d \subset A = \{e_n\}$. This inclusion is possible only if $\{a, e_n\} \subset D \subset H_n$ and $A_d = \{d^{-1}\} \subset H_n$ for each $d \in D$. Then the family $\mathcal{A}$ contains two disjoint sets $\{e_n\}$ and $\{a^{-1}\}$ which is not possible as $\mathcal{A}$ is linked.

2. The group H_n is isomorphic to the group $C_2 \times C_2$. Then we can take two distinct elements a, b generating the group H_n, and consider the maximal linked system $\square = \langle \{e_{n-1}, a\}, \{e_{n-1}, b\}, \{e_{n-1}, ab\}, \{a, b, ab\} \rangle$.

We claim that the element $\square$ is not regular in the semigroup $\lambda(X)$. Assuming the opposite, find a maximal linked system $\mathcal{F} \in \lambda(X)$ with $\square * \mathcal{F} * \square = \square$. Then for the set $\{a, b, ab\} \in \square$ we can find a set $A \in \square * \mathcal{F}$ and a family $\{D_x\}_{x \in A}$ of minimal subsets of $\square$ such that $\{a, b, ab\} \subset \bigcup_{x \in A} x * D_x$. This inclusion is possible only if $D_x = \{a, b, ab\}$ for all $x \in A$. The inclusion $A * \{a, b, ab\} = \bigcup_{x \in A} x * D_x \subset \{a, b, ab\}$ implies that $A = \{e_n\}$. Now for the set $A \in \square * \mathcal{F}$, find a minimal set $S \in \square$ and a family $\{F_s\}_{s \in S} \subset \mathcal{F}$ such that $\bigcup_{s \in S} s * F_s \subset A = \{e_n\}$. This inclusion is possible only if $S = \{a, b, ab\}$ and $F_s = \{s\} \subset H_n$ for each $d \in D$. Then the family $\mathcal{F}$ contains disjoint sets $\{s\}$, $s \in S$, which is not possible as $\mathcal{F}$ is linked. $\qquad \square$

Claim 5.9. *If $n \geq 3$, then the group H_n is trivial.*

Proof. Assume that H_n is not trivial. By Claim 5.8, $|H_n| = 2$. Fix a generator a of the cyclic group H_n and observe that the maximal linked systems $\square = \langle\{e_{n-2}, a\}, \{e_{n-2}, e_{n-1}\}, \{e_{n-2}, e_n\}, \{a, e_{n-1}, e_n\}\rangle$ and $\Delta = \langle\{e_{n-1}, a\}, \{e_{n-1}, e_n\}, \{a, e_n\}\rangle$ are non-commuting idempotents of the semigroup $\lambda(X)$ because

$$\square * \Delta = \langle\{e_{n-1}, e_{n-2}\}, \{e_{n-1}, a\}, \{e_{n-1}, e_n\}, \{e_{n-2}, a, e_n\}\rangle \neq \square = \Delta * \square.$$

$\qquad \square$

Claim 5.10. *If $n \geq 2$, then $|H_1| \leq 2$.*

Proof. Assume that $|H_1| > 2$ and chose two distinct elements $a, b \in H_1 \backslash E$. We claim that the maximal linked system

$$\Delta = \langle\{a, b\}, \{a, e_2\}, \{b, e_2\}\rangle$$

is not regular element of $\lambda(X)$. Assuming the opposite, find a maximal linked system $\mathcal{F} \in \lambda(X)$ such that $\Delta * \mathcal{F} * \Delta = \Delta$. Replacing $\mathcal{F}$ by $e_2 * \mathcal{F}$, if necessary, we can assume that $\mathcal{F} \in \lambda(H_1 \cup H_2)$. For the set $\{a, e_2\} \in \Delta$, find a set $A \in \Delta * \mathcal{F}$ and a family $\{D_x\}_{x \in A} \subset \Delta$ such that $\bigcup_{x \in A} x * D_x \subset \{a, e_2\}$. This inclusion implies that $A = \{e_2\} \in \Delta * \mathcal{F}$, which is not possible. $\qquad \square$

Now we are able to finish the proof of Theorem 1.1. If $n = |E| \geq 3$, then the semigroup X is isomorphic to L_n or to $C_2 \sqcup L_{n-1}$ by Claims 5.7— 5.10. If $n = |E| = 1$, then X is a group of cardinality $|X| \leq 4$, isomorphic to one of groups: $L_1, C_2, C_3, C_4, C_2 \times C_2$.

