Algebra and Discrete Mathematics RESEARCH ARTICLE
Volume **19** (2015). Number 2, pp. 254–269

Ultrafilters on G-spaces

O. V. Petrenko, I. V. Protasov

Abstract. For a discrete group G and a discrete G-space X, we identify the Stone-Čech compactifications βG and βX with the sets of all ultrafilters on G and X, and apply the natural action of βG on βX to characterize large, thick, thin, sparse and scattered subsets of X. We use G-invariant partitions and colorings to define G-selective and G-Ramsey ultrafilters on X. We show that, in contrast to the set-theoretical case, these two classes of ultrafilters are distinct. We consider also universally thin ultrafilters on ω, the T-points, and study interrelations between these ultrafilters and some classical ultrafilters on ω.

By a G-space, we mean a set X endowed with the action $G \times X \to X : (g, x) \mapsto gx$ of a group G. All G-spaces are supposed to be transitive: for any $x, y \in X$, there exists $g \in G$ such that $gx = y$. If $X = G$ and the action is the group multiplication, we say that X is a regular G-space.

Several intersting and deep results in combinatorics, topological dynamics and topological algebra, functional analysis were obtained by means of ultrafilters on groups (see [5–7, 12, 27, 28]).

The goal of this paper is to systematize some recent and prove some new results concerning ultrafilters on G-spaces, and point out the key open problems.

In sections 1,2 and 3, we keep together all necessary definitions of filters, ultrafilters and the Stone-Čech compactification βX of the discrete space X. We extend the action of G on X to the action of βG on βX, characterize the minimal invariant subsets of βX, define the corona $\check{X}$ of X and the ultracompanions of subsets of X.

2010 MSC: 05D10, 22A15, 54H20.

Key words and phrases: G-space, ultrafilters, ultracompanion, G-selective ultrafilter, G-Ramsey ultrafilter, T-point, ballean, asymorphism.

In section 4, we give ultrafilter charecterizations of large, thick, thin, sparse and scattered subsets of X.

In section 5, we use G-invariant partitions and colorings to define G-selective and G-Ramsey ultrafilters on X, and show that, in contrast to the set-theoretical case, these two classes are essentially different.

In section 6, we use countable group of permutatious of $\omega = \{0, 1, \dots\}$ to define thin ultrafilters on ω. We prove that some classical ultrafilters on ω (for example, P- and Q-points) are thin ultrafilters.

We conclude the paper, showing in section 7, how all above results can be considered and interpreted in the frames of general asymptology.

1. Filters and ultrafilters

A family $\mathcal{F}$ of subsets of a set X is called a *filter* if $X \in \mathcal{F}, \varnothing \notin \mathcal{F}$ and

$$A, B \in \mathcal{F}, A \subseteq C \Rightarrow A \cap B \in \mathcal{F}, C \in \mathcal{F}$$

The family of all fillters on X is partially ordered by inclusion $\subseteq$. A filter $\mathcal{U}$ that is maximal in this ordering is called an *ultrafilter*. Equivalentely, $\mathcal{U}$ is ultrafilter if $A \cup B \in \mathcal{U}$ implies $A \in \mathcal{U}$ or $B \in \mathcal{U}$. This characteristic of ultrafilters plays the key role in the Ramsey Theory: to prove that, under any finite partition of X, at least one cell of the partition has a given property, it suffices to construct an ultrafilter $\mathcal{U}$ such that each member of $\mathcal{U}$ has this property.

An ultrafilter $\mathcal{U}$ is called *principal* if $\{x\} \in \mathcal{U}$ for some $x \in X$. Non-principal ultrafilters are called *free* and the set of all free ultrafilters on X is denoted by X^*.

We endow a set X with the discrete topology. The Stone-Čech compactification βX of X is a compact Hausdorff space such that X is a subspace of βX and any mapping $f : X \to Y$ to a compact Hausdorff space Y can be extended to the continuous mapping $f^\beta : \beta X \to Y$. To work with βX, we take the points of βX to be the ultrafilters on X, with the points of X identified with the principal ultrafilters, so $X^* = \beta X \setminus X$.

The topology of βX can be defined by stating that the sets of the form $\overline{A} = \{p \in \beta X : A \in p\}$, where A is a subset of X, are base for the open sets. For a filter φ on X, the set $\overline{\varphi} = \{\overline{A} : A \in \varphi\}$ is closed in βX, and each non-empty closed subset of βX is of the form $\overline{\varphi}$ for an appropriate filter φ on X.

2. The action of βG on βX

Given a G-space X, we endow G and X with the discrete topologies and use the universal property of the Stone-Čech compactification to define the action of βG on βX.

Given $g \in G$, the mapping $x \mapsto gx : X \to \beta X$ extends to the continuous mapping

$$p \mapsto gp : \beta X \to \beta X.$$

We note that $gp = \{gP : P \in p\}$, where $gP = \{gx : x \in P\}$.

Then, for each $p \in \beta X$, we extend the mapping $g \mapsto gp : G \to \beta X$ to the continuous mapping

$$q \mapsto qp : \beta G \to \beta X.$$

Let $q \in \beta G$ and $p \in \beta X$. To describe a base for the ultrafilter $qp \in \beta X$, we take any element $Q \in q$ and, for every $g \in Q$, choose some element $P_g \in p$. Then $\bigcup_{g \in Q} gP_g \in qp$ and the family of subsets of this form is a base for qp.

