Algebra and Discrete Mathematics RESEARCH ARTICLE
Volume 13 **(2012)**. Number 1. pp. 92 – 106

On hypercentral fyzzy groups

L. A. Kurdachenko, K. O. Grin, N. A. Turbay

To memory of D.I. Zaitsev

Abstract. In an arbitrary fuzzy group we construct the upper central series and consider some its properties. In particular, the characterization of nilpotent fuzzy group has been obtained.

Let G be a group with a multiplicative binary operation denoted by juxtaposition and identity e. We recall that a fuzzy subset $\gamma : G \to [0, 1]$ is said to be a *fuzzy group on* G (see, for example, [1, S 1.2]), if it satisfies the following conditions:

(FSG 1) $\gamma(xy) \geq \gamma(x) \wedge \gamma(y)$ for all $x, y \in G$,

(FSG 2) $\gamma(x^{-1} \geq \gamma(x))$ for every $x \in G$.

Here and everywhere we adopt the usual convention on the operator *wedge* $\wedge$(and on the operator *vee* $\vee$). If W is a subset of [0, 1], then denote by $\bigwedge W$ the greatest lower bound of W and denote by $\bigvee W$ the least upper bound of W. If $W = \{a, b\}$, then, as usual, instead of $\bigwedge W$ we will write $a \wedge b$, and instead of $\bigvee W$ we will write $a \vee b$. We assume that the least upper bound of the empty set is 0, and the greatest lower bound of the empty set is 1.

However we remark that we deliberately replace the standard expression a fuzzy subgroup of G by a fuzzy group on G in order to avoid possible

2000 Mathematics Subject Classification: *20N25, 08A72, 03E72.*

Key words and phrases: *Fuzzy group; fuzzy subgroup; upper central series, lower central series, nilpotent fuzzy group; support subgroup; characteristic function.*

misunderstanding in the sequel and to emphasize that a fuzzy group is in fact a function defined on a group G. For example, if γ, κ are the fuzzy group on G and $\gamma \subseteq \kappa$, occurs, we will say that γ *is a fuzzy subgroup of* κ and denote this by $\gamma \preccurlyeq \kappa$.

Fuzzy group theory, as well as other fuzzy algebraic structures, was introduced very soon after the beginning of fuzzy set theory. Many basic results of the theory were collected in the book [1]. From our point of view, these results are not always systematized, and the methodology and the research tools seem to be at an initial stage. The obtain results apply to the different fields, but almost everywhere they have an initial character. The development of a complete theory of fuzzy groups has not happened yet. The first natural task here appears to be the description of all fuzzy subgroups of a given fuzzy group, defined on G. The second main task is the investigation of the structure of a fuzzy group on G based on its algebraic properties. One of the important concept not only in group theory, but also in the whole algebra is the notion of nilpotency. It was introduced for fuzzy groups too(see, [1, Chapters 3.2] and the papers [2], [3], [4]). This definition was given with use of the lower central series. However there are other definitions in abstract group theory. One of them is also important, based on the consept of upper central series. In fuzzy group theory the upper central series haven't constructed. In this paper we fill up this gap. More concretely, in the paper we give construction of the upper central series for arbitrary fuzzy group defined on a group G, and give general definition of nilpotent group, which is similar to one in abstract group theory.

Let μ and ν be two fuzzy groups on G, we define the operation $\circledcirc$ on them by

$$(\mu \circledcirc \nu)(x) = \bigvee_{y,z \in G, yz=x} (\mu(y) \wedge \nu(z)).$$

Note that $(\mu \circledcirc \nu)(x) = \bigvee_{y \in G}(\mu(y) \wedge \nu(y^{-1}x)) = \bigvee_{z \in G}(\mu(xz^{-1}) \wedge \nu(z))$.

Let γ, κ be the fuzzy group on G. It is said that γ and κ *are permute*, if $\gamma \circledcirc \kappa = \kappa \circledcirc \gamma$. At this point, it is worth mentioning that in general the product of two fuzzy subgroups is not a fuzzy subgroup. Actually, the product $\gamma \circledcirc \kappa$ is a fuzzy subgroup if and only if the fuzzy subgroups γ, κ are permute (see, for example, [1, 4.3]).

Recall the following definition. If X is a set, for every subset Y of X and every $a \in [0, 1]$ we define a fuzzy subset $\chi(Y, a)$ as follows:

$$\chi(Y, a) = \begin{cases} a, x \in Y, \\ 0, x \notin Y. \end{cases}$$

Clearly $\chi(H, a)$ is a fuzzy group on G for every subgroup H of G. If $Y = \{y\}$, then instead of $\chi(\{y\}, a)$ we will write shorter $\chi(y, a)$. A fuzzy subset $\chi(y, a)$ is called a *fuzzy point* (or *fuzzy singleton*).

Proposition 1. *Let G be a group. Then the following assertion holds:*

(i) *the operation $\circledcirc$ is associative;*

(ii) *the function $\chi(e, 1)$ is an identity element of an operation $\circledcirc$, moreover, if γ is a fuzzy group on G and $\lambda \preccurlyeq \gamma$, then $\lambda \circledcirc \chi(e, \gamma(e)) = \chi(e, \gamma(e)) \circledcirc \lambda = \lambda$;*

(iii) $(\chi(y, a) \circledcirc \lambda)(x) = a \wedge \lambda(y^{-1}x)$ *for all elements $x, y \in G$;*

(iv) $(\lambda \circledcirc \chi(y, a))(x) = a \wedge \lambda(xy^{-1})$ *for all elements $x, y \in G$;*

(v) $(\chi(y, a) \circledcirc \chi(u, b))(yu) = a \wedge b$ *and* $(\chi(y, a) \circledcirc \chi(u, b))(x) = 0$*, if $x \neq yu$. In other words, $(\chi(y, a) \circledcirc \chi(u, b)) = \chi(yu, a \wedge b)$, in particular, $(\chi(y, a) \circledcirc \chi(u, a)) = \chi(yu, a)$.*

Proof. It suffices to apply the results of [1, 1.2] to prove (i) and (ii).

(iii) Let x be an arbitrary element of G. If $z \neq y$, then $\chi(y, a)(z) = 0$, so we have

$$(\chi(y, a) \circledcirc \lambda)(x) = \bigvee_{z \in G}(\chi(y, a)(z) \wedge \lambda(z^{-1}x)) = \chi(y, a)(y) \wedge \lambda(y^{-1}x) = a \wedge \lambda(y^{-1}x).$$

In particular, if $u \in X$, $b \in [0, 1]$, then $(\chi(y, a) \circledcirc \chi(u, b))(x) = a \wedge \chi(u, b)(y^{-1}x)$. Recall that $\chi(u, b)(y^{-1}x) = b$ if $y^{-1}x = u$ or $x = yu$, and $\chi(u, b)(y^{-1}x) = 0$ if $y^{-1}x \neq u$ or $x \neq yu$, thus

$$(\chi(y, a) \circledcirc \chi(u, b))(x) = \begin{cases} a \wedge b, & if x = yu, \\ 0, & if x \neq yu. \end{cases}$$

Hence we obtained (v).

