Algebra and Discrete Mathematics SURVEY ARTICLE
Volume 13 **(2012)**. Number 1. pp. 59 – 91

On the contribution of D.I. Zaitsev to the Theory of Infinite Groups

Vladimir V. Kirichenko, Leonid A. Kurdachenko, Javier Otal and Igor Ya. Subbotin

Abstract. We survey the most outstanding contributions due to D.I. Zaitsev in the Theory of Infinite Groups.

Introduction

In the spring of 2012, the 70^{th} anniversary of the birth of Dmitry I. Zaitsev (1942-1990) will happen. D.I Zaitsev passed away very young, two years before his 50^{th} anniversary. But his life in mathematics is very long. His main interests were wide and affected all areas of modern Group Theory. His work greatly influenced the development of the Theory of Infinite Groups; actually many of his results are now classical and have been included in monographs and surveys. The methods developed by him found numerous followers who have successfully employed them.

In this survey-paper we want to review some of D.I. Zaitsev's main results as well as to trace their impact and their later development. Obviously we cannot reflect all aspects of his work. In particular, we almost do not touch his results on modules over group rings: they have been adequately presented in the monographs [28, 29]. Besides we deal with the

The three last authors were supported by Proyecto MTM2010-19938-C03-03 of MICINN, the Government of Aragón (Spain) and FEDER funds from European Union

2010 Mathematics Subject Classification: *20F99.*

Key words and phrases: *Soluble group; stable soluble group; radical group; generalized radical group; minimax group; factorized group; splitting theorems; chain conditions; weak chain conditions; minimax rank; Hirsch-Zaitsev rank.*

remarkable D.I. Zaitsev's results on factorization of groups by their subgroups only in a rather superficial way. This very wide and well developed topic is adequately reflected in other surveys and books.

Nevertheless, we hope that this review draws a fairly complete picture of the D.I. Zaitsev contributions to the Theory of Groups as well as it allows to understand better the depth of his talent.

1. Nilpotent and soluble subgroups of infinite groups

The first problems that D.I. Zatcev investigated dealt with questions associated with the study of the influence of nilpotent (respectively solvable) subgroups on the structure of generalized nilpotent (respectively solvable) groups. These tasks were part of a general theme, which was developed by the school headed by S.N. Chernikov at that time. In infinite groups, the question about the conditions for the existence of proper subgroups having certain properties arises naturally. The origin of this subject lays in the investigations related to the famous O. Schmidt problem on the existence of infinite abelian subgroups in infinite groups. On the other hand, a very rapidly developing theory of finite groups whose proper subgroups have some given property inspired attempts of extending these results to the infinite case. The solution of the O. Schmidt problem in the locally finite case obtained by M.I. Kargapolov [19] and P. Hall and C.R. Kulatilaka [13] had a stimulating impact on these researches. One of the first problems considered by D.I. Zatsev, was the question on the existence of nilpotent and generalized nilpotent groups having proper infinite nilpotent subgroups of a fixed class of nilpotency. Let G be a nilpotent group and denote by $nc(G)$ the nilpotency class of G. Suppose that $nc(G) = k$. Following Zaitcev [54], we say that G *is a stable nilpotent group* if every infinite subgroup of G (in particular, G itself) of nilpotency class k includes a proper infinite subgroup of nilpotency class k. The paper [54] was devoted to the analysis of stable nilpotent groups. Here are the main results of this article.

Theorem 1. *Let G be an infinite nilpotent group such that $nc(G) = k$. Then G has a proper nilpotent subgroup L such that $nc(L) = k$.*

Corollary 1. *Let G be a nilpotent group. If $nc(L) < nc(G)$ for each proper subgroup L, then G is finite.*

Corollary 2. *Let G be a torsion-free nilpotent group. Then every non-identity subgroup of G (in particular, G itself) is stable nilpotent.*

We recall that a group G is called *divisible* if $G = G^n = \langle g^n \mid g \in G\rangle$ for every positive integer n.

Corollary 3. *Let G be an infinite nilpotent group. If G has no non-identity divisible subgroups, then G is stable nilpotent.*

Corollary 4. *Let G be a periodic infinite nilpotent group. Then G is stable nilpotent if and only if G has no non-identity divisible subgroups.*

We recall that a group G is called *a Chernikov group* if G has a normal subgroup D of finite index which is a direct product of finitely many quasicyclic subgroups.

Theorem 2. *Let G be an infinite nilpotent group including a subgroup L such that $nc(L) = k > 0$. Then G has a stable nilpotent subgroup of nilpotency class k if and only if G is not Chernikov group.*

Corollary 5. *Let G be a nilpotent group. If $k > 0$ put $\mathcal{N}_k = \{H \mid H \leq G$ and $nc(H) = k\}$. Then the ordered by inclusion family $\mathcal{N}_k$ satisfies the minimal condition if and only if G is a Chernikov group.*

These results were significantly improved in [57].

Theorem 3. *Let G be a locally nilpotent group including a nilpotent subgroup L such that $nc(L) = k > 0$. Then G has a stable nilpotent subgroup of nilpotency class k if and only if G is not Chernikov group.*

A similar problem arises naturally for solvable groups. Let G be a soluble group and denote by $dl(G)$ the derived length or solubility class of G. Suppose that $dl(G) = k$. Following Zaitcev [58], we say that G *is a stable soluble group* if every infinite subgroup of G (in particular, G itself) of derived length k includes a proper infinite subgroup of derived length k. The article of D.I. Zaitsev [58] was entirely dedicated to them.

Theorem 4. *Let G be an infinite soluble group such that $dl(G) = k$. Then G has a proper subgroup L such that $dl(L) = k$.*

Corollary 6. *Let G be a soluble group. If $dl(L) < dl(G)$ for each proper subgroup L, then G is finite.*

Corollary 7. *Let G be a torsion-free soluble group. Then every non-identity subgroup of G (in particular, G itself) is stable soluble.*

Corollary 8. *Let G be an infinite soluble group. If G has no non-identity divisible subgroups, then G is stable soluble.*

In contrast to the nilpotent case, Corollary 4 does not hold in the soluble case since there exists a periodic non-abelian stable soluble group including non-identity divisible subgroups. Indeed, let $G = \langle a \rangle \mathrm{wr} C$ be the wreath product of a cyclic group $\langle a \rangle$ of prime order p by a quasicyclic p–group G. Then $G = A \leftthreetimes C$ is the semidirect product of an infinite elementary abelian normal p–subgroup by C. G is soluble, $dl(G) = 2$ and $\zeta(G) = \langle 1 \rangle$. Let H be an infinite non-abelian subgroup of G. If $HA \neq G$, then the orders of the elements of HA are bounded, and so HA (and hence H) has no non-identity divisible subgroups. By Corollary 8, H is stable. If $HA = G$, since H is non-abelian, $H \cap A \neq \langle 1 \rangle$. Since $\zeta(G) = \langle 1 \rangle$, $H \cap A$ has to be infinite. The factor-group $H/(H \cap A) \cong HA/A = G/A$ is a quasicyclic p–group. If every infinite subgroup of H is abelian, then so has to be H itself, a contradiction. Hence H has a proper infinite non-abelian subgroup and thus H is stable soluble.

Theorem 5. *Let G be a soluble group such that $dl(G) = k$. If G is not Chernikov, then G has a stable soluble subgroup L such that $dl(L) = t$ for every $1 \leq t \leq k$.*

Corollary 9. *Let G be a soluble group. Then every infinite soluble subgroup of G is stable soluble if and only if G has no quasicyclic subgroups.*

We recall that a group G is called *a radical group* if G has an ascending series whose factors are locally nilpotent groups. We remark that a radical group has an ascending series of normal subgroups with locally nilpotent factors.

Theorem 6. *Let G be a radical group including a soluble subgroup L such that $dl(L) = k > 0$. If G is not Chernikov, then G has a stable soluble subgroup of derived length k.*

Corollary 10. *Let G be a non-soluble radical group. Then for every positive integer k G has a soluble subgroup L such that $dl(L) = k$.*

2. The weak minimal and maximal conditions

At the same time, D.I. Zaitsev introduced some imperative restrictions that have been played an essential role in the development of the infinite generalized soluble group theory, namely *the weak minimal condition and the weak maximal condition.* We formulate these concepts in their most general form. Let G be a group and let $\mathfrak{M}$ be a family of subgroups of G. It is said that $\mathfrak{M}$ *satisfies the weak minimal condition* (respectively, *the* *weak maximal condition*) or that G *satisfies the weak minimal condition on $\mathfrak{M}$–subgroups* (respectively, *the weak maximal condition on $\mathfrak{M}$–subgroups*) if given an infinite descending chain (respectively, an infinite ascending chain) $\{H_n \mid n \geq 0\}$ of $\mathfrak{M}$–subgroups, there exists a number k such that the indices $|H_n : H_{n+1}|$ (respectively, the indices $|H_{n+1} : H_n|$) are finite for every $n \geq k$. Further if G satisfies the weak minimal condition on $\mathfrak{M}$–subgroups (respectively, the weak maximal condition on $\mathfrak{M}$–subgroups), in short, we will say that G *satisfies Wmin–$\mathfrak{M}$* (respectively G *satisfies Wmax–$\mathfrak{M}$*).

The weak minimal condition was introduced by D.I. Zaitsev in his papers [55, 56]. At the same time, in studying minimax groups, R. Baer arrived to the weak minimal and maximal conditions [2]. At this point, we recall that a group G is called *minimax* if G has a subnormal series

$$\langle 1 \rangle = L_0 \trianglelefteq L_1 \trianglelefteq \cdots \trianglelefteq L_n = G$$

whose factors either satisfy the minimal condition or the maximal condition. If G is a soluble–by–finite minimax group, then G has a finite subnormal series whose factors either are Chernikov or polycyclic–by–finite. Therefore soluble–by–finite minimax groups are examples of groups that satisfy both the weak minimal condition and the weak maximal condition. It is worth mentioning that, for a long time, there were no other examples known of groups satisfying conditions Wmin and Wmax.

