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Some (Hopf) algebraic properties of circulant matrices

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Abstract. We study some (Hopf) algebraic properties of circulant matrices, inspired by the fact that the algebra of circulant $n \times n$ matrices is isomorphic to the group algebra of the cyclic group with n elements. We introduce also a class of matrices that generalize both circulant and skew circulant matrices, and for which the eigenvalues and eigenvectors can be read directly from their entries.

Introduction

The starting point of this paper was a result in [1], that arose during the study of certain multiplicatively closed lattices and so called Brandt algebras in (twisted) group rings of cyclic groups: it asserts that for a twisted group ring $\mathbb{R}_F\mathbb{Z}_3$ (where F is an arbitrary map, not necessarily a two-cocycle) there exist three forms q_1, q_2, q_3, concretely determined, such that any element $x \in \mathbb{R}_F\mathbb{Z}_3$ satisfies the polynomial equation $x^3 - q_1(x)x^2 + q_2(x)x - q_3(x) = 0$. Moreover, $q_3(x)$ is given by the determinant of a 3×3 matrix (which, if F is trivial, is a circulant matrix) and, if F is a two-cocycle, then q_1 and q_2 are related in a certain (specific) way.

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We wanted to see to what extent this kind of results may be generalized from $n = 3$ to arbitrary n. It turns out that, even if analogous forms $q_1, \ldots, q_n$ may exist, not much could be said about them. Thus, we restricted our study to the case when the map F is trivial (that is, to ordinary group rings) and, slightly more general, to the case when F is a two-cocycle that is trivial in the second cohomology group. We were thus led to consider circulant matrices as well as a certain class of generalized circulants. By using circulants, we were able to prove that forms $q_1, \ldots, q_n$ exist on the group ring $\mathbb{C}\mathbb{Z}_n$ and have some properties that generalize the case $n = 3$. This is done by using the well known algebra isomorphism between the algebra of $n \times n$ circulants and the group algebra of the cyclic group $\mathbb{Z}_n$. We found useful to give this result a Hopf algebraic interpretation, obtaining along the way a result stating that the algebra of $n \times n$ matrices "factorizes" (in a certain sense) as the "product" between the algebra of $n \times n$ circulants and the algebra of $n \times n$ diagonal matrices. Since the group ring is a Hopf algebra, the algebra of $n \times n$ circulants becomes also a Hopf algebra. We have written down its Hopf structure and we found that the antipode looks particularly nice: the antipode applied to a circulant matrix is simply the transpose of it. The comultiplication Δ does not look too good, but we could prove however that if C is a circulant $n \times n$ matrix then $\Delta(C)$, regarded as an $n^2 \times n^2$ matrix, is block circulant with circulant blocks. We present also a certain result (and a concrete example) concerning some lattices in the algebra of circulant matrices.

In the last section of the paper we introduce a class of generalized circulants, as follows. Denote the elements of $\mathbb{Z}_n$ by $e_1 = 1, e_2, \ldots, e_n$; for a given map $\mu : \mathbb{Z}_n \to \mathbb{C}^*$, with $\mu(e_1) = 1$, denote $\mu(e_i) = \mu_i$ for all $i \in \{2, \ldots, n\}$. For $c_1, \ldots, c_n \in \mathbb{C}$, denote by $circ(c_1, \ldots, c_n; \mu_2, \ldots, \mu_n)$ the $n \times n$ matrix with $c_1, c_2, \ldots, c_n$ in the first row, c_1 on the main diagonal and entry $c_{j-i+1}\frac{\mu_i \mu_{j-i+1}}{\mu_j}$ in any other position (i, j) (we put $\mu_1 = 1$). Certainly, $circ(c_1, \ldots, c_n) = circ(c_1, \ldots, c_n; 1, \ldots, 1)$, so the matrices of the type $circ(c_1, \ldots, c_n; \mu_2, \ldots, \mu_n)$ generalize circulant matrices.They form an algebra, denoted by $C^n_{\mathbb{C}}(\mu)$, which is isomorphic to the twisted group algebra with two-cocycle induced by μ (this is actually how we arrived at these matrices). From general results we know that this twisted group ring is isomorphic to the ordinary group ring, so $C^n_{\mathbb{C}}(\mu)$ is isomorphic to the algebra of $n \times n$ circulants. But these matrices generalize not only circulants, but also skew circulant matrices: indeed, a skew circulant matrix $scirc(c_1, \ldots, c_n)$ turns out to be $circ(c_1, \ldots, c_n; \sigma, \sigma^2, \ldots, \sigma^{n-1})$, where

$\sigma = cos(\frac{\pi}{n}) + isin(\frac{\pi}{n})$. Moreover, for a matrix $circ(c_1, \ldots, c_n; \mu_2, \ldots, \mu_n)$ we are able to give an explicit formula for its eigenvalues (and eigenvectors) that can be read directly from its entries; this formula is a common generalization for the formulae (cf. [3]) giving the eigenvalues for circulant and skew circulant matrices.

1. Preliminaries

We work over a base field $\mathbb{K}$ of characteristic zero (so all matrices have entries in $\mathbb{K}$ and all algebras and Hopf algebras are over $\mathbb{K}$). Sometimes we will need to choose $\mathbb{K}$ to be the field $\mathbb{C}$ of complex numbers, and we will specify this every time we do it.

When we work with $n \times n$ circulant matrices or with elements in the cyclic group of order n, all subscripts will be considered mod n. For a matrix A we denote by A^T its transpose.

A circulant matrix of order n is an $n \times n$ matrix with the property that each row is obtained from the previous row by rotating once to the right. Obviously such a matrix is completely determined by its first row. If the first row is $(c_1, c_2, \ldots, c_n)$ we will denote the associated circulant matrix by $circ(c_1, c_2, \ldots, c_n)$. Note that the (i, j) entry of this matrix is c_{j-i+1}. For instance, for $n = 3$, the matrix $circ(c_1, c_2, c_3)$ is

$$\begin{pmatrix} c_1 & c_2 & c_3 \\ c_3 & c_1 & c_2 \\ c_2 & c_3 & c_1 \end{pmatrix}$$

If we denote by $C_{\mathbb{K}}^n$ the set of circulant $n \times n$ matrices, then $C_{\mathbb{K}}^n$ is an n dimensional subalgebra of the algebra $M_n(\mathbb{K})$ of $n \times n$ matrices over $\mathbb{K}$.

If $\mathbb{K} = \mathbb{C}$, the field of complex numbers, and $C = circ(c_1, c_2, \ldots, c_n)$ is a complex circulant matrix, then the eigenvalues and eigenvectors of C may be written down explicitely (cf. [3]): if we denote by $p_C(X) = c_1 + c_2X + \ldots + c_nX^{n-1}$ and $\omega = cos(\frac{2\pi}{n}) + isin(\frac{2\pi}{n})$, then the eigenvalues of C are the scalars $\lambda_j = p_C(\omega^{j-1})$, for $1 \leq j \leq n$, the eigenvector of λ_j being the vector $x_j = (1, \omega^{j-1}, \omega^{2(j-1)}, \ldots, \omega^{(n-1)(j-1)})^T$.

For more properties of circulant matrices we refer to [3], while for terminology, notation etc. concerning Hopf algebras we refer to [4].