The proof of (iv) is similar. □

Let G be a group, γ be a fuzzy group on G. Then *the center* $\mathfrak{z}(\gamma)$ *of* γ is an union of all fuzzy points $\chi(z, \gamma(z))$ such that $\chi(z, \gamma(z)) \circledcirc \chi(y, \gamma(y)) = \chi(y, \gamma(y)) \circledcirc \chi(z, \gamma(z))$ for every $\chi(y, \gamma(y)) \subseteq \gamma$.

A fuzzy group γ is called *abelian*, if $\mathfrak{z}(\gamma) = \gamma$.

We observed that in the theory of fuzzy groups the term "abelian" is used in different senses. We will use it in the traditional aspect.

Lemma 1. *Let G be a group, γ be a fuzzy group on G. Then the fuzzy point $\chi(x, a)$ and $\chi(y, b)$ are permutes if and only if $xy = yx$.*

Proof. Suppose that $\chi(x, a) \circledcirc \chi(y, b) = \chi(y, b) \circledcirc \chi(x, a)$. By Proposition 1 we have

$$\chi(x, a) \circledcirc \chi(y, b) = \chi(xy, a \wedge b) \text{ and } \chi(y, b) \circledcirc \chi(x, a) = \chi(yx, b \wedge a),$$

so we obtain $xy = yx$. Then

$$\chi(x, a) \circledcirc \chi(y, b) = \chi(xy, a \wedge b) = \chi(yx, b \wedge a) = \chi(y, b) \circledcirc \chi(x, a)$$

□

Corollary 1. *Let G be a group, γ be a fuzzy group on G. Then $\mathfrak{z}(\gamma)$ is an union of all fuzzy points $\chi(z, \gamma(z))$ such that $z \in \zeta(G)$. In particular, $\mathbf{Supp}(\mathfrak{z}(\gamma)) = \zeta(\mathbf{Supp}(\gamma))$.*

Proof. Suppose that $\chi(z, \gamma(z)) \in \mathfrak{z}(\gamma)$. Then $\chi(z, \gamma(z)) \circledcirc \chi(y, \gamma(y)) = \chi(y, \gamma(y)) \circledcirc \chi(z, \gamma(z))$ for every element $y \in \mathbf{Supp}(\gamma)$. By Lemma 1 it follows that $zy = yz$ for every element $y \in \mathbf{Supp}(\gamma)$. This means that $z \in \zeta(\mathbf{Supp}(\gamma))$.

Conversely, assume that $z \in \zeta(\mathbf{Supp}(\gamma))$. Then $zy = yz$ for each element $y \in \mathbf{Supp}(\gamma)$. Using again Lemma 1 we obtain that

$$\chi(z, \gamma(z)) \circledcirc \chi(y, \gamma(y)) = \chi(y, \gamma(y)) \circledcirc \chi(z, \gamma(z)),$$

which follows that $\chi(z, \gamma(z)) \in \mathfrak{z}(\gamma)$. □

Corollary 2. *Let G be a group, γ be a fuzzy group on G. Then γ is abelian if and only if* $\mathbf{Supp}(\gamma)$ *is abelian.*

Let G be a group, $x, y \in G$, $a, b \in [0, 1]$. Then a product $\chi(x^{-1}, a) \circledcirc \chi(y^{-1}, b) \circledcirc \chi(x, a) \circledcirc \chi(y, b)$ is called a *commutator of* $\chi(x, a)$ *and* $\chi(y, b)$ and will denoted by $[\chi(x, a), \chi(y, b)]$.

Lemma 2. *Let G be a group, $x, y \in G$, $a, b \in [0, 1]$. Then*

(i) $(\chi(x, a) \circledcirc \gamma \circledcirc \chi(x^{-1}, a))(y) = a \wedge \gamma(x^{-1}yx)$,

(ii) $\chi(x^{-1}, a) \circledcirc \chi(y^{-1}, b) \circledcirc \chi(x, a) \circledcirc \chi(y, b) = \chi([x, y], a \wedge b)$.

Proof. (i) Using Proposition 1 we obtain

$$(\chi(x, a) \circledcirc (\gamma \circledcirc \chi(x^{-1}, a)))(y) =$$
$$\bigvee_{u,v,z \in G, uvz = y} \chi(x, a)(u) \wedge (\gamma(v) \wedge \chi(x^{-1}, a)(z)) =$$
$$\chi(x, a)(x) \wedge \gamma(x^{-1}yx) \wedge \chi(x^{-1}, a)(x^{-1}) = a \wedge \gamma(x^{-1}yx) \wedge a = a \wedge \gamma(x^{-1}yx).$$

(ii) Using Proposition 1 we obtain

$$\chi(x^{-1}, a) \circledcirc \chi(y^{-1}, b) = \chi(x^{-1}y^{-1}, a \wedge b) \text{ and}$$
$$\chi(x, a) \circledcirc \chi(y, b) = \chi(xy, a \wedge b),$$

so that

$$\chi(x^{-1}, a) \circledcirc \chi(y^{-1}, b) \circledcirc \chi(x, a) \circledcirc \chi(y, b) = \chi(x^{-1}y^{-1}, a \wedge b) \circledcirc \chi(xy, a \wedge b) =$$
$$\chi(x^{-1}y^{-1}xy, (a \wedge b) \wedge (a \wedge b)) = \chi([x, y], a \wedge b).$$

□

Corollary 3. *Let G be a group, γ be a fuzzy group on G. Then the fuzzy point $\chi(x, a)$ and $\chi(y, b)$ of γ are permutes if and only if*

$$[\chi(x, a), \chi(y, b)] \subseteq \chi(e, \gamma(e)).$$

Proof. Suppose that $\chi(x, a) \circledcirc \chi(y, b) = \chi(y, b) \circledcirc \chi(x, a)$. By Lemma 2 we have $[\chi(x, a), \chi(y, b)] = \chi([x, y], a \wedge b)$. Using Lemma 1 we obtain that $[x, y] = e$. Since $\chi(x, a) \subseteq \gamma$. $\chi(y, b) \subseteq \gamma$, $a = \chi(x, a)(x) \leq \gamma(x)$ and similarly $b \leq \gamma(y)$. Since γ is a fuzzy group, $\gamma(x) \leq \gamma(e), \gamma(y) \leq \gamma(e)$. It follows that $a \wedge b \leq \gamma(e)$. Thus

$$[\chi(x, a), \chi(y, b)](e) = [\chi(x, a), \chi(y, b)]([x, y]) = a \wedge b \leq \gamma(e),$$
$$[\chi(x, a), \chi(y, b)](z) = 0 \text{ whenever } z \neq e.$$

It follows that $[\chi(x, a), \chi(y, b)] \subseteq \chi(e, \gamma(e))$.

Conversely, suppose that $[\chi(x, a), \chi(y, b)] \subseteq \chi(e, \gamma(e))$. Using Lemma 2 we obtain that $[x, y] = e$, which follows that $xy = yx$. An application of Lemma 1 proves that $\chi(x, a)$ and $\chi(y, b)$ are permutes. □

Let G be a group, γ, η be the fuzzy groups of G. Then a *fuzzy commutator subgroup* $[\gamma, \eta]$ is a fuzzy group generated by all commutators $[\chi(x, \gamma(x)), \chi(y, \eta(y))]$ where $x \in \mathbf{Supp}(\gamma)$, $y \in \mathbf{Supp}(\eta)$.