It is natural to think that the first step to consider in this setting is the investigation of groups with the weak minimal condition and the weak maximal condition for all subgroups, that is the conditions Wmin and Wmax respectively. For locally soluble-by–finite groups the description of such groups have been obtained in the papers of D.I. Zaitsev [56, 60].

Theorem 7. *Let G be a locally (soluble-by–finite) group. Then G satisfies Wmin (respectively, Wmax) if and only if G is soluble-by–finite minimax.*

The above result naturally raises the following question: *does the class of groups satisfying Wmin coincide with the class of groups satisfying Wmax?* The answer to this question is negative. Actually there exists an uncountable group satisfying Min which does not satisfy Wmax: see [45, Theorem 35.2].

By the same time, the study of groups with the weak minimal condition and the weak maximal condition for abelian subgroups (the conditions Wmin–ab and Wmax–ab respectively) has been initiated. In 1964, Yu.I.

Merzlyakov [42] published his famous widely known construction of non-radical locally polycyclic groups whose abelian subgroups are finitely generated while the special ranks of abelian subgroups are not bounded. These groups are not minimax but satisfy Wmin-ab and Wmax-ab. Although the original proof by Y.I. Merzlyakov contained a gap, the result was successfully fixed in the later paper [43]. Therefore, it became clear that in the class of radical groups, one can obtain some meaningful results. R. Baer [2] and D.I. Zaitsev [59] obtained the description of the groups satisfying Wmin–ab and Wmax–ab in the class of radical groups. We note that an abelian group satisfies Wmin–ab (respectively, Wmax–ab) if and only if it is minimax. Therefore the main result of the papers [2] and [59] can be formulated in the following way.

Theorem 8. *Let G be a radical group. Then every abelian subgroup of G is minimax if and only if G is soluble–by–finite minimax.*

This result can be largely extended. A group G is said to be *generalized radical* if G has an ascending series whose factors are locally nilpotent or locally finite. Hence a generalized radical group G either has an ascendant locally nilpotent subgroup or an ascendant locally finite subgroup and so its locally nilpotent radical $Lnr(G)$ of is non-identity or G has a non-identity locally finite normal subgroup. Thus every generalized radical group has an ascending series of normal subgroups with locally nilpotent or locally finite factors.

Theorem 9. *Let G be a generalized radical group. Then every abelian subgroup of G is minimax if and only if G is soluble–by–finite minimax.*

It could be said that the weak minimal and weak maximal conditions for non-abelian subgroups (the conditions Wmin–(non-ab) and Wmax–(non-ab) respectively) are in certain sense dual of the conditions Wmin–ab and Wmax–ab, respectively. Locally (soluble–by–finite) groups with this conditions were described by D.I. Zaitsev in the paper [61].

Theorem 10. *Let G be a locally (soluble–by–finite) group. Then G satisfies Wmin–(non-ab) if and only if G is soluble–by–finite minimax.*

For groups satisfying the weak maximal condition for non-abelian subgroups the situation was more difficult. Although a locally soluble group satisfying the minimal condition for non–abelian subgroups is Chernikov (S.N. Chernikov [4]), the class of groups in which the set $\mathcal{L}_{\text{non-ab}}(G)$ of all non-abelian subgroups satisfies the maximal condition (Max–(non-ab)) is not exhausted by abelian groups and groups satisfying Max. For example, the wreath product $G = \langle a\rangle \mathrm{wr} \langle g\rangle$ of a cyclic group $\langle a\rangle$ of prime order by an infinite cyclic group $\langle g\rangle$ does not satisfy Max but satisfies Max–(non-ab). Actually groups satisfying Max–(non-ab) were considered by D.I. Zaitsev and L.A. Kurdachenko in the paper [80].

Theorem 11. *Let G be a locally (soluble–by–finite) group and suppose that G is not polycyclic–by–finite. Then G satisfies Max–(non–ab) if and only if G has an abelian normal subgroup A such that*

(1) $A = C_G(A)$;

(2) *G/A is a finitely generated torsion-free abelian–by–finite group; and*

(3) *For every $g \in G \setminus A$, the $\mathbb{Z}\langle g\rangle$–module A is finitely generated.*

After the death of D.I. Zaitsev, the description of the groups satisfying Wmax–(non-ab) was achieved by L.S. Kazarin, L.A. Kurdachenko and I.Ya. Subbotin [20]. As we will see below from the results of this work, the situation here is even more complicated.

Theorem 12. *Let G be a locally finite group. Then G satisfies Max–(non-ab) if and only if G either is abelian or Chernikov.*

Let G be a group and A be an abelian normal subgroup of G. We denote by $a_G(A)$ the G–invariant subgroup of A satisfying the following conditions:

(i) $a_G(A)$ has an ascending series of G–invariant subgroups whose factors are G–chief; and

(ii) $A/a_G(A)$ has no non-identity minimal G–invariant subgroups.

Let R be a ring and A be an R–module. We recall that A is said to be *minimax* if A has a finite series of submodules, every factor of which is either artinian or noetherian.

Theorem 13. *Let G be a non-abelian generalized radical group, A be a maximal abelian normal subgroup of G and T be the periodic part of A. Suppose that either $a_G(A)$ does not include T or $r_0(A)$ is infinite. Then G satisfies Wmax–(non-ab) if and only if the following conditions hold:*

(1) *G/A is finitely generated torsion-free abelian-by–finite; and*

(2) *For every $g \in G \setminus A$, the $\mathbb{Z}\langle g\rangle$–module A is minimax.*

Here and elsewhere, $r_0(G)$ denotes the 0–rank of G defined with other ranks in the next section.

Theorem 14. *Let G be a non-abelian generalized radical group, A be a maximal abelian normal subgroup of G and T be the periodic part of A. Suppose that A is non-minimax, $T \leq a_G(A)$ and $r_0(A)$ is finite. Then G satisfies Wmax–(non-ab) if and only if the following conditions hold:*

(1) *A/T is minimax;*

(2) *G/A is torsion-free;*

(3) *$G/A = L$ has a normal subgroup $K = H/A$ of finite index such that either K is abelian minimax or $K = C \leftthreetimes D$, where $C = C_K(C)$ is abelian minimax and $D = C_K(D)$ is finitely generated abelian; and*

(4) *For every $g \in G \setminus A$, the $\mathbb{Z}\langle g\rangle$–module T is artinian.*

If the maximal normal abelian subgroup A is minimax, then the above results take a particularly pleasing form.

Theorem 15. *Let G be a non-abelian generalized radical group, A be a maximal abelian subgroup of G. Suppose that A is minimax. Then G satisfies Wmax–(non-ab) if and only if G is soluble–by–finite minimax.*

After the death of D.I. Zaitsev, the investigations of groups with the weak minimal and maximal conditions for distinct natural families of subgroups have been continued intensively. We mention some examples. The groups satisfying the weak minimal condition (respectively, the weak maximal condition) on non-normal subgroups were considered by L. A. Kurdachenko and V. E. Goretskii in [27]. Further, the groups in which the family of all non-(almost normal) subgroups satisfies the weak minimal condition (respectively, the weak maximal condition) were studied by G. Cutolo and L.A. Kurdachenko [7], the groups for which the family of all non-subnormal subgroups satisfies the weak minimal condition (respectively, the weak maximal condition) were studied by L.A. Kurdachenko and H. Smith in the papers [33, 34, 35], the groups in which the family of all non-nilpotent subgroups satisfies the weak maximal condition were investigated by L.A. Kurdachenko, P. Shumyatskii and I.Ya. Subbotin in [32] and by L.A. Kurdachenko and N. N. Semko in [31]. By its way, the groups satisfying the dual weak minimal condition for non-nilpotent subgroups were considered in the paper of M.R. Dixon, M.J. Evans, and H. Smith [8]; more detailed information about this topic can be found in the survey of M.R. Dixon and I.Ya. Subbotin [11]. Finally, we mention that the weak minimal and maximal conditions have shown to be very effective in the investigation of infinite dimensional linear groups (see the surveys [9, 44]).

A special place among the weak minimal and weak maximal conditions is taken by the weak minimal and weak maximal conditions for normal subgroups, the condition Wmin–n and Wmax–n. D.I. Zaitsev was the first author who realized the importance of these conditions and encouraged to the second author of this survey in the study of groups satisfying Wmin-n and Wmax-n. It is worth noting that the minimal and maximal conditions for normal subgroups are significantly different from the minimal and maximal conditions for subgroups. The description of the groups satisfying Wmin-n and Wmax-n were obtained for locally nilpotent and metabelian groups only. At the same time, B. Hartley [16] constructed an uncountable soluble group G satisfying Min–n. All this shows that the classes of locally nilpotent and metabelian groups are the first cases of classes of groups in which one can try the study of the weak minimal and weak maximal conditions for normal subgroups.

In this setting, we mention that the conditions Wmin–n and Wmax–n for locally nilpotent groups were studied by L.A. Kurdachenko [21, 22, 23, 24]. As it is well known, in locally nilpotent groups Min–n implies Min and Max–n implies Max. It is not the case for the conditions Wmin–n and Wmax–n. However L.A. Kurdachenko [21] was able to establish the following result.

Theorem 16. *Let G be a locally nilpotent group satisfying Wmin–n or Wmax–n. Then we have*

(1) *If G is periodic, then G is Chernikov;*

(2) *if G is a torsion-free group, then G is minimax; and*

(3) *if G is residually finite, then G is minimax.*

In the papers [21, 22], some examples of non-minimax locally nilpotent groups satisfying Wmin–n and Wmax–n were constructed.