2. Forms associated to circulant matrices

We consider the group $\mathbb{Z}_n$ (cyclic group of order n) and we denote its elements by $e_i = \widehat{i-1}$, for $i \in \{1, 2, \ldots, n\}$, so we have $e_ie_j = e_{i+j-1}$ and

$e_i^{-1} = e_{n-i+2}$, for all $i, j \in \{1, 2, \ldots, n\}$ (subscripts mod n, according to our convention).

We consider the group algebra $\mathbb{K}\mathbb{Z}_n$ over the base field $\mathbb{K}$. The following result is well known:

Theorem 2.1. *The group algebra $\mathbb{K}\mathbb{Z}_n$ and the algebra $C^n_{\mathbb{K}}$ of circulant $n \times n$ matrices over $\mathbb{K}$ are isomorphic, an explicit isomorphism being defined by $g : \mathbb{K}\mathbb{Z}_n \to C^n_{\mathbb{K}}$, $g(c_1e_1 + \ldots + c_ne_n) = circ(c_1, c_2, \ldots, c_n)$, for all $c_1, \ldots, c_n \in \mathbb{K}$.*

Corollary 2.2. *Every element $x = c_1e_1 + \ldots + c_ne_n \in \mathbb{K}\mathbb{Z}_n$ is a solution to a certain polynomial $X^n - q_1(x)X^{n-1} + \ldots + (-1)^n q_n(x)$. Here q_1 is a linear form called the trace of x and q_n is an n-form called the norm of x. Moreover, $q_1(x) = tr[circ(c_1, \ldots, c_n)] = nc_1$ and $q_n(x) = det[circ(c_1, \ldots, c_n)]$.*

Proof. The polynomial is exactly the characteristic polynomial of the circulant matrix corresponding to x via the isomorphism $g : \mathbb{K}\mathbb{Z}_n \simeq C^n_{\mathbb{K}}$. □

Remark 2.3. Via the isomorphism $\mathbb{K}\mathbb{Z}_n \simeq C^n_{\mathbb{K}}$, we can also consider the forms $q_1, \ldots, q_n$ as being defined on $C^n_{\mathbb{K}}$.

Definition 2.4. For $x \in \mathbb{K}\mathbb{Z}_n$, we define the *conjugate* of x by the formula

$$\overline{x} = (-1)^{n+1}x^{n-1} + (-1)^n q_1(x)x^{n-2} + \ldots + q_{n-1}(x) \in \mathbb{K}\mathbb{Z}_n. \tag{2.1}$$

As a consequence of Theorem 2.1 and Corollary 2.2, we immediately obtain:

Corollary 2.5. *An element $x \in \mathbb{K}\mathbb{Z}_n$ is invertible if and only if $q_n(x) \neq 0$, and in this case its inverse is defined by the formula $x^{-1} = \frac{\overline{x}}{q_n(x)}$.*

Corollary 2.6. *An element $x = c_1e_1 + \ldots + c_ne_n \in \mathbb{C}\mathbb{Z}_n$ is noninvertible if and only if there exists y a root of unity of order n such that $c_1 + c_2y + \ldots + c_ny^{n-1} = 0$.*

Proof. We know that $q_n(x) = det[circ(c_1, \ldots, c_n)]$, and since we are over $\mathbb{C}$ here we know that $q_n(x)$ is the product of the eigenvalues of $circ(c_1, \ldots, c_n)$, which are given, as we mentioned before, by $\{p_C(\omega^{j-1})\}$, for all $1 \leq j \leq n$, where $p_C(X) = c_1 + c_2X + \ldots + c_nX^{n-1}$ and $\omega = cos(\frac{2\pi}{n}) + isin(\frac{2\pi}{n})$, so the result follows. □

We will present explicit formulae for the forms q_i for $n=3$ and $n=4$. The case $n=3$ was studied in detail in [1], where the formulae for q_1, q_2, q_3 have been found. Namely, for $x = c_1e_1 + c_2e_2 + c_3e_3 \in \mathbb{K}\mathbb{Z}_3$, we have:

$$q_1(x) = 3c_1, \ \ q_2(x) = 3c_1^2 - 3c_2c_3, \ \ q_3(x) = c_1^3 + c_2^3 + c_3^3 - 3c_1c_2c_3.$$

Moreover, the following properties have been found in [1]:

Proposition 2.7. *If $x, y \in \mathbb{K}\mathbb{Z}_3$, then $q_1(\overline{x}) = q_2(x)$ and $q_2(x+y) = q_2(x) + q_2(y) + q_1(x)q_1(y) - q_1(xy)$.*

Similarly, by performing explicit computations, one can show that for $x = c_1e_1 + c_2e_2 + c_3e_3 + c_4e_4 \in \mathbb{K}\mathbb{Z}_4$, we have:

$$\begin{aligned}
q_1(x) &= 4c_1,\\
q_2(x) &= 6c_1^2 - 4c_2c_4 - 2c_3^2,\\
q_3(x) &= 4c_1^3 - 8c_1c_2c_4 - 4c_1c_3^2 + 4c_2^2c_3 + 4c_3c_4^2,\\
q_4(x) &= c_1^4 - c_2^4 + c_3^4 - c_4^4 - 2c_1^2c_3^2 - 4c_1^2c_2c_4 +\\
&\quad + 4c_1c_2^2c_3 + 4c_1c_3c_4^2 + 2c_2^2c_4^2 - 4c_2c_3^2c_4,
\end{aligned}$$

and also by explicit computations using these formulae one can prove:

Proposition 2.8. *If $x, y \in \mathbb{K}\mathbb{Z}_4$, then $q_1(\overline{x}) = q_3(x)$ and $q_2(x+y) = q_2(x) + q_2(y) + q_1(x)q_1(y) - q_1(xy)$.*

Actually, the above results admit generalizations to arbitrary n, at least over $\mathbb{C}$:

Proposition 2.9. *If $x \in \mathbb{C}\mathbb{Z}_n$, then $q_1(\overline{x}) = q_{n-1}(x)$.*

Proof. We denote by $s_0, s_1, s_2, \ldots, s_n$ the elementary symmetric polynomials in n variables $X_1, X_2, \ldots, X_n$, defined by $s_0 = 1$, $s_1 = X_1 + \ldots + X_n$, $s_2 = \sum_{i<j} X_iX_j, \ldots, s_n = X_1X_2\ldots X_n$. Also, for any $1 \le k \le n$, we consider the polynomial p_k defined by $p_k = X_1^k + X_2^k + \ldots + X_n^k$. These polynomials are related by Newton's identities:

$$ks_k = \sum_{i=1}^{k} (-1)^{i-1} s_{k-i} p_i, \quad \forall\, 1 \le k \le n.$$

In particular, for $k = n-1$, we have

$$(n-1)s_{n-1} = \sum_{i=1}^{n-1} (-1)^{i-1} s_{n-1-i} p_i. \tag{2.2}$$

Let now $x = c_1e_1 + \ldots + c_ne_n \in \mathbb{C}\mathbb{Z}_n$ and consider the circulant matrix $circ(c_1, \ldots, c_n)$, whose eigenvalues will be denoted by $\lambda_1, \ldots, \lambda_n$. It is clear that, for any $1 \le i \le n$, we have $q_i(x) = s_i(\lambda_1, \ldots, \lambda_n)$. On the other hand, for any $1 \le k \le n$, the eigenvalues of the circulant matrix associated to x^k are $\lambda_1^k, \ldots, \lambda_n^k$ and so we have $q_1(x^k) = \lambda_1^k + \ldots + \lambda_n^k = p_k(\lambda_1, \ldots, \lambda_n)$. Now we compute, directly from the formula (2.1):