Proposition 2. *Let G be a group,γ, η be the fuzzy groups on G. Then the following assertion hold:*

(i) *a fuzzy commutator subgroup $[\gamma, \eta]$ is an union of all fuzzy points $\chi([x_1, y_1] \dots [x_n, y_n], \gamma(x_1) \wedge \dots \wedge \gamma(x_n) \wedge \eta(y_1) \wedge \dots \wedge \eta(y_n))$ such that $x_1, \dots, x_n \in \mathbf{Supp}(\gamma)$, $y_1, \dots, y_n \in \mathbf{Supp}(\eta)$,*

(ii) $\mathbf{Supp}([\gamma, \eta]) = [\mathbf{Supp}(\gamma), \mathbf{Supp}(\eta)]$.

Proof. (i) Every element of $[\gamma, \eta]$ has a form

$$[\chi(x_1,\gamma(x_1)),\chi(y_1,\eta(y_1))]\dots[\chi(x_n,\gamma(x_n)),\chi(y_n,\eta(y_n))].$$

Using Lemma 2 and Proposition 1 we obtain

$$[\chi(x_1,\gamma(x_1)),\chi(y_1,\eta(y_1))]\dots[\chi(x_n,\gamma(x_n)),\chi(y_n,\eta(y_n))] =$$
$$\chi([x_1,y_1],\gamma(x_1)\wedge\eta(y_1))\dots\chi([x_n,y_n],\gamma(x_n)\wedge\eta(y_n)) =$$
$$\chi([x_1,y_1]\dots[x_n,y_n],\gamma(x_1)\wedge\eta(y_1)\wedge\dots\wedge\gamma(x_n)\wedge\eta(y_n)).$$

(ii) follows immediately from (i). □

Corollary 4. *Let G be a group, γ be a fuzzy group on G. Then the center $\mathfrak{z}(\gamma)$ includes a fuzzy subgroup κ of γ if and only if $[\kappa,\gamma]\preccurlyeq\chi(e,\gamma(e))$.*

Proof. Suppose that $\kappa\preccurlyeq\mathfrak{z}(\gamma)$ .Then the fuzzy points $\chi(x,\kappa(x))$ and $\chi(y,\gamma(y))$ are permutes for every $x\in\mathbf{Supp}(\kappa)$ and $y\in\mathbf{Supp}(\gamma)$. Corollary 3 shows that in this case $[\chi(x,\kappa(x)),\chi(y,\gamma(y))]\subseteq\chi(e,\gamma(e))$. Since $[\kappa,\gamma]$ generated by the commutators $[\chi(x,\kappa(x)),\chi(y,\gamma(y))]$, we obtain an inclusion $[\kappa,\gamma]\preccurlyeq\chi(e,\gamma(e))$.

Conversely, suppose that $[\kappa,\gamma]\preccurlyeq\chi(e,\gamma(e))$. Then

$$[\chi(x,\kappa(x)),\chi(y,\gamma(y))]\subseteq\chi(e,\gamma(e))$$

for every $x\in\mathbf{Supp}(\kappa)$ and $y\in\mathbf{Supp}(\gamma)$. Corollary 3 shows that in this case fuzzy points $\chi(x,\kappa(x))$ and $\chi(y,\gamma(y))$ are permutes. Since it is valid for each $y\in\mathbf{Supp}(\gamma)$, $\chi(x,\kappa(x))\in\mathfrak{z}(\gamma)$. It follows that $\kappa\preccurlyeq\mathfrak{z}(\gamma)$. □

Corollary 5. *Let G be a group, γ be a fuzzy group on G. Then γ is abelian if and only if $[\gamma,\gamma]=\chi(e,\gamma(e))$.*

Recall that if γ,κ are the fuzzy groups on G and $\kappa\preccurlyeq\gamma$, then it is said that κ *is a normal fuzzy subgroup of* γ, if $\kappa(yxy^{-1})\geq\kappa(x)\wedge\gamma(y)$ for every elements $x,y\in G$ [1, 1.4]. We denote this fact by $\kappa\trianglelefteq\gamma$. We need a following criteria of normality.

Proposition 3. *Let G be a group and γ,κ be the fuzzy groups on G. Suppose that $\kappa\preccurlyeq\gamma$. Then κ is a normal fuzzy subgroup of γ if and only if*

$$\chi(x,\gamma(x))\circledcirc\kappa\circledcirc\chi(x^{-1},\gamma(x))\preccurlyeq\kappa$$

for every elements $x\in G$.

Proof. Suppose first that κ is a normal fuzzy subgroup of γ . Let $y\in G$ and consider a product $\chi(y,\gamma(y))\circledcirc\kappa\circledcirc\chi(y^{-1},\gamma(y))$. Let x be an arbitrary element of G. From Lemma 2 we obtain

$$(\chi(y,\gamma(y))\circledcirc\kappa\circledcirc\chi(y^{-1},\gamma(y)))(x)=\gamma(y)\wedge\kappa(y^{-1}xy).$$

Put $u = y^{-1}xy$, then $x = y(y^{-1}xy)y^{-1} = yuy^{-1}$, so that

$$(\chi(y,\gamma(y)) \circledcirc \kappa \circledcirc \chi(y^{-1},\gamma(y)))(yuy^{-1}) = \gamma(y) \wedge \kappa(u).$$

Since $\kappa(u) \wedge \gamma(y) \leq \kappa(yuy^{-1})$, we obtain

$$(\chi(y,\gamma(y)) \circledcirc \kappa \circledcirc \chi(y^{-1},\gamma(y)))(yuy^{-1}) \leq \kappa(yuy^{-1}),$$

that is

$$(\chi(y,\gamma(y)) \circledcirc \kappa \circledcirc \chi(y^{-1},\gamma(y)))(x) \leq \kappa(x).$$

Since it is valid for every element $x \in G$,

$$\chi(y,\gamma(y)) \circledcirc \kappa \circledcirc \chi(y^{-1},\gamma(y)) \preccurlyeq \kappa.$$

Conversely, suppose that $\chi(y,\gamma(y)) \circledcirc \kappa \circledcirc \chi(y^{-1},\gamma(y)) \preccurlyeq \kappa$. for each $y \in G$. Let x be an arbitrary element of G. Put $z = yxy^{-1}$, then $x = y^{-1}zy$. We have

$$(\chi(y,\gamma(y)) \circledcirc \kappa \circledcirc \chi(y^{-1},\gamma(y)))(z) \leq \kappa(z).$$

Lemma 2 shows that $(\chi(y,\gamma(y)) \circledcirc \kappa \circledcirc \chi(y^{-1},\gamma(y)))(z) = \gamma(y) \wedge \kappa(y^{-1}zy)$. Then $\gamma(y) \wedge \kappa(y^{-1}zy) \leq \kappa(z)$, that is $\gamma(y) \wedge \kappa(x) \leq \kappa(yxy^{-1})$. □

Using a concept of fuzzy commutator subgroups, we can obtain a following criteria of normality for fuzzy subgroups.