A group G is called $\mathfrak{F}$–perfect if G has no proper subgroups of finite index. We note that the subgroup generated by all $\mathfrak{F}$–perfect subgroups of a group G is likewise $\mathfrak{F}$–perfect. Consequently every group G has a unique maximal $\mathfrak{F}$–perfect, which is normal and denoted by $\mathfrak{F}(G)$. In the sequel, we denote by $Tor(G)$ the unique maximal torsion (i.e. periodic) normal subgroup of the group G, *the periodic part of* G. If G is a locally nilpotent that satisfies Wmin–n or Wmax–n, then $\mathfrak{F}(G) \leq Tor(G)$.

Theorem 17. *([22]) Let G be a locally nilpotent group satisfying Wmin–n. Then we have*

(1) *$Tor(G)$ satisfies the minimal condition for G–invariant subgroups (the condition Min–G);*

(2) *$\mathfrak{F}(G)$ is a periodic divisible abelian subgroup; and*

(3) *G is soluble.*

Theorem 18. *([24]) A locally nilpotent group satisfying Wmin–n is countable.*

Concerning the groups satisfying Wmax–n we have.

Theorem 19. *([23]) Let G be a locally nilpotent group satisfying Wmax–n. Then we have*

(1) *$\mathfrak{F}(G)$ is a Chernikov subgroup;*

(2) *G is hypercentral if and only if it is soluble.*

Theorem 20. *([24]) Let G be a hypercentral group satisfying Wmax–n. Then we have.*

(1) *G satisfies Wmin–n;*

(2) *$Tor(G)/\mathfrak{F}(G)$ is nilpotent of finite exponent; and*

(3) *there exists some minimax subgroup A such that $G = Tor(G)A$.*

In the paper [25], L.A. Kurdachenko studied the conditions Wmin–n and Wmax–n in some extensions of the class of locally nilpotent groups. We mention that soluble groups satisfying the weak minimal condition for normal subgroups were investigated by D.I. Zaitsev, L.A. Kurdachenko and A.V. Tushev [82], L.A. Kurdachenko and A.V. Tushev [36, 37], M. Karbe [17], M. Karbe and L.A. Kurdachenko [18], and L.A. Kurdachenko [26]. We quote here some interesting results achieved in these researches.

Theorem 21. *([82]) Let G be a metanilpotent group satisfying Wmin–n. If G is torsion-free, then G is minimax.*

Theorem 22. *([82, 17]) Let G be a soluble group satisfying Wmin–n. If G is periodic, then G satisfies Min–n.*

Theorem 23. *([18]) Let G be a hyperabelian group satisfying Wmin–n. If G is residually finite, then G is minimax.*

A module A is said to be *just non-artinian* if A is not artinian but every proper factor-module of A is artinian. Let G be a metabelian group satisfying Wmin–n and put $A = Tor([G, G])$. Then we may think of A as a $\mathbb{Z}(G/A)$–module. Under this meaning, it can be shown that this module has a finite series of submodules in which the first factor is artinian and the other are just non-artinian. The structure of artinian modules over an abelian group of finite special rank were described by L.A. Kurdachenko [26] (but see also the book [29, Chapter 14]).

3. Restrictions related to some ranks of groups

In the Theory of Infinite Groups, some important numerical invariants come from minimax groups. The first of them was introduced by D.I. Zaitsev in the study of groups with weak minimal condition [56]. In this paper, D.I., the expression *index of minimality* was used, although it turned out unsuitable and did not adequately reflect the situation. Later on, in the paper [81], D.I. Zaitsev proposed another expression, namely *the minimax rank* that we recall here.

Let G be a group and let

$$\langle 1 \rangle = H_0 \leq H_1 \leq \cdots \leq H_{n-1} \leq H_n = G$$

be a finite chain of subgroups of G. Put $\mathcal{E} = \{H_j \mid 0 \leq j \leq n\}$. Denote by $il(\mathcal{E})$ the number of the links $H_j \leq H_{j+1}$ such that the index $|H_{j+1} : H_j|$ is infinite. The group G is said *to have finite minimax rank* $r_{mm}(G) = m$ if $il(\mathcal{E}) \leq m$ for every finite chain of subgroups $\mathcal{E}$ and there exists a finite chain $\mathcal{D}$ for which this number is exactly m. Otherwise, it is said that G *has infinite minimax rank.* If G is a finite group, clearly $r_{mm}(G) = 0$.

Let $H \leq K$ be subgroups of a group G. We say that *the link $H \leq K$ is infinite* if the index $|K : H|$ is infinite and that *the link $H \leq K$ is minimal infinite* if it is infinite and for every subgroup L such that $H \leq L \leq K$ one of the indexes $|L : H|$ or $|K : L|$ is finite.

Let G have finite minimax rank and let $\mathcal{D}$ be a finite chain of subgroups such that $il(\mathcal{D}) = r_{mm}(G)$. Let $H \leq K$ be a link of this chain such that $|K : H|$ is infinite. Let L be a subgroup of G such that $H < L < K$ and $|K : L|$ and $|L : H|$ are both infinite. The chain $\mathcal{D}_1 = \mathcal{D} \cup \{L\}$ is finite and $il(\mathcal{D}_1) = il(\mathcal{D}) + 1$, which contradicts the choice of $\mathcal{D}$. Thus every link $H \leq K$ of $\mathcal{D}$ of infinite index is minimal infinite, in the sense it has no proper refinements whose links have all infinite index.

The next result quotes some interesting properties of the minimax rank that were obtained in the paper [56].

Proposition 1. *Let G be a group, H be a subgroup of G and L be a normal subgroup of G. Then we have*

(1) *If G has finite minimax rank, then*

  (1a) *H has finite minimax rank $r_{mm}(H) \leq r_{mm}(G)$;*

  (1b) *if H has finite index, then $r_{mm}(G) = r_{mm}(H)$;*

  (1c) *G/L has finite minimax rank $r_{mm}(G/L) = r_{mm}(G) - r_{mm}(L)$; and*

  (1d) *if L is finite, then $r_{mm}(G) = r_{mm}(G/L)$.*

(2) *if L and G/L have finite minimax rank, then G has finite minimax rank too.*

As the following result from [56] shows, the connection between this rank and the minimax groups is very close.

Theorem 24. *Let G be a soluble–be–finite group. Then G has finite minimax rank if and only if G is minimax.*

From the latter, the following question naturally appears: *Is the minimax rank of every minimax group finite?* The answer is negative. Indeed, as we mentioned above, there exists a periodic uncountable group G satisfying Min (see [45, Theorem 35.2]). This group G has an ascending chain

$$\langle 1 \rangle = D_0 \leq D_1 \leq \cdots D_\alpha \leq D_{\alpha+1} \leq \cdots D_\gamma = G,$$

where $\gamma = \omega_1$ is the first uncountable ordinal. In particular, for every integer $n \geq 0$, the group G has a finite chain $\mathcal{E}$ such that $il(\mathcal{E}) = n$. This shows that G has infinite minimax rank.

Let G be an abelian group. A subset X of G is said to be $\mathbb{Z}$*–independent* or simply *independent* if whenever $x_1^{k_1} \cdots x_n^{k_n} = 1$ holds with $n \geq 1$, $x_1, \cdots, x_n \in X$ and $k_1, \cdots, k_n \in \mathbb{Z}$, then we have $x_1^{k_1} = \cdots = x_n^{k_n} = 1$. It is easy to see that the number of elements in a maximal independent subset consisting of elements of infinite order is an invariant of G called *the* $\mathbb{Z}$*–rank of* G or *the torsion-free rank of* G and denoted by $r_{\mathbb{Z}}(G)$. Clearly, $r_{\mathbb{Z}}(G) = r_{\mathbb{Z}}(G/Tor(G))$. It readily follows that an abelian group G has finite $\mathbb{Z}$–rank r if and only if $G/Tor(G)$ is isomorphic to a subgroup of the additive group $\overbrace{\mathbb{Q} \oplus \cdots \oplus \mathbb{Q}}^{r}$.

Let p be a prime and let n be a positive integer. If P is an abelian p–group, then *the n–layer of P* is the subgroup given by

$$\Omega_n(P) = \{a \in P \mid a^{p_n} = 1\}.$$

For every $n \geq 1$, it is clear that $\Omega_{n+1}(P)/\Omega_n(P)$ is an elementary abelian p–group, which can therefore be thought of as a vector space over the prime field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ of p elements. By definition, *the p–rank of P* is the $\mathbb{F}_P$–dimension of $\Omega_1(P)$. More generally, *the p–rank* of an abelian group G is by definition the p–rank of the Sylow p–subgroup of G. It is not hard to see that an abelian p–group G has finite p–rank r if and only if every elementary abelian section U/V of G has order at most p^r and there exists an elementary abelian section A/B of G such that $|A/B| = p^r$. It is said that G has *finite section p–rank* $r_p(G) = r$ if every elementary abelian p–section of G is finite of order at most p^r and there is an elementary abelian p–section A/B of G such that $|A/B| = p^r$. Similarly, a group G is said to have *finite section* 0*–rank* $r_0(G) = r$ if every torsion-free abelian section U/V of G satisfies $r_{\mathbb{Z}}(U/V) \leq r$ and there exists a torsion-free abelian section A/B such that $r_{\mathbb{Z}}(U/V) = r$. These concepts were initially introduced for soluble groups by A.I. Maltsev [41] and D.J.S. Robinson [47, §6.1], although they used a different terminology. We remark that if a group G has finite section p–rank for some prime p, then G has finite section 0–rank and $r_0(G) \leq r_p(G)$.