$$
\begin{aligned}
q_1(\overline{x}) &= (-1)^{n+1}q_1(x^{n-1}) + (-1)^n q_1(x)q_1(x^{n-2}) + \\
&\ldots - q_{n-2}(x)q_1(x) + nq_{n-1}(x) \\
&= (-1)^{n+1}p_{n-1}(\lambda_1, \ldots, \lambda_n) + (-1)^n s_1(\lambda_1, \ldots, \lambda_n)p_{n-2}(\lambda_1, \ldots, \lambda_n) + \\
&\ldots + s_{n-3}(\lambda_1, \ldots, \lambda_n)p_2(\lambda_1, \ldots, \lambda_n) - s_{n-2}(\lambda_1, \ldots, \lambda_n)p_1(\lambda_1, \ldots, \lambda_n) + \\
&+ ns_{n-1}(\lambda_1, \ldots, \lambda_n) \\
&\overset{(2.2)}{=} s_{n-1}(\lambda_1, \ldots, \lambda_n) = q_{n-1}(x),
\end{aligned}
$$

finishing the proof. □

Proposition 2.10. *If $x, y \in \mathbb{C}\mathbb{Z}_n$, then $q_2(x + y) = q_2(x) + q_2(y) + q_1(x)q_1(y) - q_1(xy)$.*

Proof. We denote by $\lambda_1, \ldots, \lambda_n$ (respectively $\mu_1, \ldots, \mu_n$) the eigenvalues of the circulant matrix corresponding to x (respectively y). Then the eigenvalues of the circulant matrix corresponding to $x + y$ (respectively xy) are $\lambda_1 + \mu_1, \ldots, \lambda_n + \mu_n$ (respectively $\lambda_1\mu_1, \ldots, \lambda_n\mu_n$). So, we have:

$$
\begin{aligned}
q_2(x+y) &= \sum_{i<j}(\lambda_i + \mu_i)(\lambda_j + \mu_j) \\
&= \sum_{i<j}\lambda_i\lambda_j + \sum_{i<j}\mu_i\mu_j + \sum_{i<j}\lambda_i\mu_j + \sum_{i<j}\mu_i\lambda_j \\
&= q_2(x) + q_2(y) + \sum_{i<j}\lambda_i\mu_j + \sum_{i<j}\mu_i\lambda_j,
\end{aligned}
$$

$$
\begin{aligned}
q_1(x)q_1(y) - q_1(xy) &= (\sum_{i=1}^{n}\lambda_i)(\sum_{j=1}^{n}\mu_j) - \sum_{k=1}^{n}\lambda_k\mu_k \\
&= \sum_{i<j}\lambda_i\mu_j + \sum_{i<j}\mu_i\lambda_j,
\end{aligned}
$$

hence indeed we have $q_2(x + y) = q_2(x) + q_2(y) + q_1(x)q_1(y) - q_1(xy)$. □

3. Hopf-algebraic properties of circulant matrices

Let H be a finite dimensional Hopf algebra. We denote as usual by $\rightharpoonup$ the left regular action of H on H^* defined by $(h \rightharpoonup \varphi)(h') = \varphi(h'h)$, for all $h, h' \in H$ and $\varphi \in H^*$. It is well known that with this action H^* becomes a left H-module algebra, so we can consider the smash product $H^*\#H$ (which is sometimes called in the literature the *Heisenberg double* of H^*). It is also well known (see [4], p. 162) that $H^*\#H$ is isomorphic as an algebra to the endomorphism algebra $End(H^*)$, an explicit algebra isomorphism being defined by

$$\lambda : H^*\#H \simeq End(H^*), \quad \lambda(\varphi\#h)(\psi) = \varphi * (h \rightharpoonup \psi),$$

for all $\varphi, \psi \in H^*$ and $h \in H$, where $*$ is the convolution product in H^* defined by $(\varphi * \psi)(h) = \varphi(h_1)\psi(h_2)$, where we used the Sweedler-type notation $\Delta(h) = h_1 \otimes h_2$ for the comultiplication of H. In particular, the restrictions of λ to H^* and H define embeddings of H^* and H into $End(H^*)$ (as algebras), defined respectively by

$$\lambda_{H^*} : H^* \to End(H^*), \quad \lambda_{H^*}(\varphi)(\psi) = \varphi * \psi,$$
$$\lambda_H : H \to End(H^*), \quad \lambda_H(h)(\psi) = h \rightharpoonup \psi.$$

If we identify $End(H^*)$ with $M_n(\mathbb{K})$, where $n = dim(H)$, we obtain algebra embeddings of H and H^* into $M_n(\mathbb{K})$. Our aim is to see how these embeddings look like if we take $H = \mathbb{K}\mathbb{Z}_n$ with its usual Hopf algebra structure.

Proposition 3.1. *The image of $(\mathbb{K}\mathbb{Z}_n)^*$ in $M_n(\mathbb{K})$ via the above embedding is the algebra $D_{\mathbb{K}}^n$ of diagonal $n \times n$ matrices. The image of $\mathbb{K}\mathbb{Z}_n$ in $M_n(\mathbb{K})$ via the above embedding is the algebra $C_{\mathbb{K}}^n$ of circulant matrices and the embedding $\lambda_{\mathbb{K}\mathbb{Z}_n}$ coincides with the algebra isomorphism $g : \mathbb{K}\mathbb{Z}_n \simeq C_{\mathbb{K}}^n$ defined in Theorem 2.1.*

Proof. We consider the basis $\{e_1, \ldots, e_n\}$ in $\mathbb{K}\mathbb{Z}_n$ as before and its dual basis $\{p_1, \ldots, p_n\}$ in $(\mathbb{K}\mathbb{Z}_n)^*$, defined by $p_i(e_j) = \delta_{ij}$ for all $i, j \in \{1, \ldots, n\}$. Thus, the algebra structure of $(\mathbb{K}\mathbb{Z}_n)^*$ is defined on this basis by $p_ip_j = \delta_{ij}p_j$ and $p_1 + \ldots + p_n = 1$. Moreover, we identify $End((\mathbb{K}\mathbb{Z}_n)^*) \equiv M_n(\mathbb{K})$ via this basis, that is, if $f \in End((\mathbb{K}\mathbb{Z}_n)^*)$ and $f(p_j) = \sum_{i=1}^n a_{ij}p_i$, we identify f with the matrix $M_f = (a_{ij})_{1 \leq i,j \leq n}$.