Proposition 4. *Let G be a group. Let ν, κ be the fuzzy groups on G, and suppose that $\nu \preccurlyeq \kappa$. Then ν is normal in κ if and only if $[\nu, \kappa] \preccurlyeq \nu$.*

Proof. Suppose that ν is a normal fuzzy subgroup of κ. Let $x \in \mathbf{Supp}(\nu)$, $y \in \mathbf{Supp}(\kappa)$, then by Proposition 3 we obtain an inclusion

$$\chi(y^{-1},\kappa(y)) \circledcirc \chi(x,\nu(x)) \circledcirc \chi(y,\kappa(y)) \subseteq \nu.$$

Lemma 2 implies that

$$[\chi(x,\nu(x)),\chi(y,\kappa(y))] = \\ \chi(x^{-1},\nu(x)) \circledcirc \chi(y^{-1},\kappa(y)) \circledcirc \chi(x,\nu(x)) \circledcirc \chi(y,\kappa(y)) \subseteq \nu$$

Since it is valid for each pair x, y, where $x \in \mathbf{Supp}(\nu)$, $y \in \mathbf{Supp}(\kappa)$, then $[\nu, \kappa] \preccurlyeq \nu$.

Conversely, suppose that $[\nu, \kappa] \preccurlyeq \nu$. Let $x \in \mathbf{Supp}(\nu)$, $y \in \mathbf{Supp}(\kappa)$, then

$$[\chi(x,\nu(x)),\chi(y,\kappa(y))] = \\ \chi(x^{-1},\nu(x)) \circledcirc \chi(y^{-1},\kappa(y)) \circledcirc \chi(x,\nu(x)) \circledcirc \chi(y,\kappa(y)) \subseteq \nu.$$

Since ν is a fuzzy group, $\chi(x,\nu(x)) \circledcirc [\chi(x,\nu(x)),\chi(y,\kappa(y))] \subseteq \nu$. But

$$\chi(x,\nu(x))\circledcirc[\chi(x,\nu(x)),\chi(y,\kappa(y))]=$$
$$\chi(x,\nu(x))\circledcirc(\chi(x^{-1},\nu(x))\circledcirc\chi(y^{-1},\kappa(y))\circledcirc\chi(x,\nu(x))\circledcirc\chi(y,\kappa(y)))=$$
$$(\chi(x,\nu(x))\circledcirc\chi(x^{-1},\nu(x)))\circledcirc(\chi(y^{-1},\kappa(y))\circledcirc\chi(x,\nu(x))\circledcirc\chi(y,\kappa(y)))=$$
$$\chi(xx^{-1},\nu(x)\wedge\nu(x))\circledcirc(\chi(y^{-1},\kappa(y))\circledcirc\chi(x,\nu(x))\circledcirc\chi(y,\kappa(y)))=$$
$$\chi(e,\nu(x))\circledcirc(\chi(y^{-1},\kappa(y))\circledcirc\chi(x,\nu(x))\circledcirc\chi(y,\kappa(y))).$$

Proposition 1 shows that

$$\chi(y^{-1},\kappa(y))\circledcirc\chi(x,\nu(x))\circledcirc\chi(y,\kappa(y))=\chi(y^{-1}xy,\kappa(y)\wedge\nu(x)\wedge\kappa(y))=$$
$$\chi(y^{-1}xy,\kappa(y)\wedge\nu(x)).$$

Therefore

$$\chi(e,\nu(x))\circledcirc(\chi(y^{-1},\kappa(y))\circledcirc\chi(x,\nu(x))\circledcirc\chi(y,\kappa(y)))=$$
$$\chi(e,\nu(x))\circledcirc\chi(y^{-1}xy,\kappa(y)\wedge\nu(x))=\chi(e(y^{-1}xy),\nu(x)\wedge\kappa(y)\wedge\nu(x))=$$
$$\chi(y^{-1}xy,\kappa(y)\wedge\nu(x)).$$

In other words,

$$\chi(y^{-1},\kappa(y))\circledcirc\chi(x,\nu(x))\circledcirc\chi(y,\kappa(y))=\chi(e,\nu(x))\circledcirc(\chi(y^{-1},\kappa(y))\circledcirc$$
$$\chi(x,\nu(x))\circledcirc\chi(y,\kappa(y)))=\chi(x,\nu(x))\circledcirc[\chi(x,\nu(x)),\chi(y,\kappa(y))]\subseteq\nu.$$

It follows that ν is a normal fuzzy subgroup of κ. □

Corollary 6. *Let G be a group, γ be a fuzzy group on G. Then every fuzzy subgroup of the center $\mathfrak{z}(\gamma)$ is normal in γ.*

Proof. Suppose that κ is a fuzzy subgroup of $\mathfrak{z}(\gamma)$. Let $x\in\mathbf{Supp}(\kappa)$ and $y\in\mathbf{Supp}(\gamma)$. Then Lemma 2 shows that $[\chi(x,\kappa(x)),\chi(y,\gamma(y))]=\chi([x,y],\kappa(x)\wedge\gamma(y))$. An inclusion $\kappa\preccurlyeq\mathfrak{z}(\gamma)$ together with Lemma 1 shows that $[x,y]=e$. We remark that $\kappa(x)\wedge\gamma(y)\le\kappa(x)\le\kappa(e)$. Thus we have

$$[\chi(x,\kappa(x)),\chi(y,\gamma(y))]([x,y])=\kappa(x)\wedge\gamma(y)\le\kappa(x)\le\kappa(e)=\kappa([x,y]);$$
$$[\chi(x,\kappa(x)),\chi(y,\gamma(y))](z)=0\le\kappa(z)\text{ whenever }z\ne[z,y].$$

In other words,

$$[\chi(x,\kappa(x)),\chi(y,\gamma(y))](u)\le\kappa(u)\text{ or }[\chi(x,\kappa(x)),\chi(y,\gamma(y))]\subseteq\kappa.$$

Since it is valid for every $x\in\mathbf{Supp}(\kappa)$, $y\in\mathbf{Supp}(\gamma)$, $[\gamma,\kappa]\preccurlyeq\kappa$. Proposition 4 proves that κ is normal in γ. □

Now we will construct an upper central series of a fuzzy group γ. Put $\mathfrak{z}_1(\gamma)=\mathfrak{z}(\gamma)$. Without loss of generality me may assume that $\mathbf{Supp}(\gamma)=G$. Then Corollary 1 shows that $\mathbf{Supp}(\mathfrak{z}(\gamma))=\zeta(G)=Z_1$. Consider a factor – group G/Z_1 and let $\phi:G\to G/Z_1$ be a natural epimorphism. We can extend ϕ to the mapping Φ from the set $\mathcal{F}(G)$ of all fuzzy groups on G

to the set $\mathcal{F}(G/Z_1)$ of all fuzzy groups on G/Z_1 (see, for example, [1, 1.1.11]). Define the function $\gamma^{\curlywedge} : G/Z_1 \to [0,1]$ by the following rule: $\gamma^{\curlywedge}(gZ_1) = \bigvee_{z\in\zeta(G)} \gamma(gz)$. In other words, $\gamma^{\curlywedge} = \Phi(\gamma)$. Then $\gamma^{\curlywedge}$ is a fuzzy group on G/Z_1 (see, for example, [1, Theorem 1.3.13]). Put $\mathfrak{z}(\gamma^{\curlywedge}) = \lambda$ and define the function $\mathfrak{z}_2(\gamma) : G \to [0,1]$ in a following way. Consider a preimage $\lambda_\curlyvee$ of λ, that is the function $\lambda_\curlyvee : G \to [0,1]$, defined by the rule $\lambda_\curlyvee(x) = \lambda(xZ_1)$, $x \in G$. Put now $\mathfrak{z}_2(\gamma) = \lambda_\curlyvee \cap \gamma$.