Proposition 2. *Let G be a soluble–by–finite group. If G has finite minimax rank, then G has finite section p–rank for every prime p. In particular, G has finite section* 0*–rank.*

The next rank is known as *the torsion-free rank* or 0*–rank* of a group, and it is tightly associated with D.I. Zaitsev's research activity. Actually, he was the first mathematician who frequently used it and showed its efficiency in solving several problems. The source of this numerical invariant comes from the study of polycyclic–by–finite groups initiated by K.A. Hirsch in the paper [14], when it was called *the Hirsch number* of a polycyclic–by–finite group. Later on, it was realized that there were broader classes of groups where this same invariant could be used. For example, for polyrational groups (see below) where this invariant was originally called *the rational rank* of the group, a term due to D.I. Zaitsev [62]. The former concept was further extended by D.I. Zaitsev [67] to locally polycyclic–by–finite groups and to arbitrary groups in [75].

In the paper [10], the torsion-free rank was generalized. The generalization made there had its starting point in the following fact. Let G be a group and suppose that

$$\Sigma_1 : \langle 1 \rangle = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_\alpha \trianglelefteq G_{\alpha+1} \trianglelefteq \cdots G_\gamma = G$$

and

$$\Sigma_2 : \langle 1 \rangle = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_\beta \trianglelefteq H_{\beta+1} \trianglelefteq \cdots H_\delta = G,$$

are two ascending series of subgroups whose factors are either infinite cyclic or periodic. If the number of infinite cyclic factors of the series Σ_1 is exactly r, then the number of infinite cyclic factors of Σ_2 is also r.

A group G is said to have *finite Hirsch-Zaitsev rank* $r_{hz}(G) = r$ if G has an ascending series whose factors are either infinite cyclic or periodic and the number of infinite cyclic factors is exactly r. If G has an ascending series with periodic and infinite cyclic factors and the set of infinite cyclic factors is infinite, then it is said that G has *infinite Hirsch-Zaitsev rank*. Otherwise, *G has no Hirsch-Zaitsev rank*.

We remark that a torsion-free abelian group of 0–rank 1 is locally cyclic and it is isomorphic to a subgroup of the full rational group $\mathbb{Q}$. That is the reason why a torsion-free abelian group of 0–rank 1 is also called *a rational group*. More generally, a group G is called *polyrational* if it has a finite subnormal series whose factors are torsion-free locally cyclic groups. Polyrational groups are examples of groups with finite Hirsch-Zaitsev rank.

The structure of locally generalized radical groups of finite Hirsch-Zaitsev rank was described in the paper [10].

Theorem 25. *Let G be a locally generalized radical group that has finite Hirsch–Zaitsev rank. Then G has a finite chain of normal subgroups*

$$T \leq L \leq K \leq S \leq G$$

such that the following conditions hold.

(1) *T is locally finite and G/T is soluble–by–finite;*

(2) *L/T is a torsion-free nilpotent group;*

(3) *K/L is a finitely generated torsion-free abelian group; and*

(4) *G/K is finite and S/K is the soluble radical of G/K.*

Moreover, if $r_{hz}(G) = r$, then there exist functions f_1 and f_2 such that $|G/K| \leq f_2(r)$ and $dl(S/T) \leq f_1(r)$.

Corollary 11. *Let G be a locally radical group of finite Hirsch-Zaitsev rank. Then $G/Tor(G)$ has a polyrational normal subgroup of finite index.*

We add the following local version of Theorem 25.

Theorem 26. *Let G be a group that satisfies the following conditions:*

(i) *If L is a finitely generated subgroup of G, then the factor-group $L/Tor(L)$ is a locally generalized radical group; and*

(ii) *there is a positive integer r such that $r_{hz}(L) \leq r$ for every finitely generated subgroup L.*

Then $G/Tor(G)$ has a soluble normal subgroup $D/Tor(G)$ of finite index. Moreover, G has finite Hirsch-Zaitsev rank r and there exists a functions f_3 such that $|G/D| \leq f_3(r)$.

We also note the following result.

Corollary 12. *Let G be a soluble–by–finite group. If G has finite minimax rank, then G has finite Hirsch-Zaitsev rank.*

It is worth mentioning that the minamax rank of a soluble–by–finite minimax group G can be calculated using the formula:

$$r_{mm}(G) = r_{hz}(G) + \sum_p m_p(G),$$

where $m_p(G)$ is the number of quasicyclic p–factors of a subnormal series of G.

Other results by D.I. Zaitsev works were linked to another important numerical invariant, namely *the special rank of a group*. Historically, this is the oldest concept among other ranks. As we are going to see, it is a natural extension of the concept of the dimension of a vector space. If A is a vector space of finite dimension k over a field F and B is a subspace of A, then it is well-known that B is finite dimensional and that dimension of B is at most k. Similarly, it is a well-known consequence of the theorems of structure of finitely generated abelian groups that if G is an abelian group with k generators and H is a subgroup of G, then H is finitely generated and has at most k generators. Thus a subgroup H of a finitely generated abelian group G is also finitely generated and the minimal number of generators of H is at most the minimal number of generators of G. This is no longer true. For example the standard restricted wreath product $\mathbb{Z}\mathrm{wr}\mathbb{Z}$ of two infinite cyclic groups is a 2–generator group but its base group is infinitely generated. Thus a subgroup of a finitely generated group need not even be finitely generated. It is also well-known that a subgroup of a finitely generated nilpotent group is finitely generated. However, even in this case, it is possible to find a subgroup with more generators than the original group has. For example, the wreath product $\mathbb{Z}_p\mathrm{wr}\mathbb{Z}_p$, where p is an odd prime. This group has two generators but its base group has p generators. The special rank corrects these anomalies.

It is said that a group G has *finite special rank* $r(G) = r$ if every finitely generated subgroup of G can be generated by r elements and r is the least positive integer with this property. The special rank of a group G is sometimes called *the Prüfer rank of* G or simply *the rank of* G. Actually the introduction of this rank is due to H. Prüfer in the paper [46] for groups of rank 1. In its more general approach (and also in the the term special rank) the introduction is due to A.I. Maltsev [40]. Later on, in the paper [1] the name Prüfer rank appeared.

If G is a locally generalized radical group of finite Hirsch-Zaitsev rank, then $G/Tor(G)$ has finite special rank and $r_{hz}(G) \leq r(G/Tor(G))$. Conversely, if G is a locally generalized radical group of finite special rank r, then G has Hirsch-Zaitsev rank at most r. In this setting D.I. Zaitsev [62] was able to prove the following important result.

Theorem 27. *Let G be a polyrational group. Then the Hirsch-Zaitsev rank of G coincides with the special rank of G.*

For finitely generated generalized soluble groups, the finiteness of certain ranks often implies the finiteness of the minimax rank. D.J.S. Robinson was the first who drew attention on this matter. In his paper [48], he was able to prove the following interesting result.

Theorem 28. *Let G be a finitely generated soluble group. If G has finite special rank, then G has finite minimax rank.*

We mention that in the paper [62], D.I. Zaitsev proved this result for a torsion-free finitely generated soluble group. In the paper [50], D.J.S. Robinson obtained the following generalization of this result.

Theorem 29. *Let G be a finitely generated soluble group. If G has finite section p–rank for all prime p, then G has finite minimax rank.*

The proof of Robinson [50] of this result made use of homology techniques. A purely group-theoretical proof of this result was given in [30].

4. On the existence of supplements and complements to some normal subgroups

We start by recalling the standard terminology on the topic.

Let G be a group and H be a proper subgroup of G. A proper subgroup K is said to be *a supplement to* H (in G) if $G = HK$. A supplement K to H is said to be *a complement to* H (in G) if further $H \cap K = \langle 1 \rangle$. If H is a normal subgroup of G and H has a complement K, then it is said that G *splits over* H and denote this by $G = H \rtimes K$. If all complements to H are conjugate, then it is said that G *conjugately splits over* H.

In the Theory of Groups, the criteria of the existence of complements and supplements to some types of subgroups play a key role. For finite groups, the situation is more or less optimal. In fact, there are a sufficient number of effective criteria for the existence of complements and supplements, such as the theorems of Maschke, Gaschütz, Schenkman, the Frattini argument, and so on. For infinite groups, the situation is much more complicated. For example, the averaging operation is often used as a technical tool in finite groups, but it cannot always be employed in infinite groups. This also applies to many other techniques. So often the conditions of existence of complements and supplements in infinite groups are associated with finiteness conditions. One source of questions about the existence of complements for different subgroups is the paper of P. Hall [12], in which he described the finite groups whose subgroups are complemented. Such groups are called *completely factorized*. The topics related to the study of groups with a given system of complemented subgroups were traditional for the group-theoretical school headed by S.N. Chernikov, and have been developing by more than 50 years. As it was proved by P. Hall, a finite group G is completely factorized if and only if G is supersoluble and its Sylow p–subgroups are elementary abelian for every prime p. Nevertheless, in the paper [5], some examples of infinite supersoluble groups with elementary abelian Sylow p–subgroups for all prime p with some non complemented subgroups were constructed. This raised the study of groups whose normal subgroups have complements (*normally factorized groups*). Interest to the groups with complemented normal subgroups associated with the fact that the class of these groups includes as a subclass of the groups whose lattice of all subgroups is the lattice with complements (see the survey of M. Curzio [6]). Moreover, it is unclear whether the class of groups in which the lattice of all subgroups is a lattice with complements (K–*groups*) and the class of all normally factorized groups coincide.

The papers [63, 64, 68] by D.I. Zaitsev are devoted to the study of normally factorized groups. The concept of LC–group is formulated in the first of these papers. A group G is said to be *an* LC–*group* if its finitely generated subgroups are completely factorized. The following results from [63] enlighten the role playing by the LC–subgroups in the normally factorized groups.