By the construction, for every $g \in G$, the mapping $p \mapsto gp : \beta X \to \beta X$ is continuous and, for every $p \in \beta X$, the mapping $q \mapsto qp : \beta G \to \beta X$ is continuous. In the case of the regular G-space X, $X = G$, we get well known (see [7]) extention of multiplication from G to βG making βG a compact right topological semigroup. For plenty applications of the semigroup βG to combinatorics and topological algebra see [6, 7, 12, 28]. It should be marked that, for any $q, r \in \beta G$, and $p \in \beta X$, we have $(qr)p = q(rp)$ so semigroup βG acts on βX.

Now we define the main technical tool for study of subsets of X by means of ultrafilters.

Given a subset A of X and an ultrafilter $p \in \beta X$ we define the *p-companion* of A by

$$A_p = \{\overline{A} \cap Gp\} = \{gp : g \in G, A \in gp\}.$$

Systematically, p-companions will be used in section 4. Here we demonstrate only one appication of p-companion to characterize minimal invariant subsets of βX. A closed subset S of βX is called *invariant* if $g \in G$ and $p \in S$ imply $gp \in S$. Clearly, S is invariant if and only if $(\beta G)p \subseteq S$ for each $p \in S$. Every invariant subset S of βX contains minimal by inclusion invariant subset. A subset M is minimal invariant if and only if $M = (\beta G)p$ for each $p \in S$. In the case of the regular G-space, the minimal invariant subsets coincide with minimal left ideals of βG so the following theorem generalizes Theorem 4.39 from [7].

Theorem 2.1. *Let X be a G-space and let $p \in \beta X$. Then $(\beta G)p$ is minimal invariant if and only if, for every $A \in p$, there exists a finite subset F of G such that $G = FA_p$.*

Proof. We suppose that $(\beta G)p$ is a minimal invariant subset and take an arbitary $r \in \beta G$. Since $(\beta G)rp = (\beta G)p$ and $p \in (\beta G)p$, there exists $q_r \in \beta G$ such that $q_r rp = p$. Since $A \in q_r rp$, there exists $x_r \in G$ such that $A \in x_r rp$ so $x_r^{-1}A \in rp$. Then we choose $B_r \in r$ such that $\overline{x_r^{-1}A} \supseteq \overline{B_r}p$ and consider the open cover $\{\overline{B_r} : r \in \beta G\}$ of βG. By compactness of βG, there is its finite subcover $\{\overline{B_{r_1}}, \dots, \overline{B_{r_n}}\}$, so $G = B_{r_1} \cup \ldots \cup B_{r_n}$. We put $F^{-1} = \{x_{r_1}, \dots, x_{r_n}\}$. Then $G = (FA)_p$ and it suffices to observe that $(FA)_p = FA_p$.

To prove the converse statement, we suppose on the contrary that $(\beta G)p$ is not minimal and choose $r \in \beta G$ such that $p \notin (\beta G)rp$. Since $(\beta G)rp$ is closed in βX, there exists $A \in p$ such that $\overline{A} \cap (\beta G)rp = \varnothing$. It follows that $A \notin qrp$ for every $q \in \beta G$. Hence, $G \setminus A \in qrp$ for each $q \in \beta G$ and, in particular, $x(G \setminus A) \in rp$ for each $x \in G$. By the assumption, $gA_p \in r$ for some $g \in G$ so $A \in g^{-1}rp$, $gA \in rp$ and we get a contradiction. □

3. Dynamical equivalences and coronas

For an infinite discrete G-space, we define two basic equivalences on the space X^* of all free ultrafilter on X.

Given any $r, q \in X^*$, we say that r, q are *parallel* (and write $r \parallel q$) if there exists $g \in G$ such that $q = gr$. We denote by $\sim$ the minimal (by inclusion) closed in $X^* \times X^*$ equivalences on X^* such that $\parallel \subseteq \sim$. The quotient $X^*/\sim$ is a compact Hausdorff space. It is called the corona of X and is denoted by $\check{X}$.

For every $p \in X^*$, we denote by $\check{p}$ the class of the equivalence $\sim$ containing p, and say that $p, q \in X^*$ are corona equivalent if $\check{p} = \check{q}$. To detect whether two ultrafilters $p, q \in X^*$ are corona equivalent, we use G-slowly oscillating functions on X.

A function $h : X \to [0, 1]$ is called *G-slowly oscillating* if, for any $\varepsilon > 0$ and finite subset $K \subset G$, there exists a finite subset F of X such that

$$diam\ h(Kx) < \varepsilon,$$

for each $x \in X \setminus F$, where $diam\ h(Kx) = \sup\{|h(y) - h(z)| : y, z \in Kx\}$.

Theorem 3.1. *Let $q, r \in X^*$. Then $\check{q} = \check{r}$ if and only if $h^\beta(r) = h^\beta(q)$ for every G-slowly oscillating function $h : X \to [0, 1]$.*

For more detailed information on dynamical equivalences and topologies of coronas see [14] and [1,13,17,19].

In the next section, for a subset A of X and $p \in X^*$, we use the *corona p-companion* of A

$$A_{\check{p}} = A^* \cap \check{p}.$$

4. Diversity of subsets of G-spaces

For a set S, we use the standard notation $[S]^{<\omega}$ for the family of all finite subsets of S.

Let X be a G-space, $x \in X, A \subseteq X, K \in [G]^{<\omega}$. We set

$$B(x, K) = Kx \cup \{x\}, B(A, K) = \bigcup_{a \in A} B(a, K),$$

and say that $B(x, K)$ is a *ball of radius* K around x. For motivation of this notation, see the section 7.

Our first portion of definitions concerns the upward directed properties: $A \in \mathcal{P}$ and $A \subseteq B$ imply $B \in \mathcal{P}$.