Proposition 5. *Let G be a group, γ be a fuzzy group on G. Then $\mathfrak{z}_2(\gamma)$ is a normal fuzzy subgroup of γ. Moreover, $\Phi(\mathfrak{z}_2(\gamma)) \subseteq \mathfrak{z}(\gamma^{\curlywedge})$.*

Proof. Let x, y are the arbitrary elements of G, then $\lambda_\curlyvee(xy) = \lambda(xyZ_1)$. As we saw above, λ is a fuzzy group on G/Z_1, therefore

$$\lambda(xyZ_1) = \lambda(xZ_1yZ_1) \geq \lambda(xZ_1) \wedge \lambda(yZ_1) = \lambda_\curlyvee(x) \wedge \lambda_\curlyvee(y),$$

so that $\lambda_\curlyvee(xy) \geq \lambda_\curlyvee(x) \wedge \lambda_\curlyvee(y)$. Similarly $\lambda_\curlyvee(x^{-1}) = \lambda(x^{-1}Z_1) = \lambda((xZ_1)^{-1}) = \lambda(xZ_1) = \lambda_\curlyvee(x)$, which shows that $\lambda_\curlyvee$ is a fuzzy group on G. Since the intersection of fuzzy groups is fuzzy group (see, for example, [1, Theorem 1.2.13]), $\mathfrak{z}_2(\gamma)$ is a fuzzy group on G. By its definition, $\mathfrak{z}_2(\gamma) \preccurlyeq \gamma$.

Let x, y again be the arbitrary elements of G, consider $\mathfrak{z}_2(\gamma)(yxy^{-1})$. We have $\mathfrak{z}_2(\gamma)(yxy^{-1}) = (\lambda_\curlyvee \cap \gamma)(yxy^{-1}) = \lambda_\curlyvee(yxy^{-1}) \wedge \gamma(yxy^{-1})$. Furthermore,

$$\lambda_\curlyvee(yxy^{-1}) = \lambda(yxy^{-1}Z_1) = \lambda(yZ_1)\lambda(xZ_1)\lambda(y^{-1}Z_1) = \lambda(yZ_1)\lambda(xZ_1)\lambda((yZ_1)^{-1}).$$

By Corollary 6 λ is a normal fuzzy subgroup of $\gamma^{\curlywedge}$. It follows that

$$\lambda(yZ_1)\lambda(xZ_1)\lambda((yZ_1)^{-1}) \geq \lambda(xZ_1) \wedge \gamma^{\curlywedge}(yZ_1).$$

Since $\gamma^{\curlywedge}(yZ_1) = \bigvee_{z\in\zeta(G)} \gamma(yz)$, $\gamma^{\curlywedge}(yZ_1) \geq \gamma(y)$, so that

$$\lambda_\curlyvee(yxy^{-1)} \geq \lambda_\curlyvee(x) \wedge \gamma(y).$$

In turn it follows that

$$\mathfrak{z}_2(\gamma)(yxy^{-1}) = \lambda_\curlyvee(yxy^{-1}) \wedge \gamma(yxy^{-1}) \geq \lambda_\curlyvee(x) \wedge \gamma(y) \wedge \gamma(x) = (\lambda_\curlyvee(x) \wedge \gamma(x)) \wedge \gamma(y) = (\lambda_\curlyvee \cap \gamma)(x) \wedge \gamma(y) = \mathfrak{z}_2(\gamma)(x) \wedge \gamma(y).$$

Hence $\mathfrak{z}_2(\gamma)$ is a normal fuzzy subgroup of γ.

Furthermore,

$$\Phi(\mathfrak{z}_2(\gamma)) = \Phi(\lambda_\curlyvee \cap \gamma) \subseteq \Phi(\lambda_\curlyvee) \subseteq \Phi(\Phi^{-1}(\lambda)).$$

Using Theorem 1.2.12 of a book [1], we obtain $\Phi(\Phi^{-1}(\lambda) \subseteq \lambda$, which shows that $\Phi(\mathfrak{z}_2(\gamma)) \subseteq \lambda$. □

Lemma 3. *Let G be a group, γ be a fuzzy group on G. Let K be a normal subgroup of G, $\phi : G \to G/K$ a natural epimorphism and Φ a natural extension of ϕ to the mapping from the set $\mathcal{F}(G)$ of all fuzzy groups on G to the set $\mathcal{F}(G/K)$ of all fuzzy groups on G/K. If $u \in G$, $a \in [0,1]$, then $\Phi(\chi(u,a)) = \chi(uK,a)$.*

Proof. By definition of a mapping Φ we have

$$\Phi(\chi(u,a))(gK) = \bigvee_{z\in K} \chi(u,a)(gz).$$

We recall that $\chi(u,a)(gz) = 0$ if $gz \neq u$. If $gz = u$, then $gK = uK$ and

$$\Phi(\chi(u,a))(gK) = \chi(u,a)(u) = a.$$

On the other hand,

$$\chi(uK,a)(gK) = 0 \text{ if } gK \neq uK \text{ and } \chi(uK,a)(uK) = a.$$

This shows that $\Phi(\chi(u,a)) = \chi(uK,a)$. □

Proposition 6. *Let G be a group, γ be a fuzzy group on G. Then*

$$\mathbf{Supp}(\mathfrak{z}_2(\gamma)) = \zeta_2(\mathbf{Supp}(\gamma))$$

the second hypercenter of $\mathbf{Supp}(\gamma)$.

Proof. Without loss of generality we can suppose that $\mathbf{Supp}(\gamma) = G$. Let $g \in G$, $u \in \mathbf{Supp}(\mathfrak{z}_2(\gamma))$, and consider now a commutator

$$[\chi(u,\mathfrak{z}_2(\gamma)(u)), \chi(g,\gamma(g))].$$

By Lemma 2 $[\chi(u,\mathfrak{z}_2(\gamma)(u)), \chi(g,\gamma(g))] = \chi([u,g],\mathfrak{z}_2(\gamma)(u) \wedge \gamma(g))$. By Lemma 3

$$\Phi(\chi([u,g],\mathfrak{z}_2(\gamma)(u) \wedge \gamma(g))) = \chi([u,g]Z_1,\mathfrak{z}_2(\gamma)(u) \wedge \gamma(g)).$$

Using again Lemma 2, we obtain

$$\chi([u,g]Z_1,\mathfrak{z}_2(\gamma)(u) \wedge \gamma(g)) = [\chi(uZ_1,\mathfrak{z}_2(\gamma)(u)), \chi(gZ_1,\gamma(g))].$$

Since $\gamma^{\curlywedge}(gZ_1) = \bigvee_{z\in\zeta(G)} \gamma(gz)$ and $e \in Z_1$, then $\gamma^{\curlywedge}(gZ_1) \geq \gamma(g)$, so that $\chi(gZ_1,\gamma(g)) \subseteq \chi(gZ_1,\gamma^{\curlywedge}(gZ_1))$. An inclusion $\Phi(\mathfrak{z}_2(\gamma)) \subseteq \lambda$ shows that $[\chi(uZ_1,\mathfrak{z}_2(\gamma)(u)), \chi(gZ_1,\gamma(g))] \subseteq [\lambda,\gamma^{\curlywedge}]$. The choice of λ together with Corollary 4 shows that

$$[\lambda,\gamma^{\curlywedge}] \preccurlyeq \chi(eZ_1,\gamma^{\curlywedge}(eZ_1)).$$

In particular, $\chi([u,g]Z_1, \mathfrak{z}_2(\gamma)(u) \wedge \gamma(g)) \subseteq \chi(eZ_1, \gamma^{\curlywedge}(eZ_1))$, which follows that $Z_1 = [u,g]Z_1 = [uZ_1, gZ_1]$. In turn, it follows that $u \in \zeta_2(G)$, the second hypercenter of G.