Theorem 30. *Let G be a normally factorized group. Then the following assertions hold:*

(1) *The subgroup generated by a finite family of normal completely factorized subgroups is a normal completely factorized subgroup;*

(2) *G has an unique maximal normal locally completely factorized subgroup (that is called the LC–radical of G);*

(3) *If G has an ascending series of normal subgroups*

$$\langle 1\rangle = L_0 \leq L_1 \leq \cdots L_\gamma = G$$

whose factors are locally supersoluble, then these factors are LC–subgroups, and hence G has an upper LC–radical series;

(4) *if G is locally finite, then the LC–radical of G contains every ascendant LC–subgroup;*

(5) *if G is locally finite, then G has the locally supersoluble radical L that coincides with its LC–radical and contains every ascendant locally supersoluble subgroup; and*

(6) *if the LC–radical of G has finite index in G, then it is a completely factorized subgroup.*

Theorem 31. *Suppose that G has a finite series of normal subgroups whose factors are locally supersoluble. Then G is normally factorized if and only if G is a K–group.*

In the paper [63], some examples of normally factorized groups in which the LC–radical is not completely factorized were shown.

In some sense, the paper [64] is dual to the paper [63].

Theorem 32. *Let G be a normally factorized group. Then the following assertions hold:*

(1) *If H is a normal subgroup of G, then H has LC–residual;*

(2) *the LC–residual of G is finite if and only if G has a normal subgroup L of finite index such that every subgroup of L has a complement in G; and*

(3) *if H is a normal subgroup of G of finite index, then H is normally factorized.*

Corollary 13. *Let G be a normally factorized group. Then the following assertions hold:*

(1) *If H is a finite normal subgroup of G, then H is normally factorized;*

(2) *if H is a subnormal subgroup of finite index in G, then H is normally factorized;*

(3) *if G is soluble and the chief factors of G are finite, then every normal subgroup of G is normally factorized; and*

(4) *if G is soluble and residually finite, then every normal subgroup of G is normally factorized.*

Corollary 14. *Let G be a metabelian normal factorized group. Then every normal subgroup of G is normally factorized.*

We proceed to formulate the problem of the complementarity in a general form. Let $\mathfrak{X}$ be a class of groups. If G is a group, then it is said that a subgroup H of G has an *$\mathfrak{X}$–complement* in G or *h is $\mathfrak{X}$–complemented* in G if there exists a subgroup K such that $G = HK$ and $H \cap K \in \mathfrak{X}$ (D.I. Zaitsev [66]). For example, if $\mathfrak{X} = \mathfrak{I}$ is the identity class, then the $\mathfrak{X}$–complementarity is the ordinary complementarity. A group G is said to be *normally $\mathfrak{X}$–factorized* if every normal subgroup of G has an $\mathfrak{X}$–complement. At the first sight, it appears that this topic could admit many variations. However, D.I. Zaitsev pointed in the paper [66] that this diversity is not so great by the following reasons.

- If G a soluble group, then [66, Theorem 2] showed that G is normally $\mathfrak{X}$–complementarity if and only if for every subgroup H of G there exists a subgroup K such that $G = \langle H, K \rangle$ and $H \cap K \in \mathfrak{X}$.

- A soluble normally $\mathfrak{X}$–factorized group is normally $\mathfrak{Y}$–factorized, where $\mathfrak{Y}$ is the class of all polycyclic $\mathfrak{X}$–groups ([66, Theorem 2]).

Consequently, it suffices to study only two cases: $\mathfrak{X} = \mathfrak{F}$ the class of all finite groups and $\mathfrak{X} = \mathfrak{G}^o$ the class of all finitely generated groups. In the first case we have *the normally $\mathfrak{F}$–factorized groups* and in the second one *the normally FG–factorized groups.*

From the results of [65], it follows that the class of normally $\mathfrak{F}$–factorized groups contains the class of Chernikov groups. We point out that the study of the normally $\mathfrak{F}$–factorized groups there carried out, required the implementation of techniques from the Theory of Modules. Given a group G and a class of groups $\mathfrak{X}$, a $\mathbb{Z}G$–module A is said to be *X–factorized* if for each submodule B of A there exists a submodule C such that $A = B + C$ and the additive group of $B \cap C$ belongs to the class $\mathfrak{X}$. The structure of $\mathfrak{X}$–factorized modules was described by D.I. Zaitsev in the paper [66], and we are quoting it now. We recall that a module A is said to be *quasifinite* if every proper submodule of A is finite.

Theorem 33. *Let G be a group and $\mathfrak{X}$ be a class of groups,. Then a $\mathbb{Z}G$–module A is $\mathfrak{X}$–factorized if and only if A contains a submodule B such that the additive group of B belongs to $\mathfrak{X}$ and we have a decomposition $A/B = C_1 \oplus C_2$, where C_1 is a semisimple submodule, $C_2 = \bigoplus_{\lambda \in \Lambda} E_\lambda$ with the E_λ quasifinite submodules and the set Λ finite.*

With the aid of the latter, D.I. Zaitsev [66] was able to obtain the following characterization of the soluble normally $\mathfrak{X}$–factorized groups.

Theorem 34. *Let $\mathfrak{X}$ be a class of groups and G be a soluble group. Then G is normally $\mathfrak{X}$–factorized if and only if G has two set of subgroups $\{H_1, \cdots, H_n\}$ and $\{K_1, \cdots, K_n\}$ satisfying the following conditions:*

(1) $G = H_1 \cdots H_n$;

(2) $\langle 1 \rangle = K_1 \trianglelefteq K_2 \trianglelefteq \cdots \trianglelefteq K_n$ *and the factors of this series are finitely generated abelian $\mathfrak{X}$–groups;*

(3) *for every $1 \leq j \leq n$, $K_j \leq H_j$ and K_j is a polycyclic $\mathfrak{X}$–group; and*

(4) *for every $1 \leq j \leq n$, K_j and H_j are normal subgroups of $G_j = \langle H_j, \cdots, H_n \rangle$, H_j/K_j is abelian and the $\mathbb{Z}G_j$–module H_j/K_j is $\mathfrak{X}$–factorized.*

In the same paper [66], the latter result was detailed for the case of normally $\mathfrak{F}$–factorized groups. Note that every normally $\mathfrak{F}$–factorized group has an unique normal subgroup $K(G)$ such that all chief factors of $G/K(G)$ are finite.

Theorem 35. *Let G be a soluble normally $\mathfrak{F}$–factorized group and suppose that the subgroup $K(G)$ is abelian. Then $G = K(G) \leftthreetimes L$, the $\mathbb{Z}G$–module $K(G)$ is semisimple, and L is a normally $\mathfrak{F}$–factorized group with finite chief factors.*

Theorem 36. *Let G be a soluble normally $\mathfrak{F}$–factorized group and suppose that the chief factors of G are finite. Then $G = CL$ and the following conditions hold:*

(1) *C is a normal Chernikov subgroup of G;*

(2) *L is a normally $\mathfrak{F}$–factorized group with finite chief factors; and*

(3) *the intersection of all subgroups of L having finite index is finite.*

Theorem 37. *Let G be a soluble normally $\mathfrak{F}$–factorized group and suppose that G is residually finite. Then $G = (A_1 \leftthreetimes (A_2 \leftthreetimes (\cdots \leftthreetimes A_n) \leftthreetimes K$ and the following conditions hold:*

(1) *The subgroups $A_1, \cdots, A_n$ are abelian and K is finite;*

(2) *for every $1 \le j \le n$, if $G_j = \langle K, A_j, \cdots, A_n\rangle$, then A_j is a semisimple $\mathbb{Z}G_j$–module; and*

(3) *$A_j \cap G_j = \langle 1\rangle$.*

In particular, $G = L \leftthreetimes K$, where $L = A_1 \cdots A_n$ is a normally factorized subgroup and K is finite.

In the paper [66], the description of the nilpotent and abelian–by–nilpotent FG–factorized groups was achieved. Later on, in the paper [70], D.I. Zaitsev significantly detailed the description of the $\mathfrak{F}$–factorized and the FG–factorized groups. He was able to reduce these problems to the following case: the group $G = A \leftthreetimes L$, where A is a semisimple $\mathbb{Z}G$–module and the group L has a series of normal subgroups

$$\langle 1\rangle \le F \le K \le L$$

with F and L/K finite and there exists an easy detailed description of K. The case in which all normal subgroups are complemented or $\mathfrak{X}$–complemented is quite seldom. Nevertheless, the search of complements and supplements to normal subgroups is very important. One of the first results here was obtained by D.I. Zaitsev in the paper [67].

If A is an abelian group, then we put $\Lambda_n(A) = \{a \in A \mid a^n = 1\}$.

Theorem 38. *Let G be a group and A be an abelian normal subgroup of G. Suppose that H is a subgroup of G satisfying the following conditions:*

(i) *The index $m = |G : H|$ is finite;*

(ii) *H includes A and moreover $H = AC$ for some subgroup C; and*

(iii) *$A \cap C \leq \Lambda_n(A)$ for some positive integer n.*

If $A = A^m$, then G has a subgroup K such that $G = AK$ and $A \leq K \leq \Lambda_{nm}(A)$.

The next results appeared in [67].