A subset A of a G-space X is called

- *large* if there exists $K \in [G]^{<\omega}$ such that $X = KA$;
- *thick* if, for every $K \in [G]^{<\omega}$, there exists $a \in A$ such, that $Ka \subseteq A$;
- *prethick* if there exists $F \in [G]^{<\omega}$ such that FA is thick.

In the dynamical terminology [7], large and prethick subsets are known as syndedic and piecewise syndedic subsets.

Theorem 4.1. *For a subset A of an infinite G-space X, the following statements hold:*

(i) *A is large if and only if $A_p \neq \varnothing$ for each $p \in X^*$;*

(ii) *A is thick if and only if, there exists $p \in X^*$ such that $A_p = Gp$.*

Proof. (i) We suppose that A is large and choose $F \in [G]^{<\omega}$ such that $X = FA$. Given any $p \in X^*$, we choose $g \in F$ such that $gA \in p$. Then $A \in g^{-1}p$ and $A_p \neq \varnothing$.

To prove the converse statement, for every $p \in X^*$, we choose $g_p \in G$ such that $A \in g_p p$ so $g_p^{-1}A \in p$. We consider an open covering of X^* by the subsets $\{g_p^{-1}A^* : p \in X^*\}$ and choose its finite subcovering $g_{p_1}^{-1}A^*, \ldots, g_{p_n}^{-1}A^*$. Then the set $H = X \setminus (g_{p_1}^{-1}A^* \cup \ldots \cup g_{p_n}^{-1}A^*)$ is finite.

We choose $F \in [G]^{<\omega}$ such that $H \subset FA$ and $\{g_{p_1}^{-1}, \dots, g_{p_n}^{-1}\} \subset F$. Then $X = FA$ so A is large.

(ii) We note that A is thick if and only if $X \setminus A$ is not large and apply (i). □

Theorem 4.2. *A subset A of an infinite G-space X is prethick if and only if there exists $p \in X^*$ such that $A \in p$ and $(\beta G)p$ is a minimal invariant subsets of βX.*

Proof. The theorem was proved for regular G-spaces in [7, Theorem 4.40]. This proof can be easily adopted to the general case if we use Theorem 2.1 in place of Theorem 4.39 from [7]. □

Corollary 4.1. *For every finite partition of a G-space X, at least one cell of the partition is prethick.*

Remark 4.1. For a subset A of an infinite G-space X, we set

$$\Delta(A) = \{g \in G : g^{-1}A \cap A \text{ is infinite}\}.$$

Let $\mathcal{P}$ be a finite partition of X. We take $p \in X^*$ such that the set $(\beta G)p$ is minimal invariant and choose $A \in \mathcal{P}$ such that $A \in p$. By Theorem 2.1, A_p is large in G. If $g \in A_p$ then $g^{-1}A \in p$ and $A \in p$. Hence, $g^{-1}A \cap A$ is infinite, so $A_p \subseteq \Delta(A)$ and $\Delta(A)$ is large.

In fact, this statement can be essentially strengthened: there is a function $f : \mathbb{N} \to \mathbb{N}$ such that, for every n-partition $\mathcal{P}$ of a G-space X, there are $A \in \mathcal{P}$ and $F \subset G$ such that $G = F\Delta(A)$ and $|F| \leqslant f(n)$. This is an old open problem (see the surveys [2, 22] whether the above statement is true with $f(n) = n$).

In the second part of the section, we consider the downward directed properties $A \in \mathcal{P}$, $B \subseteq A$ imply $B \in P$) and present some results from [3, 23] A subset A of a G-space X is called

- *thin* if, for every $F \in [G]^{<\omega}$, there exists $K \in [X]^{<\omega}$, such that $B_A(a, F) = \{a\}$ for each $a \in A \setminus K$, where $B_A(a, F) = B(a, F) \cap A$;
- *sparse* if, for every infinite subset Y of X, there exists $H \in [G]^{<\omega}$ such that, for every $F \in [G]^{<\omega}$, there is $y \in Y$ such that $B_A(y, F) \setminus B_A(y, H) = \varnothing$;
- *scattered* if, for every infinite subset Y of X, there exists $H \in [G]^{<\omega}$, such that, for every $F \in [G]^{<\omega}$, there is $y \in Y$ such that $B_Y(a, F) \setminus B_Y(a, H) = \varnothing$.

Theorem 4.3. *For a subset A of a G-space X, the following statements hold:*

(i) *A is thin if and only if $|A_p| \leqslant 1$ for each $p \in X^*$;*

(ii) *A is sparse if and only if A_p is finite for every $p \in X^*$;*

Let $(g_n)_{n\in\omega}$ be a sequence in G and let $(x_n)_{n\in\omega}$ be a sequence in X such that

(1) $\{g_0^{\varepsilon_0} \dots g_n^{\varepsilon_n} x_n : \varepsilon_i \in \{0,1\}\} \cap \{g_0^{\varepsilon_0} \dots g_m^{\varepsilon_m} x_m : \varepsilon_i \in \{0,1\}\} = \varnothing$ for all distinct $m, n \in \omega$;

(2) $|\{g_0^{\varepsilon_0} \dots g_n^{\varepsilon_n} x_n : \varepsilon_i \in \{0,1\}\}| = 2^{n+1}$ for every $n \in \omega$.

We say that a subset Y of X is a *piecewise shifted FP-set* if there exist $(g_n)_{n\in\omega}$, $(x_n)_{n\in\omega}$ satisfying (1) and (2) such that

$$Y = \{g_0^{\varepsilon_0} \dots g_n^{\varepsilon_n} x_n : \varepsilon_n \in \{0,1\}, n \in \omega\}.$$

For definition of an FP-set in a group see [7].