By using the formula for λ_{H^*} given above, we can easily see that $\lambda_{(\mathbb{K}\mathbb{Z}_n)^*}(p_k)(p_j) = p_kp_j = \delta_{jk}p_k$, for all $1 \leq j, k \leq n$, that is $\lambda_{(\mathbb{K}\mathbb{Z}_n)^*}(p_k)$ coincides via the identification $End((\mathbb{K}\mathbb{Z}_n)^*) \equiv M_n(\mathbb{K})$ with the matrix having 1 in the (k, k) position and 0 elsewhere. Thus, for an arbitrary element $x = d_1 p_1 + \ldots + d_n p_n \in (\mathbb{K}\mathbb{Z}_n)^*$, with $d_1, \ldots, d_n \in \mathbb{K}$, we have $M_{\lambda_{(\mathbb{K}\mathbb{Z}_n)^*}(x)} = diag(d_1, \ldots, d_n)$, the diagonal matrix with entries $d_1, \ldots, d_n$, q.e.d.

On the other hand, by using the formula for λ_H given above, we can see that $\lambda_{\mathbb{K}\mathbb{Z}_n}(e_k)(p_j) = e_k \rightharpoonup p_j$, for all $1 \leq j, k \leq n$, and since we have $(e_k \rightharpoonup p_j)(e_i) = p_j(e_i e_k) = p_j(e_{k+i-1}) = \delta_{j,k+i-1}$, for all $1 \leq i, j, k \leq n$, we obtain $e_k \rightharpoonup p_j = p_{j-k+1(mod\ n)}$, that is $\lambda_{\mathbb{K}\mathbb{Z}_n}(e_k)(p_j) = p_{j-k+1(mod\ n)}$, for all $1 \leq j, k \leq n$, and this means exactly that $M_{\lambda_{\mathbb{K}\mathbb{Z}_n}(e_k)} = circ(0, 0, \ldots, 1, \ldots, 0)$, where 1 is in the k^{th} position. Thus, for an arbitrary element $y = c_1 e_1 + \ldots + c_n e_n \in \mathbb{K}\mathbb{Z}_n$, with $c_1, \ldots, c_n \in \mathbb{K}$, we have $M_{\lambda_{\mathbb{K}\mathbb{Z}_n}(y)} = circ(c_1, \ldots, c_n)$, q.e.d. □

Let us recall some facts from [2]. If X is an (associative unital) algebra (with multiplication denoted by $x \otimes y \mapsto xy$ for all $x, y \in X$) and A, B are subalgebras of X, we say that X factorizes as $X = AB$ if the map $A \otimes B \to X$, $a \otimes b \mapsto ab$, for all $a \in A$, $b \in B$, is a linear isomorphism. This is equivalent to saying that there exists a so called *twisting map* $R : B \otimes A \to A \otimes B$ such that X is isomorphic as an algebra to the so called *twisted tensor product* $A \otimes_R B$.

It is well known that any smash product (such as $(\mathbb{K}\mathbb{Z}_n)^* \# \mathbb{K}\mathbb{Z}_n$) is a particular case of a twisted tensor product. Since $(\mathbb{K}\mathbb{Z}_n)^* \simeq D^n_{\mathbb{K}}$ and $\mathbb{K}\mathbb{Z}_n \simeq C^n_{\mathbb{K}}$ as algebras, we can conclude:

Proposition 3.2. *The algebra of $n \times n$ matrices factorizes as $M_n(\mathbb{K}) = D^n_{\mathbb{K}} C^n_{\mathbb{K}}$.*

Remark 3.3. Assume that $\mathbb{K} = \mathbb{C}$. In this case, it is known that the Hopf algebra $\mathbb{C}\mathbb{Z}_n$ is selfdual, that is $\mathbb{C}\mathbb{Z}_n$ is isomorphic as a Hopf algebra to $(\mathbb{C}\mathbb{Z}_n)^*$. If we consider $\omega = cos(\frac{2\pi}{n}) + isin(\frac{2\pi}{n})$, an explicit isomorphism $\phi : \mathbb{C}\mathbb{Z}_n \simeq (\mathbb{C}\mathbb{Z}_n)^*$ is defined by $\phi(e_i) = \sum_{j=1}^{n} \omega^{(i-1)(j-1)} p_j$, for all $i \in \{1, \ldots, n\}$, where $\{e_1, \ldots, e_n\}$ is the standard basis in $\mathbb{C}\mathbb{Z}_n$ and $\{p_1, \ldots, p_n\}$ is its dual basis in $(\mathbb{C}\mathbb{Z}_n)^*$. We consider the algebra isomorphisms $\mathbb{C}\mathbb{Z}_n \simeq C^n_{\mathbb{C}}$ and $(\mathbb{C}\mathbb{Z}_n)^* \simeq D^n_{\mathbb{C}}$ and thus we obtain an algebra isomorphism $\psi : C^n_{\mathbb{C}} \simeq D^n_{\mathbb{C}}$. If $C = circ(c_1, \ldots, c_n)$ is a circulant matrix, an easy computation shows that $\psi(C) = diag(\lambda_1, \ldots, \lambda_n)$, where $\{\lambda_1, \ldots, \lambda_n\}$ are the eigenvalues of C defined by $\lambda_j = c_1 + c_2\omega^{j-1} + c_3\omega^{2(j-1)} + \ldots + c_n\omega^{(n-1)(j-1)}$, for all $j \in \{1, \ldots, n\}$. In particular, this shows immediately the known fact that if X, Y are circulant matrices with eigenvalues $\lambda_1, \ldots, \lambda_n$ and respectively $\beta_1, \ldots, \beta_n$, then the circulant matrix $X+Y$ has eigenvalues $\lambda_1+\beta_1,\ldots,\lambda_n+\beta_n$ and the circulant matrix XY has eigenvalues $\lambda_1\beta_1,\ldots,\lambda_n\beta_n$.

We denote by P_n the *fundamental* circulant $n\times n$ matrix, defined by $P_n=circ(0,1,0,\ldots,0)$. It has the property that $P_n^n=I_n$ and, if $C=circ(c_1,\ldots,c_n)$, then $C=c_1I_n+c_2P_n+c_3P_n^2+\ldots+c_nP_n^{n-1}$. If we denote as before by $\{e_1,\ldots,e_n\}$ the standard basis of $\mathbb{K}\mathbb{Z}_n$, we obviously have $e_2^2=e_3$, $e_2^3=e_4$, $\ldots$, $e_2^{n-1}=e_n$. Thus, since g is an algebra map and $g(e_2)=P_n$, we have, for all $2\leq i\leq n$, $g(e_i)=g(e_2^{i-1})=P_n^{i-1}$. Recall also that, for any $1\leq i\leq n$, the inverse of e_i in $\mathbb{K}\mathbb{Z}_n$ is e_{n-i+2}.

We look again at the algebra isomorphism $g:\mathbb{K}\mathbb{Z}_n\simeq C_{\mathbb{K}}^n$. Since $\mathbb{K}\mathbb{Z}_n$ is a Hopf algebra, we can transfer its structure to $C_{\mathbb{K}}^n$ via g, and thus $C_{\mathbb{K}}^n$ becomes a Hopf algebra. We will write down its counit, comultiplication and antipode.

Let $C=circ(c_1,\ldots,c_n)$ be a circulant matrix. It is easy to see that the transferred counit is defined by $\varepsilon:C_{\mathbb{K}}^n\rightarrow\mathbb{K}$, $\varepsilon(C)=c_1+c_2+\ldots+c_n$.