Corollary 15. *Let G be a locally finite group and A be a Chernikov normal subgroup of G. Then A has an $\mathfrak{F}$–complement in G if and only if every Sylow p–subgroup of A has an $\mathfrak{F}$–complement in every Sylow p–subgroup of G.*

Corollary 16. *Let G be a locally finite group.*

(1) *If the Sylow p–subgroups of G are abelian, then every Chernikov normal subgroup of G has an $\mathfrak{F}$–complement in G;*

(2) *if the Sylow p–subgroups of G are abelian–by–finite, then every Chernikov normal subgroup of G has an $\mathfrak{F}$–complement in G; and*

(3) *if the Sylow p–subgroups of G are finite–by–abelian, then every Chernikov normal subgroup of G has an $\mathfrak{F}$–complement in G.*

Other results are intended to search complements to the most regular residuals of a group. These results fall in two types: the first refers to the search of complements in some extensions of the class of finite groups, while the others consider the case in which this residual satisfies some finiteness condition. As an example of the first type results we can bring a generalization of the following result about polycyclic–by–finite groups. It is known that every polycyclic–by–finite group G has a normal subgroup K such that $[K, K]$ is nilpotent. A. Learner in the paper [38] reinforced this claim by proving that K is a product of $[K, K]$ and some nilpotent subgroup. One of the class of groups extensions of the class of polycyclic groups is the class of soluble A_3–groups or groups having a finite subnormal series whose factors are abelian that either are Chernikov groups or torsion-free locally cyclic groups (A.I. Maltsev [41]). For these groups, D.I. Zaitsev proved in the paper [69] the following result.

Theorem 39. *Let G be a soluble A_3–group. Then G has a normal subgroup H of finite index such that $H = KL$, where the subgroups K and L are nilpotent and K is normal in G.*

In the same article by D.I. Zaitsev [69] and in another paper by J.C. Lennox and D.J.S. Robinson [39], the following generalization of this theorem was obtained.

Theorem 40. *Let G be a group and H be a normal soluble A_3–subgroup of G. If G/H is nilpotent–by–finite, then G has a nilpotent subgroup K such that HK has finite index in G.*

An entire series of D.I. Zaitsev's papers [71, 73, 75, 77, 78, 79] is dedicated to problems of the existence of supplements to some known residuals. Here we deal with the case when this residual is abelian and satisfies some finiteness conditions. In a natural way this residual can be considered as a module. The problem of the existence of supplements logically becomes linked to some direct decompositions of modules. This is a separate topic, which, of course, also was initiated by D.I. Zaitsev. Since then, it was substantially expanded and extended (see the book [29, Chapter 10]).

Theorem 41. *Let G be a group and A be an abelian normal subgroup of G. Suppose that the following conditions hold:*

(i) *The $\mathbb{Z}G$–module A is noetherian;*

(ii) *the additive group of every non-zero factor-module of A is not cyclic; and*

(iii) *G/A is locally supersoluble and $G/C_G(A)$ is hypercyclic.*

Then G splits conjugately over A.

If R is a ring and G is a group, we denote by *ωRG the augmentation ideal of the group ring RG.*

Theorem 42. *Let G be a group and A be an abelian normal subgroup of G. Suppose that the following conditions hold:*

(i) *The $\mathbb{Z}G$–module A is noetherian;*

(ii) *$A = A(\omega\mathbb{Z}(G/A))$; and*

(iii) *G/A is locally nilpotent and $G/C_G(A)$ is hypercentral.*

Then G splits conjugately over A.

Theorem 43. *Let G be a group and A be an abelian normal subgroup of G. Suppose that the following conditions hold:*

(i) *The $\mathbb{Z}G$–module A is noetherian;*

(ii) *every non-zero factor-module of A is infinite; and*

(iii) *G/A is locally soluble and hyperfinite.*

Then G splits conjugately over A.

These statements have a dual statement.

Theorem 44. *Let G be a group and A be an abelian normal subgroup of G. Suppose that the following conditions hold:*

(i) *The $\mathbb{Z}G$–module A is artinian;*

(ii) *the additive group of every non-zero submodule of A is not cyclic; and*

(iii) *G/A is locally supersoluble and $G/C_G(A)$ is hypercyclic.*

Then G splits conjugately over A.

Theorem 45. *Let G be a group and A be an abelian normal subgroup of G. Suppose that the following conditions hold:*

(i) *The $\mathbb{Z}G$–module A is artinian;*

(ii) *$C_A(G) = \langle 1 \rangle$; and*

(iii) *G/A is locally nilpotent and $G/C_G(A)$ is hypercentral.*

Then G splits conjugately over A.

Theorem 46. *Let G be a group and A be an abelian normal subgroup of G. Suppose that the following conditions hold:*

(i) *The $\mathbb{Z}G$–module A is artinian;*

(ii) *every non-zero submodule of A is infinite; and*

(iii) *G/A is locally soluble and hyperfinite.*

Then G splits conjugately over A.

At this point, it is worth mentioning two results by D.J.S. Robinson [49, 51]; they also have dual statements.

Theorem 47. *Let G be a group, H a subgroup of G and A be a $\mathbb{Z}G$–module. Suppose that the following condition holds:*

(i) *A is a noetherian $\mathbb{Z}G$–module;*

(ii) *H is a normal subgroup of G;*

(iii) *H is hypercentral;*

(iv) *the upper FC–hypercenter of $G/C_G(A))$ includes $HC_G(A)/C_G(A)$; and*

(v) *$A = A(\omega\mathbb{Z}H)$.*

Then every extension E of A by G splits conjugately over A.

The concept of *upper FC–hypercenter* can be found in [29].

Theorem 48. *Let G be a group, H a subgroup of G and A be a $\mathbb{Z}G$–module. Suppose that the following condition holds:*

(i) *A is an artinian $\mathbb{Z}G$–module;*

(ii) *H is a normal subgroup of G;*

(iii) *H is locally nilpotent;*

(iv) *the upper FC–hypercenter of $G/C_G(A))$ includes $HC_G(A)/C_G(A)$; and*

(v) *$C_A(H) = \langle 1 \rangle$.*

Then every extension E of A by G splits conjugately over A.

We again ote that the second result was proved by D.J.S. Robinson using some methods of homology theory. A purely group-theoretical proof of this theorem has been presented in the book [29, Chapter 17]).

In the paper [30], some further extensiond were shown. Let G be a group and $B \leq C$ be normal subgroups of G. The factor C/B is said to be *G–central* if $G = C_G(C/B)$; otherwise, C/B is called *G–eccentric*. If A is a normal subgroup of G, *the upper G–central series of A*,

$$\langle 1 \rangle = A_0 \leq A_1 \leq \cdots \leq A_\alpha \leq A_{\alpha+1} \leq \cdots A_\gamma = A$$

is defined by the following rules: $A_1 = A \cap \zeta(G)$ and $A_{\alpha+1}/A_\alpha = A/A_\alpha \cap \zeta(G/A_\alpha)$ for every $\alpha < \gamma$. The subgroup A_γ is said to be *the upper G–hypercenter of A*. The subgroup A is said to be *G–hypercentral* if $A = A_\gamma$.

Let once more A be a normal subgroup of G. It is said that A is *G–hypereccentric* if A has an ascending series of G–invariant subgroups

$$\langle 1 \rangle = D_0 \leq D_1 \leq \cdots \leq D_\alpha \leq D_{\alpha+1} \leq \cdots D_\delta = A$$

whose factors are G–eccentric and G–chief.

Finally, we recall that a normal subgroup L of a group G is said to be *G–hyperfinite* if L has an ascending series

$$\langle 1 \rangle = L_0 \leq L_1 \leq \cdots \leq L_\alpha \leq L_{\alpha+1} \leq \cdots L_\gamma = L$$

of G–invariant subgroups whose factors are finite.

Theorem 49. *Let G be a group and A be an abelian normal subgroup of G. Suppose that G has an ascendant subgroup $H \geq A$ such that H/A is finitely generated nilpotent. If A is H–hyperfinite and H–hypereccentric, then G conjugately splits over A.*

5. Groups represented as the product of two proper subgroups

At the end of his life, D.I. Zaitsev began to consider topics related to the study of groups that can be represented as a product of two proper subgroups. As in other fields of his work, he obtained there remarkable and deep results, outlined a new approach that stimulated further fruitful development of this area. Unfortunately his premature death prevented him from executing his plans and solving the problems, the ways of reaching solutions of which he outlined. We do not intend here to review the theory of factorizable groups, we just want to show some important results obtained by D.I. Zaitsev in this area and trace their further development. We have already mentioned (see Theorem 39) that D.I Zatsev had obtained some natural factorizations of soluble A_3–groups. In this setting, D.I. Zaitsev initiated to the so-called *inverse problem of factorization.*

Let G be a group, A B be proper subgroups of G such that $G = AB$. The mentioned problem could formulate in general as follows: *what can we say about the structure of the group G?* For example, if A and B belong to some given class of groups $\mathfrak{X}$, is there another class of groups $\mathfrak{Y}$ (which should be near to $\mathfrak{X}$) such that $G \in \mathfrak{Y}$. A theorem by N. Ito [15] was the starting point for these investigations and for D.I. Zaitsev's research. Ito proved that a group which is a product of two abelian subgroups is metabelian. To invert this situation, the factorization problem can be detailed as follows: *what can be said about the properties of the abelian subgroups A and B that can be extended to the group $G = AB$?* This question was considered in the fundamental paper by D.I. Zaitsev [72] . D.I. Zaitsev paid attention to the following aspect. Suppose that a group G is the product of two subgroups A and B. If H is a normal subgroup of G, then

$$AH \cap BH = H(A \cap BH) = H(B \cap AH) = (A \cap BH)(B \cap AH).$$

Frequently many questions about factorized groups can be reduced to the groups of the form $G = AB = AL = BL$, where L is a normal subgroup of G. If L is abelian, then $A \cap L$ and $B \cap L$ are normal subgroups of G and so is the subgroup $D = (A \cap L)(B \cap L)$. Thus the factor-group G/D is a semidirect product of its abelian normal subgroup L/D by the image of A or B. Such a representation makes it possible to use modular techniques for factorization, which was effectively proved in [72].