Theorem 4.4. *For a subset A of a G-space X, the following statements are equivalent:*

(i) *A is scattered;*

(ii) *for every infinite subset Y of A, there exists $p \in Y^*$ such that Y_p is finite;*

(iii) *$A_p p$ is discrete in X^* for every $p \in X^*$;*

(iv) *A contains no piecewise shifted FP-sets.*

Theorem 4.5. *Let G be a countable group and let X be a G-space. For a subset A of X, the following statements hold:*

(i) *A is large if and only if $A_{\check{p}} \neq \varnothing$ for each $p \in X^*$;*

(ii) *A is thick if and only if $\check{p} \subseteq A^*$ for some $p \in X^*$;*

(iii) *A is thin if and only if $|A_{\check{p}}| \leqslant 1$ for each $p \in X^*$;*

(iv) *if $A_{\check{p}}$ is finite for each $p \in X^*$ then A is sparse;*

(v) *if, for every infinite subset Y of A, there is $p \in Y^*$ such that $Y_{\check{p}}$ is finite then A is scattered.*

Question 4.1. *Does the conversion of Theorem 4.5 (iv) hold?*

Question 4.2. *Does the conversion of Theorem 4.5 (v) hold?*

Remark 4.2. If G is an uncountable Abelian group then the corona $\check{G}$ is a singleton [13]. Thus, Theorem 4.5 does not hold (with $X = G$) for uncountable Abelian groups.

5. Selective and Ramsey ultrafilters on G-spaces

We recall (see [4]) that a free ultrafilter $\mathcal{U}$ on an infinite set X is said to be *selective* if, for any partition $\mathcal{P}$ of X, either one cell of $\mathcal{P}$ is a member of $\mathcal{U}$, or some member of $\mathcal{U}$ meets each cell of $\mathcal{P}$ in at most one point. Selective ultrafilters on ω are also known under the name *Ramsey ultrafilters* because $\mathcal{U}$ is selective if and only if, for each colorings $\chi : [\omega]^2 \to \{0,1\}$ of 2-element subsets of ω, there exists $U \in \mathcal{U}$ such that the restriction $\chi|_{[U]^2} \equiv const$.

Let G be a group, X be a G-space with the action $G \times X \to X$, $(g,x) \mapsto gx$. All G-spaces under consideration are supposed to be transitive: for any $x, y \in X$, there exists $g \in G$ such that $gx = y$. If $G = X$ and gx is the product of g and x in G, X is called a *regular G-space*. A partition $\mathcal{P}$ of a G-space X is *G-invariant* if $gP \in \mathcal{P}$ for all $g \in G$, $P \in \mathcal{P}$.

Let X be an infinite G-space. We say that a free ultrafilter $\mathcal{U}$ on X is *G-selective* if, for any G-invariant partition $\mathcal{P}$ of X, either some cell of $\mathcal{P}$ is a member of $\mathcal{U}$, or there exists $U \in \mathcal{U}$ such that $|P \cap U| \leqslant 1$ for each $P \in \mathcal{P}$.

Clearly, each selective ultrafilter on X is G-selective. Selective ultrafilters on ω exist under some additional to ZFC set-theoretical assumptions (say, CH), but there are models of ZFC with no selective ultrafilters on ω.

Let X be a G-space, $x_0 \in X$. We put $St(x_0) = \{g \in G : gx_0 = x_0\}$ and identify X with the left coset space $G/St(x_0)$. If $\mathcal{P}$ is a G-invariant partition of $X = G/S, S = St(x_0)$, we take $P_0 \in \mathcal{P}$ such that $x_0 \in P_0$, put $H = \{g \in G : gS \in P_0\}$ and note that the subgroup H completely determines $\mathcal{P}$: xS and yS are in the same cell of $\mathcal{P}$ if and only if $y^{-1}x \in H$. Thus, $\mathcal{P} = \{x(H/S) : x \in L\}$ where L is a set of representatives of the left cosets of G by H.

Theorem 5.1. *For every infinite G-space X, there exists a G-selective ultrafilter $\mathcal{U}$ on X in ZFC.*

Proof. We take $x_0 \in X$, put $S = St(x_0)$ and identify X with G/S. We choose a maximal filter $\mathcal{F}$ on G/S having a base consisting of subsets of the form A/S where A is a subgroup of G such that $S \subset A$ and $|A : S| = \infty$. Then we take an arbitrary ultrafilter $\mathcal{U}$ on G/S such that $\mathcal{F} \subseteq \mathcal{U}$.

To show that $\mathcal{U}$ is G-selective, we take an arbitrary subgroup H of G such that $S \subseteq H$ and consider a partition $\mathcal{P}_H$ of G/S determined by H.

If $|H \cap A : S| = \infty$ for each subgroup A of G such that $A/S \in \mathcal{F}$ then, by the maximality of $\mathcal{F}$, we have $H/S \in \mathcal{F}$. Hence, $H/S \in \mathcal{U}$.

Otherwise, there exists a subgroup A of G such that $A/S \in \mathcal{F}$ and $|H \cap A : S|$ is finite, $|H \cap A : S| = n$. We take an arbitrary $g \in G$ and denote $T_g = gH \cap A$. If $a \in T_g$ then $a^{-1}T_g \subseteq A$ and $a^{-1}T_g \subseteq H$. Hence, $a^{-1}T_g/S \subseteq A \cap H/S$ so $|T_g/S| \leqslant n$. If x and y determine the same coset by H, then they determine the same set T_g. Then we choose $U \in \mathcal{U}$ such that $|U \cap x(H \cap A/S)| \leqslant 1$ for each $x \in G$. Thus, $|U \cap P| \leqslant 1$ for each cell P of the partition $\mathcal{P}_H$. □

The next theorem characterizes all G-spaces X such that each free ultrafilter on X is G-selective.