We compute now the comultiplication $\Delta:C_{\mathbb{K}}^n\rightarrow C_{\mathbb{K}}^n\otimes C_{\mathbb{K}}^n$:

$$\begin{aligned}
\Delta(C) &= (g\otimes g)\circ\Delta_{\mathbb{K}\mathbb{Z}_n}\circ g^{-1}(C)\\
&= (g\otimes g)\circ\Delta_{\mathbb{K}\mathbb{Z}_n}(c_1e_1+\ldots+c_ne_n)\\
&= (g\otimes g)(c_1e_1\otimes e_1+\ldots+c_ne_n\otimes e_n)\\
&= c_1g(e_1)\otimes g(e_1)+\ldots+c_ng(e_n)\otimes g(e_n)\\
&= c_1I_n\otimes I_n+c_2P_n\otimes P_n+c_3P_n^2\otimes P_n^2+\ldots+c_nP_n^{n-1}\otimes P_n^{n-1}.
\end{aligned}$$

Note that the counit property $(\varepsilon\otimes id)\circ\Delta=id$ applied to C becomes, by using the above formulae and the fact that $\varepsilon(P_n)=1$,

$$\begin{aligned}
C &= (\varepsilon\otimes id)\circ\Delta(C)\\
&= (\varepsilon\otimes id)(c_1I_n\otimes I_n+c_2P_n\otimes P_n+c_3P_n^2\otimes P_n^2+\ldots+c_nP_n^{n-1}\otimes P_n^{n-1})\\
&= c_1I_n+c_2P_n+c_3P_n^2+\ldots+c_nP_n^{n-1},
\end{aligned}$$

that is, the counit property is equivalent to the basic property of the fundamental circulant matrix. We compute now the formula for the antipode $S:C_{\mathbb{K}}^n\rightarrow C_{\mathbb{K}}^n$:

$$\begin{aligned}
S(C) &= g\circ S_{\mathbb{K}\mathbb{Z}_n}\circ g^{-1}(C)\\
&= g\circ S_{\mathbb{K}\mathbb{Z}_n}(c_1e_1+\ldots+c_ne_n)\\
&= g(c_1e_1+c_2e_2^{-1}+\ldots+c_ne_n^{-1})\\
&= g(c_1e_1+c_2e_n+c_3e_{n-1}+\ldots+c_ne_2)
\end{aligned}$$

$$\begin{aligned} &= circ(c_1, c_n, c_{n-1}, \ldots, c_3, c_2) \\ &= circ(c_1, c_2, \ldots, c_n)^T. \end{aligned}$$

That is, $S(C)$ is just the transpose of C.

If we denote by $Q_n = circ(0, 0, \ldots, 0, 1) = P_n^T$ the transpose of P_n, then the antipode property $S(C_{(1)})C_{(2)} = \varepsilon(C)I_n$, with notation $\Delta(C) = C_{(1)} \otimes C_{(2)}$, is equivalent to the following relation involving the matrices P_n and Q_n:

$$c_1 I_n + c_2 Q_n P_n + c_3 Q_n^2 P_n^2 + \ldots + c_n Q_n^{n-1} P_n^{n-1} = (c_1 + \ldots + c_n) I_n,$$

which is obviously true because we actually have $Q_n P_n = I_n$, that is Q_n is the inverse of P_n.

Remark 3.4. It is well known that the element $x = \frac{1}{n}(e_1 + \ldots + e_n)$ is a so called integral in $\mathbb{K}\mathbb{Z}_n$, that is it satisfies the condition $hx = \varepsilon(h)x$ for all $h \in \mathbb{K}\mathbb{Z}_n$. If we write $h = c_1 e_1 + \ldots + c_n e_n$, with $c_1, \ldots, c_n \in \mathbb{K}$, then the equality $hx = \varepsilon(h)x$ may be transferred in $C_{\mathbb{K}}^n$ via the isomorphism g, and we obtain $g(h)g(x) = \varepsilon(h)g(x)$, that is

$$circ(c_1, \ldots, c_n)circ(1, 1, \ldots, 1) = (c_1 + \ldots + c_n)circ(1, 1, \ldots, 1),$$

and this is equivalent to the fact that $c_1 + \ldots + c_n$ is an eigenvalue for $circ(c_1, \ldots, c_n)$ with eigenvector $(1, 1, \ldots, 1)^T$.

4. Brandt algebras and lattices in circulat matrices

In this section we assume that the base field is $\mathbb{C}$.

Theorem 4.1. *Let $C = circ(c_1, \ldots, c_n)$ be a circulant matrix. We denote by Δ the comultiplication of the Hopf algebra $C_{\mathbb{C}}^n$ and by P_n the fundamental circulant matrix $circ(0, 1, 0, \ldots, 0)$. Then $\Delta(C)$, regarded as an $n^2 \times n^2$ matrix, is the block circulant with circulant blocks matrix $circ(c_1 I_n, c_2 P_n, \ldots, c_n P_n^{n-1})$. Moreover, the eigenvalues of $\Delta(C)$ are the eigenvalues of C, each one with multiplicity n.*

Proof. The first statement follows easily from the formula $\Delta(C) = c_1 I_n \otimes I_n + c_2 P_n \otimes P_n + c_3 P_n^2 \otimes P_n^2 + \ldots + c_n P_n^{n-1} \otimes P_n^{n-1}$. To prove the second statement, one verifies first that the block diagonal matrix that diagonalizes $\Delta(C)$ is

$$diag(\Lambda_1 + \Lambda_2 + \ldots + \Lambda_n, \Lambda_1 + \omega\Lambda_2 + \ldots + \omega^{n-1}\Lambda_n, \ldots,$$

$$\dots, \Lambda_1 + \omega^{n-1}\Lambda_2 + \dots + \omega^{(n-1)^2}\Lambda_n),$$

where $\omega = cos(\frac{2\pi}{n}) + isin(\frac{2\pi}{n})$, and

$$\Lambda_1 = c_1 I_n, \quad \Lambda_2 = c_2 diag(1, \omega, \dots, \omega^{n-1}), \dots,$$
$$\dots \Lambda_n = c_n diag(1, \omega^{n-1}, \dots, \omega^{(n-1)^2}),$$

and the eigenvalues of $\Delta(C)$ are still $p_C(1), p_C(\omega), \dots, p_C(\omega^{n-1})$, each one with multiplicity n. $\square$

Recalling that the algebra of complex circulant matrices is isomorphic to $\mathbb{C}\mathbb{Z}_n$, we can introduce the following analogue of the concept in [1]:

Definition 4.2. A set B of complex circulant matrices is called an *integral (rational) Brandt algebra* if $q_i(a), q_i(b), q_i(a+b), q_i(ab) \in \mathbb{Z}(\mathbb{Q})$, for all $a, b \in B$ and $1 \leq i \leq n$, where q_i are the forms defined in Corollary 2.2, transferred to $C^n_{\mathbb{C}}$ as in Remark 2.3.