Conversely, let $u \in \zeta_2(G)$. Then $uZ_1 \in \zeta(G/Z_1)$. Since $G = \mathbf{Supp}(\gamma)$, $\gamma(u) \neq 0$. Corollary 1 shows that fuzzy point $\chi(uZ_1, \gamma^{\curlywedge}(uZ_1))$ lies in the center λ of $\gamma^{\curlywedge}$. It follows that $\lambda(uZ_1) = \gamma^{\curlywedge}(uZ_1)$. An equation $\gamma^{\curlywedge}(uZ_1) = \bigvee_{z \in \zeta(G)} \gamma(uz)$ shows that $\gamma^{\curlywedge}(yZ_1) \geq \gamma(y)$. We have now

$$\mathfrak{z}_2(\gamma)(u) = (\lambda_\gamma \cap \gamma)(u) = \lambda_\gamma(u) \wedge \gamma(u) = \lambda(uZ_1) \wedge \gamma(u) = \\ \gamma^{\curlywedge}(uZ_1) \wedge \gamma(u) = \gamma(u),$$

which shows that $\chi(u, \gamma(u)) \subseteq \mathfrak{z}_2(\gamma)$. It follows that $\mathbf{Supp}(\mathfrak{z}_2(\gamma)) = \zeta_2(\mathbf{Supp}(\gamma))$. □

Proposition 7. *Let G be a group, γ be a fuzzy group on G. Then $[\mathfrak{z}_2(\gamma), \gamma] \preccurlyeq \mathfrak{z}(\gamma)$.*

Proof. Let $g \in G$, $u \in \mathbf{Supp}(\mathfrak{z}_2(\gamma))$, and consider now a commutator

$$[\chi(u, \mathfrak{z}_2(\gamma)(u)), \chi(g, \gamma(g))].$$

By Lemma 2 $[\chi(u, \mathfrak{z}_2(\gamma)(u)), \chi(g, \gamma(g))] = \chi([u,g], \mathfrak{z}_2(\gamma)(u)) \wedge \gamma(g))$. As in Proposition 6 we can prove an inclusion $\chi([u,g]Z_1, \mathfrak{z}_2(\gamma)(u) \wedge \gamma(g)) \subseteq \chi(eZ_1, \gamma^{\curlywedge}(eZ_1))$. Lemma 3 shows that $\Phi(\chi([u,g], \mathfrak{z}_2(\gamma)(u) \wedge \gamma(g))) = \chi([u,g]Z_1, \mathfrak{z}_2(\gamma)(u) \wedge \gamma(g))$. This equation shows that the preimage κ of $\chi(eZ_1, \gamma^{\curlywedge}(eZ_1))$ includes $[\mathfrak{z}_2(\gamma), \gamma]$. Hence a next our step must be the consideration of a fuzzy group κ . We have $\gamma^{\curlywedge}(eZ_1) = \bigvee_{z \in \zeta(G)} \gamma(ez)$. Since γ is a fuzzy group on G, $\gamma(e) \geq \gamma(z)$ for each $z \in Z_1$. It follows that $\bigvee_{z \in \zeta(G)} \gamma(ez) = \gamma(ee) = \gamma(e)$. So that $\gamma^{\curlywedge}(eZ_1) = \gamma(e)$. Let g be an arbitrary element of G, then κ contains a fuzzy point $\chi(g, \kappa(g))$. It follows that $\Phi(\chi(g, \kappa(g))) \subseteq \chi(eZ_1, \gamma(e))$. On the other hand, Lemma 3 shows that $\Phi(\chi(g, \kappa(g))) = \chi(gZ_1, \kappa(g))$, and we obtain that $\chi(gZ_1, \kappa(g)) \subseteq \chi(eZ_1, \gamma(e))$. It follows that $gZ_1 = Z_1$, which means that $g \in Z_1$. Corollary 1 shows that $\chi(g, \kappa(g)) \subseteq \mathfrak{z}_1(\gamma)$. Since $\kappa = \cup_{g \in G} \chi(g, \kappa(g))$, we obtain that $\kappa \subseteq \mathfrak{z}_1(\gamma)$. By above proved, this inclusion implies an inclusion $[\mathfrak{z}_2(\gamma), \gamma] \preccurlyeq \mathfrak{z}(\gamma)$. □

Now we can continue to construct an upper central series of a fuzzy group γ. Using for this transfinite induction. Without loss of generality me may assume that $\mathbf{Supp}(\gamma) = G$. We have already constructed the terms $\mathfrak{z}_1(\gamma) = \mathfrak{z}(\gamma)$ and $\mathfrak{z}_2(\gamma)$. Suppose that we have already constructed the terms $\mathfrak{z}_\beta(\gamma)$ for all ordinals $\beta < \alpha$. If α is a limit ordinal, then we put $\mathfrak{z}_\alpha(\gamma) = \cup_{\beta < \alpha} \mathfrak{z}_\beta(\gamma)$. Suppose now that α is a not limit ordinal, that is

$\alpha - 1 = \eta$ exists. Let $K = \zeta_\eta(G)$, the η^{th} term of an upper central series of G. Consider a factor–group G/K and let $\phi : G \to G/K$ be a natural epimorphism. We can extend ϕ to the mapping Φ from the set $\mathcal{F}(G)$ of all fuzzy groups on G to the set $\mathcal{F}(G/K)$ of all fuzzy groups on G/K (see, for example, [1, 1.1.11]). Define the function $\gamma^\curlywedge : G/K \to [0,1]$ by the following rule: $\gamma^\curlywedge(gK) = \bigvee_{z\in K} \gamma(gz)$. In other words, $\gamma^\curlywedge = \Phi(\gamma)$. Then $\gamma^\curlywedge$ is a fuzzy group on G/K (see, for example, [1, Theorem 1.3.13]). Put $\mathfrak{z}(\gamma^\curlywedge) = \lambda$ and define the function $\mathfrak{z}_\alpha(\gamma) : G \to [0,1]$ in a following way. Consider a preimage $\lambda_\curlyvee$ of λ, that is the function $\lambda_\curlyvee : G \to [0,1]$, defined by the rule $\lambda_\curlyvee(x) = \lambda(xK)$ for every $x \in G$. Put now $\mathfrak{z}_\alpha(\gamma) = \lambda_\curlyvee \cap \gamma$.