Theorem 5.2. *Let G be a group, S be a subgroup of G such that $|G : S| = \infty$, $X = G/S$. Each free ultrafilter on X is G-selective if and only if, for each subgroup T of G such that $S \subset T \subset G$, either $|T : S|$ is finite or $|G : T|$ is finite.*

Applying Theorem 2, we conclude that each free ultrafilter on an infinite Abelian group G (as a regular G-space) is selective if and only if $G = \mathbb{Z} \oplus F$ or $G = \mathbb{Z}_{p^\infty} \times F$, where F is finite, $\mathbb{Z}_{p^\infty}$ is the Prüffer p-group. In particular, each free ultrafilter on $\mathbb{Z}$ is $\mathbb{Z}$-selective.

For a G-space X and $n \geqslant 2$, a coloring $\chi : [X]^n \to \{0, 1\}$ is said to be *G-invariant* if, for any $\{x_1, \dots, x_n\} \in [X]^n$ and $g \in G$, $\chi(\{x_1, \dots, x_n\}) = \chi(\{gx_1, \dots, gx_n\})$. We say that a free ultrafilter $\mathcal{U}$ on X is *(G, n)-Ramsey* if, for every G-invariant coloring $\chi : [X]^n \to \{0, 1\}$, there exists $U \in \mathcal{U}$ such that $\chi|_{[U]^n} \equiv const$. In the case $n = 2$, we write "G-Ramsey" instead of $(G, 2)$-Ramsey.

Theorem 5.3. *For any G-space X, each G-Ramsey ultrafilter on X is G-selective.*

The following three theorems show that the conversion of Theorem 5.3 is very far from truth. Let G be a discrete group, βG is the Stone-Čech compactification of G as a left topological semigroup, $K(\beta G)$ is the minimal ideal of βG.

Theorem 5.4. *Each ultrafilter from the closure cl $K(\beta\mathbb{Z})$ is not $\mathbb{Z}$-Ramsey.*

A free ultrafilter $\mathcal{U}$ on an Abelian group G is said to be a *Schur ultrafilter* if, for any $U \in \mathcal{U}$, there are distinct $x, y \in U$ such that $x + y \in U$.

Theorem 5.5. *Each Schur ultrafilter $\mathcal{U}$ on $\mathbb{Z}$ is not $\mathbb{Z}$-Ramsey.*

A free ultrafilter $\mathcal{U}$ on $\mathbb{Z}$ is called *prime* if $\mathcal{U}$ cannot be represented as a sum of two free ultrafilters.

Theorem 5.6. *Every $\mathbb{Z}$-Ramsey ultrafilter on $\mathbb{Z}$ is prime.*

Question 5.1. *Is each $\mathbb{Z}$-Ramsey ultrafilter on $\mathbb{Z}$ selective?*

Some partial positive answers to this question are in the following two theorems.

Theorem 5.7. *Assume that a free ultrafilter $\mathcal{U}$ on $\mathbb{Z}$ has a member A such that $|g+A\cap A|\leqslant 1$ for each $g\in\mathbb{Z}\setminus\{0\}$. If $\mathcal{U}$ is $\mathbb{Z}$-Ramsey then $\mathcal{U}$ is selective.*

Theorem 5.8. *Every $(\mathbb{Z},4)$-Ramsey ultrafilter on $\mathbb{Z}$ is selective.*

All above results are from [9].

Remark 5.1. Let G be an Abelian group. A coloring $\chi:[G]^2\to\{0,1\}$ is called a PS-*coloring* if $\chi(\{a,b\})=\chi(\{a-g,b+g\})$ for all $\{a,b\}\in[G]^2$, equivalently, $a+b=c+d$ implies $\chi(\{a,b\})=\chi(\{c,d\})$. A free ultrafilter $\mathcal{U}$ on G is called a *PS-ultrafilter* if, for any PS-coloring χ of $[G]^2$, there is $U\in\mathcal{U}$ such that $\chi|_{[U]^2}\equiv$ *const*. The following statements were proved in [18], see also [6, Chapter 10].

If G has no elements of order 2 then each PS-ultrafilter on G is selective. A strongly summable ultrafilter on the countable Boolean group $\mathbb{B}$ is a PS-ultrafilter but not selective. If there exists a PS-ultrafilter on some countable Abelian group then there is a P-point in ω^*.

Clearly, an ultrafilter $\mathcal{U}$ on $\mathbb{B}$ is a PS-ultrafilter if and only if $\mathcal{U}$ is $\mathbb{B}$-Ramsey. Thus, a $\mathbb{B}$-Ramsey ultrafilter needs not to be selective, but such an ultrafilter cannot be constructed in ZFC with no additional assumptions.

6. Thin ultrafilters

A free ultrafilter $\mathcal{U}$ on ω is said to be

- *P-point* if, for any partition $\mathcal{P}$ of ω, either $A\in\mathcal{U}$ for some cell A of $\mathcal{P}$ or there exists $U\in\mathcal{U}$ such that $U\cap A$ is finite for each $A\in\mathcal{P}$;
- *Q-point* if, for any partition $\mathcal{P}$ of ω into finite subsets, there exists $U\in\mathcal{U}$ such that $|U\cap A|\leqslant 1$ for each $A\in\mathcal{P}$.