Proposition 4.3. *The set of complex circulant matrices that have integral (rational) eigenvalues is an integral (rational) Brandt algebra.*

Proof. Follows immediately from the fact that the eigenvalues of a sum (product) of circulant matrices are the sums (products) of the eigenvalues of the given matrices and by using the fact that for a circulant matrix C all $q_i(C)$ are symmetric polynomials in the eigenvalues of C. $\square$

Proposition 4.4. *The set of complex circulant matrices $C = circ(c_1, \dots, c_n)$ that have integer (rational) eigenvalues is given by*

$$(c_1, c_2, \dots, c_n)^T = \frac{1}{n} M(\lambda_1, \dots, \lambda_n)^T,$$

where $\lambda_1, \dots, \lambda_n$ are integer (rational) numbers, and the $n \times n$ matrix M has $\overline{\omega^{(i-1)(j-1)}}$ as the (i,j) entry, where $\omega = cos(\frac{2\pi}{n}) + isin(\frac{2\pi}{n})$. Moreover, if the elements λ_i satisfy the extra condition $\lambda_{k+1} = \lambda_{n-k+1}$, for all $1 \leq k \leq n-1$, then the matrix C has real entries.

Proof. If we consider the matrix $A = (a_{ij})$ such that $a_{ij} = \omega^{(i-1)(j-1)}$, it is easy to see that $A(c_1, \dots, c_n)^T = (p_C(1), p_C(\omega), \dots, p_C(\omega^{n-1}))$, and the result follows because the inverse of A is $\frac{1}{n}M$. For the second statement, just note that in the matrix M, the column $k+1$ is the conjugate of the column $n-k+1$. $\square$

Corollary 4.5. *The set of block circulants with circulant blocks $\Delta(B)$, where B is the set of circulant matrices with integral (rational) eigenvalues and Δ is the comultiplication of the Hopf algebra $C^n_{\mathbb{C}}$, is an integral (rational) Brandt algebra.*

Proof. The result follows because, as we have seen in Theorem 4.1, the eigenvalues of $\Delta(b)$, for any $b \in B$, coincide with the eigenvalues of b. $\square$

Now we can study lattices in the algebra of complex circulant matrices. This study takes us to some conditions about when a circulant matrix is obtained as integral linear combinations of some elements of this algebra. Here we will focus to the subset of circulant matrices that have integral entries.

Theorem 4.6. *We consider the lattice $\mathbb{Z}v_1+\mathbb{Z}v_2+\ldots+\mathbb{Z}v_n$ in the algebra of complex circulant $n \times n$ matrices, generated by the linearly independent vectors*

$$\begin{aligned} v_1 &= c_{11}I_n + c_{12}P_n + \ldots + c_{1n}P_n^{n-1}, \\ v_2 &= c_{21}I_n + c_{22}P_n + \ldots + c_{2n}P_n^{n-1}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ v_n &= c_{n1}I_n + c_{n2}P_n + \ldots + c_{nn}P_n^{n-1}. \end{aligned}$$

We denote by C the $n \times n$ matrix with entries c_{ij}. If all the entries of the matrix C^{-1} are integral, then any circulant matrix with integral entries belongs to this lattice, that is, may be written as $a_1v_1 + a_2v_2 + \ldots + a_nv_n$, with $a_1, \ldots, a_n \in \mathbb{Z}$.

Proof. The matrix C is invertible because we assumed that $v_1, \ldots, v_n$ are linearly independent. Since we assumed that the entries of C^{-1} are integral, it follows that each of the matrices I_n, P_n, P_n^2, $\ldots$, P_n^{n-1} may be written as an integral linear combination of the elements $v_1, \ldots, v_n$, and now the result follows because any circulant matrix $circ(c_1, \ldots, c_n)$ may be written as $circ(c_1, \ldots, c_n) = c_1I_n + c_2P_n + c_3P_n^2 + \ldots + c_nP_n^{n-1}$. $\square$

Moreover, if we consider the lattice $\mathbb{Z}\Delta(v_1) + \mathbb{Z}\Delta(v_2) + \ldots + \mathbb{Z}\Delta(v_n)$ in the set of block circulants with circulant blocks, if C^{-1} has integral entries then every block circulant with circulant blocks of the form $circ(a_1I_n, a_2P_n, \ldots, a_nP_n^{n-1})$, with $a_1, \ldots, a_n \in \mathbb{Z}$ belongs to this lattice, that is may be written as an integral linear combination of $\Delta(v_1), \ldots, \Delta(v_n)$.

Example 4.7. Consider the lattice of 3×3 circulant matrices $\mathbb{Z}v_1+\mathbb{Z}v_2+\mathbb{Z}v_3$, with

$$\begin{aligned} v_1 &= -P_3 + P_3^2, \\ v_2 &= -\frac{1}{3}I_3 + \frac{1}{3}P_3 + \frac{1}{3}P_3^2, \\ v_3 &= \frac{1}{3}I_3 + \frac{2}{3}P_3 - \frac{1}{3}P_3^2. \end{aligned}$$

The matrix of coefficients

$$\begin{pmatrix} 0 & -1 & 1 \\ -1/3 & 1/3 & 1/3 \\ 1/3 & 2/3 & -1/3 \end{pmatrix}$$

has an inverse with integral entries, so every circulant 3×3 matrix with integral entries may be written as $av_1+bv_2+cv_3$, with $a,b,c\in\mathbb{Z}$. Moreover, each matrix $circ(a_1I_3, a_2P_3, a_3P_3^2)$, with $a_1,a_2,a_3\in\mathbb{Z}$, may be written as $m_1\Delta(v_1)+m_2\Delta(v_2)+m_3\Delta(v_3)$, with $m_1,m_2,m_3\in\mathbb{Z}$.

5. A class of generalized circulants

There exist various generalizations of circulant matrices, appeared around 1980, see for instance [3], [5] and references therein. As proved by Waterhouse in [5], many of them are related to twisted group rings. In this section we will introduce a certain class of matrices that generalize both circulant and skew circulant matrices, also related to twisted group rings, having the property that their eigenvalues can be read directly from the entries of the matrices.

Consider again the cyclic group $\mathbb{Z}_n$ with elements denoted as before by $e_1,\ldots,e_n$, and $F:\mathbb{Z}_n\times\mathbb{Z}_n\to\mathbb{K}^*$ a two-cocycle (here $\mathbb{K}^*$ is the set of nonzero elements in $\mathbb{K}$), that is F satisfies

$$\begin{aligned} &F(e_1,x)=F(x,e_1)=1, \quad \forall\, x\in\mathbb{Z}_n, \\ &F(x,y)F(xy,z)=F(y,z)F(x,yz), \quad \forall\, x,y,z\in\mathbb{Z}_n. \end{aligned}$$

We can consider then the twisted group ring $\mathbb{K}_F\mathbb{Z}_n$, which is an associative algebra (with unit e_1) obtained from the group ring $\mathbb{K}\mathbb{Z}_n$ by deforming its product using F, namely $x\cdot_F y=F(x,y)xy$, for all $x,y\in\mathbb{Z}_n$.