It remains to consider the case $|E| = 2$. By Claims 5.8, 5.10, $\max\{|H_1|, |H_2|\} \leq 2$. If $|H_1| = |H_2| = 1$, then $X \cong L_2$. If $|H_1| = 1$ and $|H_2| = 2$, then $X \cong L_1 \sqcup C_2$. If $|H_1| = 2$ and $|H_2| = 1$, then $X \cong C_2 \sqcup L_1$.

Finally assume that $|H_1| = |H_2|$. For $i \in \{1, 2\}$ let a_i be the unique generator of the 2-element cyclic group H_i.

Claim 5.11. $a_2 * e_1 = a_1$.

Proof. Assuming that $a_2 * e_1 \neq a_1$, we get $a_2 * e_1 = e_1$. Then the maximal linked systems

$$\square_e = \langle \{e_1, a_1\}, \{e_1, a_2\}, \{e_1, e_2\}, \{a_1, a_2, e_2\} \rangle \text{ and}$$

$$\square_a = \langle \{a_1, e_1\}, \{a_1, e_2\}, \{a_1, a_2\}, \{e_1, e_2, a_2\} \rangle$$

are not commuting idempotents of $\lambda(X)$ (because $\square_e * \square_a = \square_a$ while $\square_a * \square_e = \square_e$). $\square$

Claim 5.11 implies that the semigroup $X = H_1 \cup H_2$ is isomorphic to $L_2 \times C_2$.

6.   Proof of Theorem 1.2

Given a semigroup X, we need to check the equivalence of the following statements:

(1) $\varphi(X)$ is a commutative Clifford semigroup;

(2) $\varphi(X)$ is an inverse semigroup;

(3) the idempotents of $\varphi(X)$ commute and $\varphi(X)$ is sub-Clifford or regular in $N_2(X)$;

(4) X is isomorphic to C_2, L_n, or $L_n \sqcup C_2$ for some $n \in \omega$.

We shall prove the implications $(4) \Rightarrow (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$.

The implication $(4) \Rightarrow (1)$ follows from Propositions 4.7, 4.8 while $(1) \Rightarrow (2) \Rightarrow (3)$ are trivial or well-known; see [13, II.1.2].

To prove that $(3) \Rightarrow (4)$, assume that idempotents of the semigroup $\varphi(X)$ commute and $\varphi(X)$ is sub-Clifford or regular in $N_2(X)$. Then the idempotents of the semigroup X commute and thus the set $E = \{e \in X : ee = e\}$ is a commutative subsemigroup of X. By analogy with Claim 5.2 we can prove:

Claim 6.1. *The semigroup X is inverse.*

Claim 6.2. *The semilattice E is linear and finite.*

Proof. Assuming that the semilattice E is not linear, we can find two non-commuting idempotents $x, y \in E$. By (the proof of) Theorem 1.1 of [4], the filter $\mathcal{F} = \langle \{x, y\} \rangle$ is not a regular element in $\upsilon(X)$ and $\mathcal{F} \neq \langle \{x, y, xy\} \rangle = \mathcal{F} * \mathcal{F} = \mathcal{F} * \mathcal{F} * \mathcal{F}$, which is not possible if the semigroup $\varphi(X)$ is sub-Clifford or regular in $N_2(X)$. So, the semilattice E is linear. Since $\beta(X) \subset \varphi(X)$, Proposition 2.1 implies that the linear semilattice E is finite. $\qquad\qquad\square$

Since X is an inverse semigroup with finite linear semilattice E, we can apply Theorem 7.5 of [6] to derive our next claim.

Claim 6.3. *The semigroup X is inverse and Clifford.*

Since the semigroup X is inverse and Clifford, the idempotents of X commute with all elements of X; see [13, II.2.6].