Clearly, $\mathcal{U}$ is selective if and only if $\mathcal{U}$ is a P-point and a Q-point. It is well known that the existence of P- or Q-points is independent of the system of axioms ZFC.

We say that a free ultrafilter $\mathcal{U}$ on ω is a *T-point* if, for every countable group G of permutations of ω, there is a thin subset $U \in \mathcal{U}$ in the G-space ω.

To give a combinatorical characterization of T-points (see [8, 9]), we need some definitions.

A covering $\mathcal{F}$ of a set X is called uniformly bounded if there exists $n \in \mathbb{N}$ such that $|\cup\{F \in \mathcal{F} : x \in F\}| \leqslant n$ for each $x \in X$.

For a metric space (X, d) and $n \in \mathbb{N}$, we denote $B_d(x, n) = \{y \in X : d(x, y) \leqslant n\}$. A metric d is called *locally finite (uniformly locally finite)* if, for every $n \in \mathbb{N}$, $B_d(x, n)$ is finite for each $x \in X$ (there exists $c(n) \in \mathbb{N}$ such that $|B_d(x, n)| \leqslant c(n)$ for each $x \in X$).

A subset A of (X, d) is called *d-thin* if, for every $n \in \mathbb{N}$ there exists a bounded subset B of X such that $B_d(a, n) \cap A = \{a\}$ for each $a \in A \setminus B$.

Theorem 6.1. *For a free ultrafilter $\mathcal{U}$ on ω, the following statement are equivalent:*

(i) *$\mathcal{U}$ is a T-point;*

(ii) *for any sequence $(\mathcal{F}_n)_{n\in\omega}$ of uniformly bounded coverings of ω, there exists $U \in \mathcal{U}$ such that, for each $n \in \omega$, $|F \cap U| \leqslant 1$ for all but finitely many $F \in \mathcal{F}_n$;*

(iii) *for each uniformly locally finite metric d on ω, there is a d-thin member $U \in \mathcal{U}$.*

We do not know if a sequence of coverings in *(ii)* can be replaced to a sequence of partitions.

Remark 6.1. By [10, Theorem 3], a free ultrafilter $\mathcal{U}$ on ω in selective if and only if, for every metric d on ω, there is a d-thin member of $\mathcal{U}$.

Remark 6.2. By [10, Theorem 8], a free ultrafilter $\mathcal{U}$ on ω is a Q-point if and only if, for every locally finite metric d on ω, there is a d-thin member of $\mathcal{U}$.

Remark 6.3. It is worth to be mentioned the following metric characterization of P-points: a free ultrafilter $\mathcal{U}$ on ω is a P-point if and only if, for every metric d on ω, either some member of $\mathcal{U}$ is bounded or there is $U \in \mathcal{U}$ such that (U, d) is locally finite.

A free ultrafilter $\mathcal{U}$ on ω is said to be a *weak P-point* (a *NWD-point*) if $\mathcal{U}$ is not a limit point of a countable subset in ω^* (for every injective mapping $f : \omega \to \mathbb{R}$, there is $U \in \mathcal{U}$ such that $f(U)$ is nowhere dense in $\mathbb{R}$). We note that a weak P-point exists in ZFC.

In the next theorem, we summarize the main results from [8].

Theorem 6.2. *Every P-point and every Q-point is a T-point. Under CH, there exists a T-point which is neither P-point, nor NWD-point, nor Q-point. For every ultrafilter $\mathcal{V}$ on ω, there exist a T-point $\mathcal{U}$ and a mapping $f : \omega \to \omega$ such that $\mathcal{V} = f^\beta(\mathcal{U})$.*

Question 6.1. *Does there exist a T-point in ZFC?*

Question 6.2. *Is every weak P-point a T-point?*

Question 6.3. *(T. Banakh). Let $\mathcal{U}$ be a free ultrafilter on ω such that, for any metric d on ω, some member of $\mathcal{U}$ is discrete in (X, d). Is $\mathcal{U}$ a T-point?*

A free ultrafilter $\mathcal{U}$ on ω is called a $T_{\aleph_0}$-point if, for each minimal well ordering $<$ of ω, there is a $d_<$-thin member of $\mathcal{U}$, where $d_<$ is the natural metric on ω defined by $<$. By Theorem 6.1, each T-point is $T_{\aleph_0}$-point.

Question 6.4. *Is every $T_{\aleph_0}$-point a T-point? Does there exist a $T_{\aleph_0}$-point in ZFC?*

Remark 6.4. An ultrafilter $\mathcal{U}$ on ω is called *rapid* if, for any partition $\{P_n : n \in \omega\}$ of ω into finite subsets, there exists $U \in \mathcal{U}$ such that $|U \cap P_n| \leqslant n$ for every $n \in \omega$. Jana Flašková (see [10, p.350]) noticed that, in contrast to Q-points, a rapid ultrafilter needs not to be a T-point.

Remark 6.5. A family $\mathcal{F}$ of infinite subsets of ω is *coideal* if $M \subseteq N, M \in \mathcal{F} \Rightarrow N \in \mathcal{F}$ and $M = N_0 \cup N_1, M \in \mathcal{F} \Rightarrow N_0 \in \mathcal{F} \vee N_1 \in \mathcal{F}$. Clearly, the family of all infinite subsets of ω is a coideal.