We have seen that $\mathbb{K}\mathbb{Z}_n$ may be embedded as an algebra in $M_n(\mathbb{K})$ (and the image of this embedding is the algebra of circulant matrices $C_{\mathbb{K}}^n$). We can do something similar for a twisted group ring $\mathbb{K}_F\mathbb{Z}_n$. Namely, define the map $\lambda_F : \mathbb{K}_F\mathbb{Z}_n \to End((\mathbb{K}_F\mathbb{Z}_n)^*)$, $\lambda_F(a)(\psi) = a \rightharpoonup \psi$, where $(a \rightharpoonup \psi)(b) = \psi(ba)$, for all $a, b \in \mathbb{K}_F\mathbb{Z}_n$ and $\psi \in (\mathbb{K}_F\mathbb{Z}_n)^*$. If we denote by $\{p_j\}$ the basis of $(\mathbb{K}_F\mathbb{Z}_n)^*$ dual to the basis $\{e_j\}$ of $\mathbb{K}_F\mathbb{Z}_n$, one can check that we have $\lambda_F(e_k)(p_j) = F(e_{j-k+1}, e_k)p_{j-k+1}$, for all $j, k \in \{1, \ldots, n\}$. If we identify as usual $End((\mathbb{K}_F\mathbb{Z}_n)^*) \cong M_n(\mathbb{K})$, we obtain an algebra embedding $\lambda_F : \mathbb{K}_F\mathbb{Z}_n \hookrightarrow M_n(\mathbb{K})$, defined as follows: if $x = c_1e_1 + \ldots + c_ne_n \in \mathbb{K}_F\mathbb{Z}_n$, then $\lambda_F(x)$ is the $n \times n$ matrix whose (i, j) entry is $c_{j-i+1}F(e_i, e_{j-i+1})$. For instance, for $n = 3$ and $n = 4$ the corresponding matrices are respectively given by

$$\begin{pmatrix} c_1 & c_2 & c_3 \\ c_3F(e_2,e_3) & c_1 & c_2F(e_2,e_2) \\ c_2F(e_3,e_2) & c_3F(e_3,e_3) & c_1 \end{pmatrix}$$

$$\begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ c_4F(e_2,e_4) & c_1 & c_2F(e_2,e_2) & c_3F(e_2,e_3) \\ c_3F(e_3,e_3) & c_4F(e_3,e_4) & c_1 & c_2F(e_3,e_2) \\ c_2F(e_4,e_2) & c_3F(e_4,e_3) & c_4F(e_4,e_4) & c_1 \end{pmatrix}$$

We denote by $C^n_{\mathbb{K}}(F)$ the image of the map λ_F; so $C^n_{\mathbb{K}}(F)$ is an algebra, isomorphic to $\mathbb{K}_F\mathbb{Z}_n$.

Assume now that the two-cocycle F is trivial in the cohomology group $H^2(\mathbb{Z}_n, \mathbb{K}^*)$, that is there exists a map $\mu : \mathbb{Z}_n \to \mathbb{K}^*$, with $\mu(e_1) = 1$ (and with notation $\mu(e_i) = \mu_i$ for all $i \in \{2, \ldots, n\}$), such that $F(e_i, e_j) = \mu(e_i)\mu(e_j)\mu(e_ie_j)^{-1}$, for all $i, j \in \{1, \ldots, n\}$. We will denote the algebra $C^n_{\mathbb{K}}(F)$ by $C^n_{\mathbb{K}}(\mu)$. Also, for $x = c_1e_1 + \ldots + c_ne_n \in \mathbb{K}_F\mathbb{Z}_n$, we will denote

$$\lambda_F(x) := \lambda_\mu(x) := circ(c_1, c_2, \ldots, c_n; \mu_2, \ldots, \mu_n),$$

which is a matrix having $c_1, c_2, \ldots, c_n$ in the first row, c_1 on the main diagonal and entry $c_{j-i+1}\frac{\mu_i\mu_{j-i+1}}{\mu_j}$ in any other position (i, j) (with the convention $\mu_1 = 1$). Obviously, we have $circ(c_1, \ldots, c_n) = circ(c_1, \ldots, c_n; 1, \ldots, 1)$. For instance, for $n = 3$, the matrix $circ(c_1, c_2, c_3; a, b)$ is

$$\begin{pmatrix} c_1 & c_2 & c_3 \\ c_3ab & c_1 & c_2a^2b^{-1} \\ c_2ab & c_3b^2a^{-1} & c_1 \end{pmatrix}$$

We aim to find the eigenvalues and eigenvectors for a matrix $circ(c_1, c_2, \ldots, c_n; \mu_2, \ldots, \mu_n)$. For this, we will rely on the known fact that for the two-cocycle F given by $F(e_i, e_j) = \mu(e_i)\mu(e_j)\mu(e_ie_j)^{-1}$ the twisted group ring $\mathbb{K}_F\mathbb{Z}_n$ is isomorphic to the group ring $\mathbb{K}\mathbb{Z}_n$, an isomorphism being defined by $\varphi : \mathbb{K}_F\mathbb{Z}_n \simeq \mathbb{K}\mathbb{Z}_n$, $\varphi(e_i) = \mu(e_i)e_i$ for all $1 \le i \le n$. Thus, we have also an algebra isomorphism $\Psi : C^n_{\mathbb{K}}(\mu) \simeq C^n_{\mathbb{K}}$ defined by $\Psi(circ(c_1, \dots, c_n; \mu_2, \dots, \mu_n)) = circ(c_1, c_2\mu_2, c_3\mu_3, \dots, c_n\mu_n)$ with inverse $\Psi^{-1}(circ(c_1, \dots, c_n)) = circ(c_1, \frac{c_2}{\mu_2}, \dots, \frac{c_n}{\mu_n}; \mu_2, \dots, \mu_n)$.

Proposition 5.1. *Assume that λ is an eigenvalue for the matrix $circ(c_1, c_2\mu_2, \dots, c_n\mu_n)$, with eigenvector $(x_1, x_2, \dots, x_n)^T$. Then λ is an eigenvalue for $circ(c_1, c_2, \dots, c_n; \mu_2, \dots, \mu_n)$ with eigenvector $(x_1, x_2\mu_2, \dots, x_n\mu_n)^T$.*

Proof. In order to be able to use the algebra isomorphism Ψ, we need to transform the equality

$$circ(c_1, c_2\mu_2, \dots, c_n\mu_n)(x_1, \dots, x_n)^T = \lambda(x_1, \dots, x_n)^T \quad (5.1)$$

into a relation between circulant matrices. We consider the circulant $circ(x_1, x_n, x_{n-1}, \dots, x_3, x_2)$, and we remark that the first column of $circ(c_1, c_2\mu_2, \dots, c_n\mu_n)circ(x_1, x_n, x_{n-1}, \dots, x_3, x_2)$ is exactly $circ(c_1, c_2\mu_2, \dots, c_n\mu_n)(x_1, \dots, x_n)^T$, and by (5.1) this column is $(\lambda x_1, \dots, \lambda x_n)^T$. On the other hand, $circ(c_1, c_2\mu_2, \dots, c_n\mu_n)circ(x_1, x_n, x_{n-1}, \dots, x_3, x_2)$ is a circulant matrix (being the product of two circulants) and since we know that its first column is $(\lambda x_1, \dots, \lambda x_n)^T$ we find out that $circ(c_1, c_2\mu_2, \dots, c_n\mu_n)circ(x_1, x_n, x_{n-1}, \dots, x_3, x_2) = circ(\lambda x_1, \lambda x_n, \lambda x_{n-1}, \dots, \lambda x_3, \lambda x_2)$. To this equality we apply the algebra map Ψ^{-1}; we obtain:

$$\begin{aligned} circ(c_1, c_2, \dots, c_n; \mu_2, \dots, \mu_n)circ(x_1, \frac{x_n}{\mu_2}, \frac{x_{n-1}}{\mu_3}, \dots, \frac{x_3}{\mu_{n-1}}, \frac{x_2}{\mu_n}; \mu_2, \dots, \mu_n) \\ = circ(\lambda x_1, \frac{\lambda x_n}{\mu_2}, \frac{\lambda x_{n-1}}{\mu_3}, \dots, \frac{\lambda x_3}{\mu_{n-1}}, \frac{\lambda x_2}{\mu_n}; \mu_2, \dots, \mu_n). \end{aligned}$$