Thus, for every ordinal α we constructed the α^{th} term $\mathfrak{z}_\alpha(\gamma)$ of an upper central series of γ. The building of an upper central series of γ come to an end on some ordinal σ. In other words, this means that if $L = \zeta_\sigma(G)$ and $\gamma^\curlywedge(gL) = \bigvee_{z\in L} \gamma(gz)$, then the center of $\gamma^\curlywedge$ is $\chi(e, \gamma^\curlywedge(e))$. Then $\mathfrak{z}_\sigma(\gamma)$ is called the *upper hypercenter of* γ and will denoted by $\mathfrak{z}_\infty(\gamma)$.

A fuzzy group γ is called *hypercentral*, if $\gamma = \mathfrak{z}_\infty(\gamma)$. Let

$$\chi(e, \gamma(e)) = \mathfrak{z}_0(\gamma) \preccurlyeq \mathfrak{z}(\gamma) = \mathfrak{z}_1(\gamma) \preccurlyeq \ldots \preccurlyeq \mathfrak{z}_\alpha(\gamma) \preccurlyeq \mathfrak{z}_{\alpha+1}(\gamma) \preccurlyeq \ldots \preccurlyeq \mathfrak{z}_\sigma(\gamma)$$

be the upper central series of γ. Using the same arguments that in the proofs of Propositions 5, 6, 7, we can obtain that every term of the upper central series is a normal fuzzy subgroup of γ. $\mathbf{Supp}(\mathfrak{z}_\alpha(\gamma)) = \zeta_\alpha(\mathbf{Supp}(\gamma))$, $[\mathfrak{z}_{\alpha+1}(\gamma), \gamma] \preccurlyeq \mathfrak{z}_\alpha(\gamma)$ for each $\alpha < \sigma$.

We can obtain a following characterization of hypercentral fuzzy group.

Theorem 1. *Let G be a group, γ be a fuzzy group on G. Then γ is hypercentral if and only if* $\mathbf{Supp}(\gamma)$ *is hypercentral.*

Proof. Again without loss of generality we can assume that

$$G = \mathbf{Supp}(\gamma).$$

Suppose that γ is hypercentral and let

$$\chi(e, \gamma(e)) = \mathfrak{z}_0(\gamma) \preccurlyeq \mathfrak{z}_1(\gamma) \preccurlyeq \ldots \preccurlyeq \mathfrak{z}_\alpha(\gamma) \preccurlyeq \mathfrak{z}_{\alpha+1}(\gamma) \preccurlyeq \ldots \preccurlyeq \mathfrak{z}_\sigma(\gamma) = \gamma$$

be the upper central series of γ. As we remarked above, $\mathbf{Supp}(\mathfrak{z}_\alpha(\gamma)) = \zeta_\alpha(\mathbf{Supp}(\gamma))$ for each $\alpha < \sigma$. In particular,

$$G = \mathbf{Supp}(\gamma) = \mathbf{Supp}(\mathfrak{z}_\sigma(\gamma)) = \zeta_\sigma(\mathbf{Supp}(\gamma)) = \zeta_\sigma(G),$$

i.e. G is hypercentral.

Conversely, assume that G is hypercentral. Then in every factor–group of G its center is non–identity. In particular, $Z_1 = \zeta(G) \neq < e >$. By

Corollary 1 $Z_1 = \mathbf{Supp}(\mathfrak{z}(\gamma))$, so that $\mathfrak{z}_1(\gamma) = \mathfrak{z}(\gamma) \neq \chi(e, \gamma(e))$. If G is abelian, then $\gamma = \mathfrak{z}(\gamma)$. In other words, γ is abelian, and all is proved. Suppose that we have already constructed the terms $\mathfrak{z}_\beta(\gamma)$ of the upper central series for all ordinals $\beta < \alpha$. If α is a limit ordinal, then $\mathfrak{z}_\alpha(\gamma) = \cup_{\beta<\alpha}\mathfrak{z}_\beta(\gamma)$. Suppose now that α is a not limit ordinal, that is $\alpha - 1 = \eta$ exists. Let $K = \zeta_\eta(G)$. Consider a factor–group G/K and let $\phi : G \to G/K$ be a natural epimorphism. We can extend ϕ to the mapping Φ from the set $\mathcal{F}(G)$ of all fuzzy groups on G to the set $\mathcal{F}(G/K)$ of all fuzzy groups on G/K (see, for example, [1, Theorem 1.1.11]). Define the function $\gamma^\curlywedge : G/K \to [0, 1]$ by the following rule: $\gamma^\curlywedge(gK) = \bigvee_{z\in K} \gamma(gz)$. In other words, $\gamma^\curlywedge = \Phi(\gamma)$. Then $\gamma^\curlywedge$ is a fuzzy group on G/K (see, for example, [1, Theorem 1.3.13]). Since G/K is a non–identity hypercentral group, then its center $\zeta(G/K) = D/K$ is non–identity. Application of Corollary 1 shows that $D/K = \mathbf{Supp}(\mathfrak{z}(\gamma^\curlywedge))$, so that $\mathfrak{z}(\gamma^\curlywedge) = \lambda \neq \chi(e, \gamma^\curlywedge(e))$, and using the above arguments we can construct the term $\mathfrak{z}_\alpha(\gamma)$ of the upper central series. If G/K is abelian, then $\gamma = \mathfrak{z}_\alpha(\gamma)$. If not, we can continue the building of an upper central series of γ. □

The following concept is dual to a concept of an upper central series. Let γ be a fuzzy group on a group G. We define the lower central series of γ by the following rule: put $\mathfrak{g}_1(\gamma) = \gamma$, $\mathfrak{g}_2(\gamma) = [\gamma, \gamma]$. Assume that we have already construct the terms $\mathfrak{g}_\beta(\gamma)$ for all ordinals $\beta < \alpha$. If α is a limit ordinal, then we put $\mathfrak{g}_\alpha(\gamma) = \cup_{\beta<\alpha}\mathfrak{g}_\beta(\gamma)$. Suppose now that α is a not limit ordinal, that is $\alpha - 1$ exists. Then put $\mathfrak{g}_\alpha(\gamma) = [\mathfrak{g}_{\alpha-1}(\gamma), \gamma]$. Thus, for every ordinal α we constructed the α^{th} term $\mathfrak{g}_\alpha(\gamma)$ of a lower central series of γ. The building of an upper central series of γ come to an end on some ordinal σ. In other words, this means that $\mathfrak{g}_\sigma(\gamma) = [\mathfrak{g}_\sigma(\gamma), \gamma]$. Then $\mathfrak{g}_\sigma(\gamma)$ is called the *lower hypocenter of* γ and will denoted by $\mathfrak{g}_\infty(\gamma)$.

A fuzzy group γ is called *hypocentral*, if $\mathfrak{g}_\infty(\gamma) \preccurlyeq \chi(e, \gamma(e))$.

As and for abstract groups, we can define for fuzzy groups the general term of central factor and central series.

Let γ be a fuzzy group on a group G and κ, ν be the normal fuzzy subgroups of γ such that $\kappa \preccurlyeq \nu$. We say that κ, ν *form a central link*, if $[\nu, \gamma] \preccurlyeq \kappa$.

As we saw above, the terms $\mathfrak{z}_\alpha(\gamma)$, $\mathfrak{z}_{\alpha+1}(\gamma)$ of the upper central series and the terms $\mathfrak{g}_{\alpha+1}(\gamma)$, $\mathfrak{g}_\alpha(\gamma)$ of the lower central series form the central links.