Following [27], we say that a coideal F is

- *P-coideal* if, for every decreasing sequence $(A_n)_{n\in\omega}$ in $\mathcal{F}$ there is $B \in \mathcal{F}$ such that $B \setminus A_n$ is finite for each $n \in \omega$;
- *Q-coideal* if, for every $A \in \mathcal{F}$ and every partition $A = \cup_{n\in\omega} F_n$ with F_n finite, there is $B \in \mathcal{F}$ such that $B \subseteq A$ and $|B \cap F_n| \leqslant 1$ for each $n \in \omega$.

We say that a coideal $\mathcal{F}$ is a *T-coideal* if, for every countable group G of permutations of ω and every $M \in \mathcal{F}$ there exists a G-thin subset $N \in \mathcal{F}$ such that $N \subseteq M$.

Generalizing the first statement in Theorem 6.2, we get: every P-coideal and every Q-coideal is a T-coideal.

Remark 6.6. We say that $\mathcal{U} \in \omega^*$ is sparse (scattered) if, for every countable group G of permutations of ω, there is sparse (scattered) in (G, w) member of $\mathcal{U}$. Clearly, T-point $\Rightarrow$ sparse ultrafilter $\Rightarrow$ scattered ultrafilter.

Question 6.5. *Does there exist sparse (scattcred) ultrafilter in ZFC? Is every weak P-point scattered ultrafilter?*

Question 6.6. *Let $\mathcal{U}$ be a free ultrafilter on ω such that, for every countable group G of permutations of ω, the orbit $\{g\mathcal{U} : g \in G\}$ is discrete in ω^*. Is $\mathcal{U}$ a weak P-point?*

7. The ballean context

Following [21,25], we say that a *ball structure* is a triple $\mathcal{B} = (X, P, B)$, where X, P are non-empty sets and, for every $x \in X$ and $\alpha \in P$, $B(x, \alpha)$ is a subset of X which is called a *ball of radius* α around x. It is supposed that $x \in B(x, \alpha)$ for all $x \in X$ and $\alpha \in P$. The set X is called the *support* of $\mathcal{B}$, P is called the set of *radii*.

Given any $x \in X, A \subseteq X$ and $\alpha \in P$ we set

$$B^*(x, \alpha) = \{y \in X : x \in B(y, \alpha)\}, \; B(A, \alpha) = \bigcup_{a \in A} B(a, \alpha)$$

A ball structure $\mathcal{B} = (X, P, B)$ is called a *ballean* if

- for any $\alpha, \beta \in P$, there exist α', β' such that, for every $x \in X$,

$$B(x, \alpha) \subseteq B^*(x, \alpha'), \; B^*(x, \beta) \subseteq B(x, \beta');$$

- for any $\alpha, \beta \in P$, there exists $\gamma \in P$ such that, for every $x \in X$,

$$B(B(x, \alpha), \beta) \subseteq B(x, \gamma);$$

A ballean $\mathcal{B}$ on X can also be determined in terms of entourages of the diagonal of $X \times X$ (in this case it is called a coarse structure [26]) and can be considered as an asymptotic counterpart of a uniform topological space.

Let $\mathcal{B}_1 = (X_1, P_1, B_1)$, $\mathcal{B}_2 = (X_2, P_2, B_2)$ be balleans. A mapping $f : X_1 \to X_2$ is called a $\prec$-*mapping* if, for every $\alpha \in P_1$, there exists $\beta \in P_2$ such that, for every $x \in X_1$, $f(B_1(x, \alpha)) \subseteq B_2(f(x), \beta)$. A bijection $f : X_1 \to X_2$ is called an *asymorphism* if f and f^{-1} are $\prec$-mappings.

Every metric space (X, d) defines the metric ballean $(X, \mathbb{R}^+, B_d)$, where $B_d(x, r) = \{y \in X : d(x, y) \leqslant r\}$. By [25, Theorem 2.1.1], a ballean (X, P, B) is metrizable (i.e. asymorphic to some metric ballean) if and only if there exists a sequence $(\alpha_n)_{n\in\omega}$ in P such that, for every $\alpha \in P$, one can find $n \in \omega$ such that $B(x, \alpha) \subseteq B(x, \alpha_n)$ for each $x \in X$.

Let G be a group, $\mathcal{I}$ be an ideal in the Boolean algebra $\mathcal{P}_G$ of all subsets of G, i.e. $\varnothing \in \mathcal{I}$ and if $A, B \in \mathcal{I}$ and $A' \subseteq A$ then $A \cup B \in \mathcal{I}$ and $A' \in \mathcal{I}$. An ideal $\mathcal{I}$ is called a *group ideal* if, for all $A, B \in \mathcal{I}$, we have $AB \in \mathcal{I}$ and $A^{-1} \in \mathcal{I}$. For construction of group ideals see [16].

For a G-space X and a group ideal $\mathcal{I}$ on G, we define the ballean $\mathcal{B}(G, X, \mathcal{I})$ as the triple $(X, \mathcal{I}, B)$ where $B(x, A) = Ax \cup \{x\}$. In the case where $\mathcal{I}$ is the ideal of all finite subsets of G, we omit $\mathcal{I}$ and return to the notation $B(x, A)$ used from the very beginning of the paper.

The following couple of theorems from [10, 15] demonstrate the tight interrelations between balleans and G-spaces.

Theorem 7.1. *Every ballean $\mathcal{B}$ with the support X is asymorphic to the ballean $\mathcal{B}(G, X, \mathcal{I})$ for some subgroup G of the group S_X of all permutations of X and some group ideal $\mathcal{I}$ on G.*

Theorem 7.2. *Every metrizable ballean $\mathcal{B}$ with the support X is asymorphic to the ballean $\mathcal{B}(G, X, \mathcal{I})$ for some subgroup G of S_X and some group ideal $\mathcal{I}$ on G with countable base such that, for all $x, y \in X$, there is $A \in \mathcal{I}$ such that $y \in Ax$.*

A ballean $\mathcal{B} = (X, P, B)$ is called *locally finite (uniformly locally finite)* if each ball $B(x, \alpha)$ is finite (for each $\alpha \in P$, there exists $n \in \mathbb{N}$ such that $|B(x, \alpha)| \leqslant n$ for every $x \in X$.

