Algebra and Discrete Mathematics
Volume 12 **(2011)**. Number 2. pp. 85 – 93

RESEARCH ARTICLE

# Fully invariant subgroups of an infinitely iterated wreath product

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Communicated by V. I. Sushchansky

Abstract. The article deals with the infinitely iterated wreath product of cyclic groups $C_p$ of prime order $p$. We consider a generalized infinite wreath product as a direct limit of a sequence of finite $n$th wreath powers of $C_p$ with certain embeddings and use its tableau representation. The main result are the statements that this group doesn't contain a nontrivial proper fully invariant subgroups and doesn't satisfy the normalizer condition.

## Introduction

A *wreath product* of permutation groups is a group-theoretical construction, which is widely used for building groups with certain special properties. Given two permutation groups $(G_1, X_1)$ and $(G_2, X_2)$, where $G_1$ acts on $X_1$ and $G_2$ acts on $X_2$, we denote their wreath product to be the permutation group $G_1 \wr G_2 = \{[g_1(x), g_2] \mid g_2 \in G_2, g_1 : X_2 \to G_1\}$, which acts on the direct product $X_1 \times X_2$ (imprimitive action). The notion of the wreath product of two groups can be easily generalized to an arbitrary finite number of factors. If all factors are isomorphic to $G$ then the corresponding wreath product is often called the *wreath power* of $G$ (or the $n$-iterated wreath product of $G$).

Given a residue field $\mathbb{Z}_p$ ($p$ is prime) we consider its additive group $C_p$ (without loss of generality we can assume that $C_p$ acts on itself by the right translations). The finite $n$th wreath power of $C_p$ (which is isomorphic to the Sylow $p$-subgroup of the finite symmetric group $S_{p^n}$) was studied

**2000 Mathematics Subject Classification:** *20B22, 20E18, 20E22.*
**Key words and phrases:** *wreath product, fully invariant subgroups.*

by L. A. Kaloujnine [7]. In particular, in [7] the author investigated the structure of finite wreath powers of cyclic permutation groups (characteristic subgroups, upper and lower central series and derived series were described). Later, similar results for wreath powers of elementary abelian groups were obtained by V. I. Sushchansky in [15].

The notion of iterated wreath product admits various generalizations in the case of an infinite number of factors. In [4] P. Hall introduced a general construction

$$W = \mathsf{wr}_{\lambda\in\Lambda} G_\lambda$$

of the "restricted" (in the sense of an action as a permutation group) wreath product of permutation groups indexed by a totally ordered set. In the same article the author used this wreath product construction to obtain the examples of characteristically simple groups.

Similar approaches can be found in papers of I. D. Ivanuta [6], W. C. Holland [5] (unrestricted wreath product of permutation groups indexed by a partially ordered set), M. Dixon and T. A Fournelle ([1] and [2]).

For example, in [6] the author adapted the Hall's general construction with factors indexed by a totally ordered set to describe the main (transitive) infinite Sylow $p$-subgroup of the finitary symmetric group. To operate with considered wreath product I. D. Ivanuta also used a tableau representation of its elements.

The normalizer condition for the direct limits of finite standard wreath products (so called Kargapolov groups) was studied in [13] by Yu. I. Merzlyakov. Also in [14] a criterion of self normalizability for some classes of subgroups of the finitary unitriangular group was established.

In this article we consider the infinite wreath product construction (denoted by $U_p^\omega$) as a direct limit of a sequence of finite $n$th wreath powers of $C_p$ with certain embeddings. We also use the so called tableau representation of $U_p^\omega$ for the study of its properties.

In the first section the generalized infinite wreath product $U_p^\omega$ is defined. Then, in section 2 we present a review of known results on the characteristic structure of $U_p^\omega$. The main results of the article are given in the last section:

1) *if $p \neq 2$ and $R$ is a fully invariant subgroup of $U_p^\omega$ then either $R = E$ (the identity subgroup), or $R = U_p^\omega$*; in other words, *$U_p^\omega$ is fully invariantly simple* (theorem 2);

2) *$U_p^\omega$ doesn't satisfy the normalizer condition* (theorem 3).

The author wishes to thank Professor V. I. Sushchansky for his advice in the preparation of this paper.

## 1. Generalized wreath product

In this section we consider a group of infinite tableaux of reduced polynomials (an approach similar to what was proposed by L.A. Kaloujnine in [7]) and then define it as a direct limit of finitely iterated wreath products (generalized wreath product).