Theorem 50. *Let G be a group and A and B abelian subgroups of G such that $G = AB$. Then we have.*

(1) *If $r_{\mathbb{Z}}(A)$ and $r_{\mathbb{Z}}(B)$ are finite, then $r_{hz}(G)$ is finite and $r_{hz}(G) \leq r_{\mathbb{Z}}(A) + r_{\mathbb{Z}}(B) - r\mathbb{Z}(A \cap B)$;*

(2) *if $r_p(A)$ and $r_p(B)$ are finite for every prime p, then $r_p(G)$ is also finite for every prime p;*

(3) *if A and B have finite special rank, then so have G; and*

(4) *if $r_{mm}(A)$ and $r_{mm}(B)$ are finite, then $r_{mm}(G)$ is finite and we have that $rmm(G) \leq r_{mm}(A) + r_{mm}(B) - r_{mm}(A \cap B)$.*

Corollary 17. *Let G be a group and A and B abelian subgroups of G such that $G = AB$.*

(1) *If A and B are periodic, then G is locally finite (O.H. Kegel);*

(2) *if A and B are periodic divisible and the Sylow p–subgroups of A and B are Chernikov for every prime p, then G is abelian;*

(3) *if A and B are isomorphic to the additive group of rational numbers, then G is abelian;*

(4) *if A and B are Chernikov subgroups, then G is Chernikov (N.F. Sesekin [52]); and*

(5) *if A and B are finitely generated, then G is polycyclic (N.F. Sesekin [53]).*

In the paper [74], the results of abelian factorizations were supplemented with the following important theorem.

Theorem 51. *Let G be a group and A and B be periodic abelian subgroups of G such that $G = AB$. Then we have.*

(1) *If π is a set of primes, A_π is the Sylow π–subgroup of A and B_π that of B, then $A_\pi B_\pi$ is the Sylow π–subgroup of G; and*

(2) *the family $\{G_p = A_p B_p \mid p \text{ prime}\}$ is a Sylow basis of G.*

In this setting, it should be mentioned that a periodic metabelian group need not to have a Sylow basis.

The inverse problem of the factorization and the theorem of N. Ito naturally lead to the following question. Suppose that the subgroups A B are similar in some way to abelian subgroups. *How a group $G = AB$ is close to a soluble group?* One of the first results in this direction was obtained by N.S. Chernikov [3].

Theorem 52. *Let G be a group and A and B central–by–finite subgroups of G such that $G = AB$. Then G is soluble–by–finite.*

Since the derived subgroup of a central–by–finite group is finite, then the above result leads to the following question. Suppose that the subgroups A B have finite derived subgroups. *Is $G = AB$ soluble-by–finite?* This problem continued in the list of research's interests of D.I. Zaitsev until the end of his life, but its solution has not been found. That is, it is not solved yet although D.I. Zaitsev [76] obtained some results in this direction.

Theorem 53. *Let G be a group and A and B subgroups of G such that $G = AB$. Suppose that A is abelian and B is an FC–group. If $\zeta(B) \neq \langle 1 \rangle$, then G has a non-identity normal subgroup K that is an FC–group. Moreover K is either abelian or finite or contained in B.*

Corollary 18. *Let G be an infinite group and A and B subgroups of G such that $G = AB$. Suppose that A is abelian and B is an FC–group. If $\zeta(B) \neq \langle 1 \rangle$, then G is not simple.*

Theorem 54. *Let G be a group and A and B subgroups of G such that $G = AB$. Suppose that A is abelian and B is an FC–group. If B is hypercentral–by–finite (B hypercentral), then G is hyperabelian–by–finite (G hyperabelian).*

Theorem 55. *Let G be a group and A and B subgroups of G such that $G = AB$. Suppose that A is abelian, B is an FC–group and $[B, B]$ is Chernikov. Then G is soluble–by–finite.*

Theorem 56. *Let G be a group and A and B subgroups of G such that $G = AB$. Suppose that A is abelian and B has finite derived subgroup. Then G is soluble–by–finite.*

D.I. Zaitsev also obtained important results on factorizations of some generalized soluble groups. Since the topic of factorization of soluble groups is a very advanced subject area and its results are published in some surveys and books, we shall not discuss them here.

References

[1] R. Baer, *Local and global hypercentrality and supersolubility I, II*, Indagationes mathematical N.**28**, 1966, pp. 93-126.

[2] R. Baer, *Polyminimaxgruppen*, Math. Ann. N.**175**, 1968, pp. 1-43.

[3] N.S. Chernikov, *On the products of almost abelian groups*, Ukrain. Math. J. N.**33**, 1981, pp. 136-138.

[4] S.N. Chernikov, *Infinite groups with given properties of their systems of infinite subgroups*, Doklady AN USSR N.**159** 1964, pp. 759-760.

[5] S.N. Chernikov, *Some condition of existence of complement to subgroups in supersoluble groups*, in *The group-theoretical researches*, Naukova Dumka, Kiev, 1978, pp. 8-15.

[6] M. Curzio, *Problemi di complementazione in theoria dei gruppi*, Symp. Math. 1967-1968, Gubbio N.**1** 1969, pp. 195-208.

[7] G. Cutolo, L.A. Kurdachenko, *Weak chain conditions for non-almost normal subgroups*, Groups 93 Galway/St.Andrews, Galway 1993, vol. 1, London Math. Soc., Lecture Notes Ser. N.**211**, 1995, pp. 120-130.

[8] M.R. Dixon, M.E. Evans, H. Smith, *Locally soluble-by-finite groups with the weak minimal conditions on non-nilpotent subgroups*, J. Algebra N.**249**, 2002, pp. 226-246.

[9] M.R. Dixon, L.A. Kurdachenko, J.M. Muñoz-Escolano, J. Otal, *Trends in infinite dimensional linear groups*, Groups St.Andrews 2009 in Bath, vol. 1, London Math. Soc. Lecture Notes Ser. N.**387**, 2011, pp. 271-282.

[10] M.R. Dixon, L.A. Kurdachenko, N.V. Polyakov, *On some ranks of infinite groups*, Ricerche Mat. N.**56** 2007, pp. 43-59.

[11] M.R. Dixon, I.Ya. Subbotin, *Groups with finiteness conditions on some subgroup systems: a contemporary stage*, Algebra and discrete Math. N.**4**, 2009, pp. 29-54.

[12] P. Hall, *Complemented groups*, J. London Math. Soc. N.**12**, 1937, pp. 201-204.

[13] P. Hall, C.R. Kulatilaka, *A property of locally finite groups*, J. London Math. Soc. N.**39**, 1964, pp. 235-239.

[14] K.A. Hirsch, *On infinite soluble groups I*, Proc. London Math. Soc. N.**44** 1938, pp. 53-60.

[15] N. Ito, *Über das Produkt von zwei abelschen Gruppen*, Math. Z. N.**62** 1955, pp. 400-401.

[16] B. Hartley, *Uncountable artinian modules and uncountable soluble groups satisfying Min-n*, Proc. London Math. Soc. N.**35**, 1977, pp. 55-75.

[17] M. Karbe, *Unendliche Gruppen mit schwachen Kettenbedingungen fur endlich erzeugte Untergruppen*, Arch. Math. N.**45**, pp. 97-110.

[18] M. Karbe, L.A. Kurdachenko, *Just infinite modules over locally soluble groups*, Arch. Math. N.**51**, 1988, pp. 401-411.

[19] M.I. Kargapolov, *On a problem of O.Yu. Schmidt*, Sibir. Math. J. N.**4**, 1963, pp. 232-235.

[20] L.S. Kazarin, L.A. Kurdachenko, I.Ya. Subbotin, *Groups saturated with abelian subgroups*, International J. Algebra and Computation N.**8**, 1998, pp. 443c466.

[21] L.A. Kurdachenko, *The groups satisfying the weak minimal and maximal conditions for normal subgroups*, Sibir. Math. J. N.**20**, 1979, pp. 1068-1075.

[22] L.A. Kurdachenko, *Locally nilpotent groups with the weak minimal condition for normal subgroups*, Sibir. Math. J. N.**25**, 1984, pp. 589-594.

[23] L.A. Kurdachenko, *Locally nilpotent groups with the weak minimal and maximal comditions for normal subgroups*, Doklady AN Ukrain. SSR. N.**8**, 1985, pp. 9-12.

[24] L.A. Kurdachenko, *The locally nilpotent groups with condition Min–∞–n*, Ukrain. Math. J. N.**42**, 1990, 303-307.

[25] , L.A. Kurdachenko, *On some classes of groups with the weak minimal and maximal conditions for normal subgroups*, Ukrain. Math. J. N.**42**, 1990, pp. 1050-1056.

[26] L.A. Kurdachenko, *Artinian modules over groups of finite rank and the weak minimal condition for normal subgroups*, Ricerche Mat. N.**44**, 1995, pp. 303-335.

[27] L.A. Kurdachenko, V.E. Goretsky, *Groups with weak minimal and maximal conditions for subgroups that are not normal*, Ukrain. Math. J. N.**41**, 1989, pp. 1705-1709.

[28] , L.A. Kurdachenko, J. Otal, I.Ya. Subbotin, *Groups with prescribed quotient groups and associated module theory*, World Scientific, Singapore, 2002.

[29] L.A. Kurdachenko, J. Otal, I.Ya. Subbotin, *Artinian modules over group rings*, Birkhauser, Basel, 2007.

[30] L.A. Kurdachenko, J. Otal, I.Ya. Subbotin, *Some criteria for existence of supplements to normal subgroups and their applications*, International J. Algebra and Computation N.**20**, 2010, pp. 689-719.