A finite series

$$\chi(e, \gamma(e)) = \kappa_0 \preccurlyeq \kappa_1 \preccurlyeq \ldots \preccurlyeq \kappa_{n-1} \preccurlyeq \kappa_n = \gamma$$

of normal fuzzy subgroups of γ is called *central*, if the fuzzy subgroups κ_{j-1},κ_j form a central link for every j, $1 \leq j \leq n$.

Theorem 2. *Let G be a group, γ be a fuzzy group on G. If γ has a finite central series*

$$\chi(e, \gamma(e)) = \kappa_0 \preccurlyeq \kappa_1 \preccurlyeq \ldots \preccurlyeq \kappa_{n-1} \preccurlyeq \kappa_n = \gamma,$$

then $\kappa_j \preccurlyeq \mathfrak{z}_j(\gamma)$, $1 \leq j \leq n$, and $\mathfrak{g}_m(\gamma) \preccurlyeq \kappa_{n-m+1}$, $1 \leq m \leq n$.

Proof. We have $\mathfrak{g}_1(\gamma) = \gamma = \kappa_n$. Suppose that we have already proved an inclusion $\mathfrak{g}_m(\gamma) \preccurlyeq \kappa_{n-m+1}$ for some m. Consider a link κ_{n-m}, κ_{n-m+1}. Since this link is central, $[\kappa_{n-m+1}, \gamma] \preccurlyeq \kappa_{n-m}$. Then $\mathfrak{g}_{m+1}(\gamma) = [\mathfrak{g}_m(\gamma), \gamma] \preccurlyeq [\kappa_{n-m+1}, \gamma] \preccurlyeq \kappa_{n-m}$.

Since $[\kappa_1, \gamma] \preccurlyeq \chi(e, \gamma(e))$, Corollary 4 yields that $\kappa_1 \preccurlyeq \mathfrak{z}_1(\gamma)$. Assume that we have already proved an inclusion $\kappa_j \preccurlyeq \mathfrak{z}_j(\gamma)$. Let $\theta = \kappa_{j+1}$. We have $[\theta, \gamma] = [\kappa_{j+1}, \gamma] \preccurlyeq \kappa_j \preccurlyeq \mathfrak{z}_j(\gamma)$. Let $x \in \mathbf{Supp}(\theta)$, y be an arbitrary element of G, then $[\chi(x, \theta(x)), \chi(y, \gamma(y))] \subseteq \mathfrak{z}_j(\gamma)$. Lemma 2 shows that $[\chi(x, \theta(x)), \chi(y, \gamma(y))] = \chi([x, y], \theta(x) \wedge \gamma(y))$. In particular, $[x, y] \in \mathbf{Supp}(\mathfrak{z}_j(\gamma)) = \zeta_j(\mathbf{Supp}(\gamma))$. Put $K = \zeta_j(\mathbf{Supp}(\gamma))$. Consider a factor–group G/K and let $\phi : G \to G/K$ be a natural epimorphism. We can extend ϕ to the mapping Φ from the set $\mathcal{F}(G)$ of all fuzzy groups on G to the set $\mathcal{F}(G/K)$ of all fuzzy groups on G/K (see, for example, [1, Theorem 1.1.11]). As above consider a function $\gamma^{\curlywedge} = \Phi(\gamma)$. Then $\gamma^{\curlywedge}$ is a fuzzy group on G/K (see, for example, [1, Theorem 1.3.13]). Lemma 4 shows that $\Phi(\chi([x, y], \theta(x) \wedge \gamma(y))) = \chi([x, y]K, \theta(x) \wedge \gamma(y))$. We noted above that $[x, y] \in K$, that is $\chi([x, y]K, \theta(x) \wedge \gamma(y)) \subseteq \chi(e, \gamma^{\curlywedge}(e))$. On the other hand, an application of Lemma 2 gives an equation

$$\chi([x, y]K, \theta(x) \wedge \gamma(y)) = \chi([xK, yK], \theta(x) \wedge \gamma(y)) = \\ [\chi(xK, \theta(x)), \chi(yK, \gamma(y))],$$

so that $[\chi(xK, \theta(x)), \chi(yK, \gamma(y))] \subseteq \chi(e, \gamma^{\curlywedge}(e))$. By Corollary 2

$$\chi(xK, \theta(x)) \subseteq \mathfrak{z}(\gamma^{\curlywedge}) = \lambda.$$

Then the preimage $\chi(xK, \theta(x))$ of $\chi(xK, \theta(x))$ lies in $\lambda_{\curlyvee} \cap \gamma = \mathfrak{z}_{j+1}(\gamma)$, where $\lambda_{\curlyvee}$ is a preimage of $\mathfrak{z}(\gamma^{\curlywedge})$. Since an inclusion $\chi(x, \theta(x)) \subseteq \mathfrak{z}_{j+1}(\gamma)$ is true for each $x \in \mathbf{Supp}(\theta)$, $\kappa_{j+1} = \theta = \cup_{x \in \mathbf{Supp}(\theta)} \chi(x, \theta(x)) \preccurlyeq \mathfrak{z}_{j+1}(\gamma)$. □

Let G be a group and γ a fuzzy group on G. Then γ is called *nilpotent*, if $\mathfrak{g}_m(\gamma) \preccurlyeq \chi(e, \gamma(e))$ for some positive integer m. If γ is nilpotent, then we say that the *nilpotency class of* γ *is* c (and denote this by $ncl(\gamma) = c$), if c is the least positive integer, having a property $\mathfrak{g}_c(\gamma) \preccurlyeq \chi(e, \gamma(e))$.

Now we can give a characterization of nilpotent fuzzy group, which used the concepts of upper central series and arbitrary central series.

Corollary 7. *Let G be a group, γ be a fuzzy group on G. Then the following assertions are equivalent:*

(i) *γ is nilpotent and $ncl(\gamma) = c$;*

(ii) *$\mathfrak{z}_c(\gamma) = \gamma$;*

(iii) *γ has a finite central series, moreover, if*

$$\chi(e, \gamma(e)) = \kappa_0 \preccurlyeq \kappa_1 \preccurlyeq \ldots \preccurlyeq \kappa_{n-1} \preccurlyeq \kappa_n = \gamma$$

is an arbitrary finite central series of γ, then $c \leq n$.

References

[1] J.N. Mordeson, K.R. Bhutani, A. Rosenfeld, *Fuzzy Group Theory*, Springer: Berlin, 2005.

[2] K.C. Gupta, B.K. Sarma, *Nilpotent fuzzy groups*,Fuzzy Sets and Systems, N.**101**, 1999, pp.167-176.

[3] J.G. Kim, *Commutative fuzzy sets and nilpotent fuzzy groups*, Information Sciences, N.**83**, 1995, pp.161-174.

[4] M. Asaad, S. Abou - Zaid, *Fuzzy subgroups of nilpotent groups*, Fuzzy Sets and Systems, N.**24**, 1993, pp.321-323.

Contact information

L. A. Kurdachenko,
K. O. Grin,
N. A. Turbay

Department of Algebra, Oles Honchar Dnipropetrovsk National University, 72 Gagarin Av., Dnepropetrovsk, Ukraine 49010
E-Mail: `lkurdachenko@i.ua`, `k.grin@mail.ru`, `nadezhda.turbay@gmail.com`

Received by the editors: 01.02.2012
and in final form 28.03.2012.