Claim 6.4. *Each subgroup H in X has cardinality $|H| \leq 2$.*

Proof. Assume X contains a subgroup H of cardinality $|H| > 2$. We lose no generality assuming that the subgroup H coincides with the maximal subgroup H_e containing the idempotent e of H. Take any subset $F \subset H$ with $|H \setminus F| = 1$ and consider the filter $\mathcal{F} = \langle F \rangle$. It follows that $\mathcal{F} \neq \langle H \rangle = \mathcal{F} * \mathcal{F} = \mathcal{F} * \mathcal{F} * \mathcal{F}$, which is forbidden if the semigroup $\lambda(X)$ is sub-Clifford. Next, we show that $\mathcal{F}$ is not regular in the semigroup $N_2(X) \supset \varphi(X)$. Assuming the opposite, find an upfamily $\mathcal{A} \in N_2(X)$ such that $\mathcal{F} = \mathcal{F} * \mathcal{A} * \mathcal{F}$. Replacing $\mathcal{A}$ by the linked upfamily $e\mathcal{A}$, we can assume that $\mathcal{A} \in N_2(eX)$. Since $eX \setminus H_e$ is an ideal in eX, Corollary 3.4 implies that the $\mathcal{F}$ is a regular element of the semigroup $N_2(H)$ and hence we can assume that $\mathcal{A} \in N_2(H)$. Then $\mathcal{F} * \mathcal{A} \in N_2(H) \subset N_2(X)$. The inclusion $F \in \mathcal{F} = \mathcal{F} * \mathcal{A} * \mathcal{F}$ implies the existence of a set $B \in \mathcal{F} * \mathcal{A}$, $B \subset H$, and a family $\{F_b\}_{b \in B} \subset \mathcal{F}$ such that $\bigcup_{b \in B} b * F_b \subset F$. Replacing F_b by the smallest possible set F generating the filter $\mathcal{F}$, we can assume that $F_b = F$ for all $b \in B$. Then we get $B * F = \bigcup_{b \in B} b * F_b \subset F$ and hence $B = \{e\}$. Since $B \in \mathcal{F} * \mathcal{A}$, for the smallest set $F \in \mathcal{F}$ and each point $x \in F$ we can find a set $A_x \subset \mathcal{A}$, $A_x \subset H$, such that $\bigcup_{x \in F} x * A_x \subset \{e\}$. It follows that $A_x = \{x^{-1}\}$ and hence the family $\mathcal{A} \supset \{A_x : x \in F\}$ is not linked, which is a desired contradiction. So, $\mathcal{F}$ is not regular in $N_2(X)$ and $\mathcal{F} \neq \mathcal{F} * \mathcal{F} = \mathcal{F} * \mathcal{F} * \mathcal{F}$, which is not possible if the semigroup $\lambda(X)$ is sub-Clifford or regular in $N_2(X)$. $\qquad\square$

By analogy with Claim 5.6 we can prove:

Claim 6.5. *The semigroup $\varphi(X)$ is regular in $N_2(X)$.*

Since the semilattice E is linear, we can write it as $E = \{e_1, \ldots, e_n\}$ where $e_i e_j = e_i \neq e_j$ for all $1 \leq i < j \leq n$. For every $i \leq n$ by $H_i = H_{e_i}$ denote the maximal subgroup of X that contains the idempotent e_i. By Claim 6.4, each subgroup H_i has cardinality $|H_i| \leq 2$ and hence is commutative. Then the inverse Clifford semigroup X also is commutative.

Claim 6.6. *For any $1 \leq i < n$ the maximal subgroup H_i is trivial.*

Proof. Assume conversely that for some $i < n$ the subgroup H_i is not trivial and take any element $a \in H_i \setminus E$. Next, consider the filter $\mathcal{F} = \langle F \rangle$ generated by the doubleton $F = \{a, e_{i+1}\}$. We claim that $\mathcal{F}$ is a non-regular element in the semigroup $N_2(X)$, which will contradict Claim 6.5.

Assuming that $\mathcal{F}$ is regular in $N_2(X)$, we can find a linked upfamily $\mathcal{A} \in N_2(X)$ such that $\mathcal{F} = \mathcal{F} * \mathcal{A} * \mathcal{F}$. Replacing $\mathcal{A}$ by $e_{i+1}\mathcal{A}$, we can assume that $\mathcal{A} \in N_2(e_{i+1}X)$. For the set $F = \{a, e_{i+1}\} \in \mathcal{F}$, find a set $B \in \mathcal{F} * \mathcal{A}$ such that $B * F \subset F$. The latter inclusion implies that $B \subset H_i \cup H_{i+1}$. The inclusion $B \in \mathcal{F} * \mathcal{A}$ implies that the intersection $B \cap H_i$ is not empty and the inclusion $B * F \subset F$ implies that $B \cap H_i = \{e_i\}$ and then $\{a, e_{i+1}\} = F \supset B * F \supset \{e_i\} * \{a, e_{i+1}\} = \{a, e_i\}$, which is a desired contradiction. □