[31] L.A. Kurdachenko, N.N. Semko, *Groups with the weak maximal condition on the non-nilpotent subgroups*, Ukrain. Math. J. N.**58**, 2006, pp. 1068-1083.

[32] L.A. Kurdachenko, P. Shumyatsky, I.Ya. Subbotin, *Groups with many nilpotent subgroups* Algebra Colloquium N.**8**, 2001, pp. 129-143.

[33] L.A. Kurdachenko, H. Smith, *Groups with the weak minimal condition for non-subnormal subgroups*, Annali Mat. N.**173**, 1997, 299-312.

[34] L.A. Kurdachenko, H. Smith, *Groups with the weak maximal condition for non-subnormal subgroups*, Ricerche Mat. N.**47**, 1998, pp. 29-49.

[35] L.A. Kurdachenko, H. Smith, *Groups with the weak minimal conditions for non-subnormal subgroups II*, Comm. Math. Univ. Carolinae N.**46**, 2005, pp. 271-278.

[36] L.A. Kurdachenko, A.V. Tushev, *Soluble groups of class two satisfying weak minimal condition for normal subgroups*, Ukrain. Math. J. N.**37**, 1985, pp. 300-306.

[37] L.A. Kurdachenko, A.V. Tushev, *On some class of groups satisfying weak minimal condition for normal subgroups*, Ukrain. Math. J. N.**37**, 1985, pp. 457-462.

[38] A. Learner, *The embedding of a class of polycyclic groups*, Proc. London. Math. Soc. N.**12**, 1962, pp. 496-510.

[39] J.C. Lennox, D.J.S. Robinson, *Soluble products of nilpotent groups*, Rend. Semin. Math. Univ. Padova. N.**62**, 1980, pp. 261-280.

[40] A.I. Maltsev, *On groups of finite rank*, Mat. Sb. N.**22**, 1948, pp. 351-352.

[41] A.I. Maltsev, *On certain classes of infinite soluble groups*, Mat. Sb. N.**28**, 1951, pp. 567-588 = English translation: Amer. Math. Soc. Translations N.**2**, 1956, pp. 1-21.

[42] Yu.I. Merzlyakov, *On locally soluble groups of finite rank*, Algebra i logika N.**3**, 1964, pp. 5-16.

[43] Yu.I. Merzlyakov, *On locally soluble groups of finite rank II*, Algebra i logika N.**8**, 1969, pp. 686-690.

[44] J.M. Muñoz-Escolano, J. Otal, N.N. Semko, *The structure of infinite dimensional linear groups satisfying certain finiteness conditions* Algebra and discrete Math. . N.**4**, 2009, pp. 120-134.

[45] A.Yu. Ol'shanskij, *Geometry of defining relations in groups*, Kluwer Acad. Publ., Dordrecht, 1991.

[46] H. Prüfer, *On Theorie der Abelschen Gruppen*, Math. Z. N.**20**, 1924, pp. 165-187.

[47] D.J.S. Robinson, *Infinite soluble and nilpotent groups*, Queen Mary College Mathematics Notes, London, 1968.

[48] D.J.S. Robinson, *A note on groups of finite rank*, Comp. Math. N.**31**, 1969, pp. 240-246.

[49] D.J.S. Robinson, *Splitting theorems for infinite groups*, Sympos. Mat. Ist. Naz. alta Mat. N.**17**, 1975, pp. 441-470.

[50] D.J.S. Robinson, *Applications of cohomology to the theory of groups*, London Math. Soc. Lecture Notes Ser. N.**71**, 1982, pp. 46-80.

[51] D.J.S. Robinson, *Cohomology of locally nilpotent groups*, J. pure applied Algebra N.**48**, 1987, pp. 281-300.

[52] N.F. Sesekin, *On the product of finitely related Abelian groups*, Siberian Math. J. . N.**6**, 1968, pp. 1427-1429.

[53] N.F. Sesekin, *On the product of finitely generated Abelian groups*, Math. Notes N.**13**, 1973, pp. 443-446.

[54] D.I. Zaitsev, *Stable nilpotent groups*, Math. Notes N.**2**, 1967, pp. 337-346.

[55] D.I. Zaitsev, *Groups satisfying the weak minimal condition*, Doklady AN USSR N.**178**, 1968, pp. 780-782.

[56] D.I. Zaitsev, *Groups satisfying the weak minimal condition*, Ukrain Math. J. N.**20**, 1968, pp. 472-482.

[57] D.I. Zaitsev, *On existence of stable nilpotent subgroups in locally nilpotent groups*, Math. Notes N.**4**, 1968, pp. 361-368.

[58] D.I. Zaitsev, *Stable soluble groups*, Izv. AN USSR, ser. Math. N.**33**, 1969, pp. 765-780.

[59] D.I. Zaitsev, *On groups satisfying weak minimal condition*, Math. Sb. N.**78**, 1969, 232-331.

[60] D.I. Zaitsev, *To the theory of minimax groups*, Ukrain Math. J. N.**23**, 1971, pp. 652-660.

[61] D.I. Zaitsev, *Groups satisfying the weak minimal condition for non-abelian subgroups*, Ukrain Math. J. N.**23**, 1971, pp. 661-665.

[62] D.I. Zaitsev, *On soluble groups of finite rank*, in *The groups with the restrictions for subgroups*, Naukova Dumks Kiev, 1971, pp. 115-130.

[63] D.I. Zaitsev, *Normal factorized groups*, in *The groups with systems of complemented subgroups*, Math. Inst. Kiev, 1972, pp. 5-34.

[64] D.I. Zaitsev, *To the theory of normal factorized groups*, in *The groups with prescribed properties of subgroups*, Math Inst. Kiev, 1973, pp. 78-104.

[65] D.I. Zaitsev, *On the complementations of the subgroups in extremal groups*, in *Investigations of the groups on prescribed properties of subgroups*, Math. Inst. Kiev, 1974, pp. 72-130.

[66] D.I. Zaitsev, *The groups with the complemented normal subgroups*, in *Some problems of group theory*, Math. Inst. Kiev, 1975, pp. 30-74.

[67] D.I. Zaitsev, *On groups with the complemented normal subgroups*, Algebra and Logic N.**14**, 1975, pp. 5-14.

[68] D.I. Zaitsev, *On the existence of direct complements in the groups with operators*, in *Researches in group theory*, Math. Inst. Kiev, 1976, pp. 26-44.

[69] D.I. Zaitsev, *On soluble groups of finite rank*, Algebra i logika N.**16**, 1977, pp. 300-312.

[70] D.I. Zaitsev, *On soluble groups with complemented normal subgroups*, in *Group theoretical researches*, Math. Inst. Kiev, 1978, pp. 77-86.

[71] D.I. Zaitsev, *The hypercyclic extensions of abelian groups*, in *The groups defined by the properties of systems of subgroups*, Math. Inst. Kiev, 1979, pp. 16-37.

[72] D.I. Zaitsev, *The products of abelian groups*, Algebra i logika N.**19**, 1980, pp. 94-106.

[73] D.I. Zaitsev, *On extensions of abelian groups*, in *The constructive describtion of groups with prescribed properties of subgroups*, Math. Inst. Kiev, 1980, pp. 16-40.

[74] D.I. Zaitsev, *The groups of operators of finite rank and their applications*, in the book *VI Symposium on group theory*, Naukova Dumka, Kiev, 1980, pp. 22-37.

[75] D.I. Zaitsev, *On splitting of extensions of the abelian groups*, in *The investigations of groups with the prescribed properties of systems of subgroups*, Math. Inst. Kiev, 1981, pp. 14-25.

[76] D.I. Zaitsev, *Theorem of Ito and products of groups*, Math. Notes N.**33**, 1983, pp. 807-818.

[77] D.I. Zaitsev, *The splitting extensions of abelian groups* in *The structure of groups and the properties of its subgroups*, Math. Inst. Kiev, 1986, pp. 22-31.

[78] D.I. Zaitsev, *The hyperfinite extensions of abelian groups*, in *The investigations of groups with the restrictions on subgroups*, Math. Inst. Kiev, 1988, pp. 17-26.

[79] D.I. Zaitsev, *On locally supersoluble extensions of abelian groups*, Ukrain. Math. J. N.**42**, 1990, pp. 908-912.

[80] D.I. Zaitsev, L.A. Kurdachenko, *Groups with the maximal condition on non-abelian subgroups*, Ukrain Math. J. N.**43**, 1991, pp. 925-930.

[81] D.I. Zaitsev, L.A. Kurdachenko, *The weak minimal and maximal conditions for subgroups in groups*, Preprint Math. Inst. Kiev, 1975, 52 pp.

[82] D.I. Zaitsev, L.A. Kurdachenko, A.V. Tushev, *The modules over nilpotent groups of finite rank*, Algebra and logic N.**24**, 1985, pp. 412-436.

Contact information

V.V. Kirichenko
Department of Mechanics and Mathematics, Kyiv National Taras Shevchenko University, Volodymyrska, 64, Kyiv, 01033, Ukraine
E-Mail: vkir@univ.kiev.ua

L.A. Kurdachenko
Department of Algebra, National University of Dnepropetrovsk, Vul. Naukova 13, Dnepropetrovsk 50, Ukraine 49050
E-Mail: lkurdachenko@gmail.com

J. Otal
Department of Mathemathics-IUMA, University of Zaragoza, 50009 Zaragoza, Spain
E-Mail: otal@unizar.es

I.Ya. Subbotin
Department of Mathematics and Natural Sciences, National University, 5245 Pacific Concourse Drive, LA, CA 90045, USA
E-Mail: isubboti@nu.edu

Received by the editors: 01.02.2012
and in final form 01.02.2012.