Theorem 7.3. *Every locally finite ballean $\mathcal{B}$ with the support X is asymorphic to the ballean $\mathcal{B}(G, X, \mathcal{I})$ for some subgroup G of S_X and some group ideal $\mathcal{I}$ on G with a base consisting of subsets compact in the topology of pointwise convergence on S_X.*

Theorem 7.4. *Every uniformly locally finite ballean $\mathcal{B}$ with the support X is asymorphic to the ballean $\mathcal{B}(G, X, [G]^{<\omega})$ for some subgroup G of S_X.*

We note that Theorem 7.4 plays the key part in the proof of Theorem 6.1.

For ultrafilters on metric spaces and balleans we address the reader to [12, 20, 24].

References

[1] T. Banakh, O. Chervak, L. Zdomskyy, *On character of points in the Higson corona of a metric space*, Comment. Math. Univ. Carolin. **54** (2013), no 2, 159–178.

[2] T. Banakh, I. Protasov, S. Slobodianiuk, *Densities, submeasures and partitions of groups*, Algebra Discrete Math. **17** (2014), no 2, 193–221.

[3] T. Banakh, I. Protasov, S. Slobodianiuk, *Scattered subsets of groups*, Ukr. Math. J. **67** (2015), 304–312.

[4] W.W. Comfort, *Ultrafilters: some old and some new results*, Bull. Amer. Math. Soc. **83**(1977), 417-455.

[5] H. Dales, A. Lau, D. Strauss, *Banach Algebras on a Semigroups and Their Compactifications*, Mem. Amer. Math. Soc., vol. 2005, 2010.

[6] M. Filali, I. Protasov, *Ultrafilters and Topologies on Groups*, Math. Stud. Monogr. Ser., Vol. 13, VNTL Publishers, Lviv, 2011.

[7] N. Hindman, D. Strauss, *Algebra in the Stone-Čech compactification: Theory and Applications*, Walter de Gruyter, Berlin, New York, 1998.

[8] O. Petrenko, I. Protasov, *Thin ultrafilters*, Notre Dame J. Formal Logic **53** (2012), 79–88.

[9] O. Petrenko, I. Protasov, *Selective and Ramsey ultrafilters on G-spaces*, Notre Dame J. Formal Logic (to appear).

[10] O.V. Petrenko, I.V. Protasov, *Balleans and G-spaces*, Ukr. Math. J. **64** (2012), 344–350.

[11] O.V. Petrenko, I.V. Protasov, *Balleans and filters*, Matem. Stud. **38** (2012), 3–11.

[12] I. Protasov, *Combinatorics of Numbers*, Math. Stud. Monogr. Ser, Vol. 2, VNTL Publisher, Lviv, 1997.

[13] I. Protasov, *Coronas of balleans*, Topology Appl. **149** (2005), 149–160.

[14] I. Protasov, *Dynamical equivalences on G^**, Topology Appl. **155** (2008), 1394–1402.

[15] I.V. Protasov, *Balleans of bounded geometry and G-spaces*, Mat. Stud. **30** (2008), 61–66.

[16] I. Protasov, *Counting Ω-ideals*, Algebra Universalis **62** (2010), 339–343.

[17] I. Protasov, *Coronas of ultrametric spaces*, Comment. Math. Univ. Carolin. **52** (2011), no 2, 303–307.

[18] I. Protasov, *Ultrafilters and partitions of Abelian groups*, Ukr. Math. J. **53** (2011), 99–107.

[19] I. Protasov, *Weak P-points in coronas of G-spaces*, Topology Appl. **159** (2012), 587–592.

[20] I. Protasov, *Ultrafilters on metric spaces*, Topology Appl. **164** (2014), 207–214.

[21] I. Protasov, T. Banakh, *Ball Structures and Colorings of Graphs and Groups*, Math. Stud. Monogr. Ser, Vol. 11, VNTL Publisher, Lviv, 2003.

[22] I. Protasov, S. Slobodianiuk, *Partitions of groups*, Math. Stud., **42** (2014), 115–128.

[23] I. Protasov, S. Slobodianiuk, *On the subset combinatorics of G-spaces*, Algebra Discrete Math. **17** (2014), 98–109.

[24] I. Protasov, S. Slobodianiuk, *Ultrafilters on balleans*, Ukr. Math. J. (to appear).

[25] I. Protasov, M. Zarichnyi, *General Asymptology*, Math. Stud. Monogr. Ser., Vol. 12, VNTL Publishers, Lviv, 2007.

[26] J. Roe, *Lectures on Coarse Geometry*, Amer. Math. Soc., Providence, R.I. 2003.

[27] S. Todorcevic, *Introduction to Ramsey Spaces*, Princeton Univ. Press, 2010.

[28] Y. Zelenyuk, *Ultrafilters and Topologies on Groups*, de Grueter, 2012.

Contact information

O. V. Petrenko,
I. V. Protasov

Department of Cybernetics,
Taras Shevchenko National University,
Volodymyrs'ka St., 64, 01601 Kyiv, Ukraine
E-Mail(s): opetrenko72@gmail.com,
i.v.protasov@gmail.com

Received by the editors: 26.06.2015
and in final form 26.06.2015.