The first column of the matrix $circ(x_1, \frac{x_n}{\mu_2}, \frac{x_{n-1}}{\mu_3}, \dots, \frac{x_3}{\mu_{n-1}}, \frac{x_2}{\mu_n}; \mu_2, \dots, \mu_n)$ is

$$(x_1, \frac{x_2}{\mu_n}F(e_2, e_n), \frac{x_3}{\mu_{n-1}}F(e_3, e_{n-1}), \dots, \frac{x_{n-1}}{\mu_3}F(e_{n-1}, e_3), \frac{x_n}{\mu_2}F(e_n, e_2))^T,$$

that is

$$(x_1, \frac{x_2}{\mu_n}\mu_2\mu_n, \frac{x_3}{\mu_{n-1}}\mu_3\mu_{n-1}, \dots, \frac{x_{n-1}}{\mu_3}\mu_{n-1}\mu_3, \frac{x_n}{\mu_2}\mu_n\mu_2)^T,$$

which is $(x_1, x_2\mu_2, \ldots, x_{n-1}\mu_{n-1}, x_n\mu_n)^T$. Similarly, we can see that the first column of the matrix $circ(\lambda x_1, \frac{\lambda x_n}{\mu_2}, \frac{\lambda x_{n-1}}{\mu_3}, \ldots, \frac{\lambda x_3}{\mu_{n-1}}, \frac{\lambda x_2}{\mu_n}; \mu_2, \ldots, \mu_n)$ is $(\lambda x_1, \lambda x_2\mu_2, \ldots, \lambda x_n\mu_n)^T$. So, the above matrix equality implies $circ(c_1, \ldots, c_n; \mu_2, \ldots, \mu_n)(x_1, x_2\mu_2, \ldots, x_{n-1}\mu_{n-1}, x_n\mu_n)^T = \lambda(x_1, x_2\mu_2, \ldots, x_n\mu_n)^T$, i.e. λ is an eigenvalue for $circ(c_1, c_2, \ldots, c_n; \mu_2, \ldots, \mu_n)$ with eigenvector $(x_1, x_2\mu_2, \ldots, x_n\mu_n)^T$. $\square$

Assume now that $\mathbb{K} = \mathbb{C}$. Since we know ([3]) the eigenvalues and eigenvectors for any circulant matrix, in particular for $circ(c_1, c_2\mu_2, \ldots, c_n\mu_n)$, we obtain immediately from this Proposition the eigenvalues and eigenvectors for $circ(c_1, c_2, \ldots, c_n; \mu_2, \ldots, \mu_n)$:

Proposition 5.2. *For the complex matrix* $C = circ(c_1, c_2, \ldots, c_n; \mu_2, \ldots, \mu_n)$ *define the polynomial* $p_C(X) = c_1 + c_2\mu_2 X + c_3\mu_3 X^2 + \ldots + c_n\mu_n X^{n-1}$. *For* $j \in \{1, 2, \ldots, n\}$ *define* $\lambda_j = p_C(\omega^{j-1})$, *where* $\omega = cos(\frac{2\pi}{n}) + isin(\frac{2\pi}{n})$. *Then* $\lambda_1, \ldots, \lambda_n$ *are the eigenvalues of* C, *and the eigenvector of* λ_j *is* $x_j = (1, \mu_2\omega^{j-1}, \mu_3\omega^{2(j-1)}, \ldots, \mu_n\omega^{(n-1)(j-1)})^T$, *for all* $1 \leq j \leq n$.

Remark 5.3. Exactly as for ordinary circulant matrices, it follows that every element $x = c_1e_1 + \ldots + c_ne_n \in \mathbb{K}_F\mathbb{Z}_n$ is a solution to a certain polynomial $X^n - q_1(x)X^{n-1} + \ldots + (-1)^n q_n(x)$, with $q_1(x) = tr[circ(c_1, \ldots, c_n; \mu_2, \ldots, \mu_n)] = nc_1$ and $q_n(x) = det[circ(c_1, \ldots, c_n; \mu_2, \ldots, \mu_n)]$. Moreover, if $\mathbb{K} = \mathbb{C}$, then for any $1 \leq i \leq n$ we have that $q_i(x) = s_i(\lambda_1, \ldots, \lambda_n)$, where s_i is the i^{th} elementary symmetric polynomial and $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $circ(c_1, \ldots, c_n; \mu_2, \ldots, \mu_n)$.

We show now that, over the field $\mathbb{C}$, the matrices of the type $circ(c_1, c_2, \ldots, c_n; \mu_2, \ldots, \mu_n)$ generalize not only circulant matrices but also skew circulant matrices. Recall from [3] that a skew circulant matrix is a circulant followed by a change in sign to all the elements below the main diagonal. Such a matrix is denoted by $scirc(c_1, \ldots, c_n)$. For example, $scirc(a, b, c)$ is the matrix

$$\begin{pmatrix} a & b & c \\ -c & a & b \\ -b & -c & a \end{pmatrix}$$

For a given n, we denote by $\sigma = cos(\frac{\pi}{n}) + isin(\frac{\pi}{n})$ and $\omega = \sigma^2 = cos(\frac{2\pi}{n}) + isin(\frac{2\pi}{n})$. With this notation, a straightforward computation (using the fact that $\sigma^n = -1$) shows:

Proposition 5.4. $scirc(c_1, \ldots, c_n) = circ(c_1, \ldots, c_n; \sigma, \sigma^2, \ldots, \sigma^{n-1})$.

Consequently, the skew circulant matrices are a subalgebra of $M_n(\mathbb{C})$ (this was noticed also in [3]), which will be denoted by $sC^n_{\mathbb{C}}$. By what we have done before it follows that we have an algebra isomorphism $\Psi : sC^n_{\mathbb{C}} \simeq C^n_{\mathbb{C}}$, $\Psi(scirc(c_1, \ldots, c_n)) = circ(c_1, \sigma c_2, \sigma^2 c_3, \ldots, \sigma^{n-1} c_n)$. Moreover, the eigenvalues of a skew circulant matrix, computed in [3], may be reobtained by applying Proposition 5.2: namely, the eigenvalues of $scirc(c_1, \ldots, c_n) = circ(c_1, \ldots, c_n; \sigma, \sigma^2, \ldots, \sigma^{n-1})$ are given by $\lambda_j = p_C(\omega^{j-1})$, for $1 \leq j \leq n$, where $p_C(X) = c_1 + c_2\sigma X + c_3\sigma^2 X^2 + \ldots + c_n\sigma^{n-1}X^{n-1}$.

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