Now we are able to finish the proof of Theorem 1.2. By Claim 6.6, all maximal subgroups H_i, $i < n$, are trivial. If the group H_n is trivial, then $X = E$ is isomorphic to the linear semilattice L_n. If H_n is not trivial, then $H_n \cong C_2$ by Claim 6.4 and X is isomorphic to the semigroup $L_{n-1} \sqcup C_2$. For $n = 1$ we get $L_0 = \emptyset$ and $L_0 \sqcup C_2 = C_2$.

7.   Proof of Theorem 1.3

Given a semigroup X we need to prove the equivalence of the following statements:

(1) $N_2(X)$ is a finite commutative Clifford semigroup;

(2) $N_2(X)$ is an inverse semigroup;

(3) idempotents of $N_2(X)$ commute and $N_2(X)$ is sub-Clifford or regular;

(4) X is isomorphic to C_2 or L_n for some $n \in \omega$.

We shall prove the implications $(4) \Rightarrow (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$.

The implication $(4) \Rightarrow (1)$ follows from Propositions 4.7(1) and 4.2 while $(1) \Rightarrow (2) \Rightarrow (3)$ are trivial or well-known; see [13, II.1.2].

To prove that $(3) \Rightarrow (4)$, assume that idempotents of the semigroup $N_2(X)$ commute and $N_2(X)$ is sub-Clifford or regular. Then the idempotents of the subsemigroups $\lambda(X)$ and $\varphi(X)$ of $N_2(X)$ also commute and these semigroups are regular in the semigroup $N_2(X)$. By Theorems 1.1 and 1.2, the semigroup X is isomorphic to one of the semigroups C_2, $L_1 \sqcup C_2$ or L_n for some $n \in \omega$. It remains to prove that X cannot be isomorphic to $L_1 \sqcup C_2 = \{0, 1, -1\}$.

This follows from the fact that the semigroup $N_2(\{0, 1, -1\})$ contains two idempotents $\Delta = \{A \subset \{0, 1, -1\} : |A| \geq 2\}$ and $\mathcal{F} = \langle\{\{0, 1, -1\}\rangle$, which do not commute because $\Delta * \mathcal{F} = \mathcal{F} \neq \langle\{0, 1\}, \{0, -1\}\rangle = \mathcal{F} * \Delta$.

8. Proof of Theorem 1.4

Given a semigroup X we need to prove the equivalence of the following statements:

(1) $\upsilon(X)$ is a finite commutative Clifford semigroup;

(2) $\upsilon(X)$ is an inverse semigroup;

(3) idempotents of $\upsilon(X)$ commute and $\upsilon(X)$ is sub-Clifford or regular;

(4) X is a finite linear semilattice, isomorphic to L_n for some $n \in \omega$.

We shall prove the implications $(4) \Rightarrow (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$.

The implication $(4) \Rightarrow (1)$ follows from Proposition 4.2 while $(1) \Rightarrow (2) \Rightarrow (3)$ are trivial.

To prove that $(3) \Rightarrow (4)$, assume that the idempotents of the semigroup $\upsilon(X)$ commute and $\upsilon(X)$ is sub-Clifford or regular. Then the idempotents of the semigroup X commute and thus the set $E = \{e \in X : ee = e\}$ is a commutative subsemigroup of X. By analogy with Claims 5.2—5.4 we can prove that the semigroup X is inverse and Clifford and the semilattice E is finite and linear.

Next, we show that each subgroup H of X is trivial. Assume conversely that X contains a non-trivial subgroup H. Then the filter $\mathcal{F} = \langle H \rangle$ and the upfamily $\mathcal{U} = \{A \subset H : A \neq \emptyset\}$ are two non-commuting idempotents in the semigroup $\upsilon(H) \subset \upsilon(X)$ (because $\mathcal{F} * \mathcal{U} = \mathcal{U} \neq \mathcal{F} = \mathcal{U} * \mathcal{F}$).

Now we see that the inverse Clifford semigroup X contains no non-trivial subgroups and hence coincides with its maximal semilattice E, which is finite and linear.

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