### 1.1. The tableau representation

Let $p$ be a prime ($p \neq 2$) and $C_p$ be the additive group of the residue field $\mathbb{Z}_p$. In other words, $C_p$ is the cyclic additive group of order $p$, which acts on itself by the right translations. Define $U_p^\omega$ as a group of infinite almost zero tableaux

$$[a_1(x_2,\ldots,x_k),\ldots,a_n(x_{n+1},\ldots,x_k),0,\ldots],\quad k,n\in\mathbb{N},\quad k>n,\tag{1}$$

where $a_i(x_{i+1},\ldots,x_k)$ is a polynomial over $\mathbb{Z}_p$ reduced (degree of each variable $\leq p-1$) modulo the ideal

$$\langle x_{i+1}^p - x_{i+1}, x_{i+2}^p - x_{i+2}, \ldots, x_k^p - x_k\rangle.$$

The group $U_p^\omega$ acts on the direct product

$$X=\prod_{i=1}^{\infty}\mathbb{Z}_p=\{(t_1,\ldots,t_m,0,\ldots)\mid t_i\in\mathbb{Z}_p, m\in\mathbb{N}\}\tag{2}$$

($X$ is the set of all almost zero sequences over $\mathbb{Z}_p$). If $u\in U_p^\omega$ and $t=(t_i)_{i=1}^\infty\in X$ then

$$t^u=(t_1+a_1(t_2,\ldots,t_k),\ldots,t_n+a_n(t_{n+1},\ldots,t_k),t_{k+1},t_{k+2},\ldots).\tag{3}$$

For simplicity, we introduce some auxiliary notation. Let

$$a_i(\overline{x}_{i+1,k})=a_i(x_{i+1},\ldots,x_k)$$

and $[u]_i = a_i(\overline{x}_{i+1,k})$ – the $i$th coordinate of the tableau $u$. Also, let $[a_i(\overline{x}_{i+1,k})]_{i=1}^\infty$ be a short notation of the tableau (1) and $f(\overline{x}^u)$ denote the reduced polynomial, which is equivalent to the polynomial

$$f(\ldots,x_j+a_j(x_{j+1},\ldots,x_k),\ldots).$$

Thus, according to (3), if $u=[a_i(\overline{x}_{i+1,k})]_{i=1}^\infty$ and $v=[b_i(\overline{x}_{i+1,k})]_{i=1}^\infty$ then

$$uv=[a_i(\overline{x}_{i+1,k})+b_i(\overline{x}^u_{i+1,k})].\tag{4}$$

If $[u]_n\neq 0$ and $[u]_i=0$ for all $i>n$ then $n$ is called the *depth of the tableau $u$*.

### 1.2. $U_p^\omega$ as a direct limit of wreath powers

Recall that a sequence $\{G_n\}_{n=1}^\infty$ of groups with a corresponding sequence

$$\{\varphi_n : G_n \to G_{n+1}\}_{n=1}^\infty$$

of embeddings is called *direct system* and denoted by $\langle G_n, \varphi_n\rangle_{n=1}^\infty$.

Let $P_n$ be a Sylow $p$-subgroup of the symmetric group $S_{p^n}$ ($p$ is prime and $n \in \mathbb{N}$). In [7] the group $P_n$ was described by L.A. Kaloujnine as a group of tableaux

$$[a_1, a_2(x_1), \ldots, a_n(x_1, x_2, \ldots, x_{n-1})], \tag{5}$$

where $a_1 \in \mathbb{Z}_p$, $a_i(x_1, \ldots, x_{i-1})$ is a polynomial (over the residue field $\mathbb{Z}_p$) reduced modulo the ideal

$$\langle x_1^p - x_1, x_2^p - x_2, \ldots, x_{i-1}^p - x_{i-1}\rangle.$$

If we denote by $a_i(x_1, \ldots, x_{i-1})$ and $b_i(x_1, \ldots, x_{i-1})$ the $i$th coordinates of tableaux $u$ and $v$ respectively then the $i$th coordinate of $u \cdot v$ can be found as follows

$$a_i(x_1, \ldots, x_{i-1}) + b_i(x_1 + a_1, \ldots, x_{i-1} + a_{i-1}(x_1, \ldots, x_{i-2})).$$

Also $P_n$ can be considered as the $n$th *wreath power* of a cyclic group $C_p$, i.e. $P_n = C_p \wr \ldots \wr C_p$ ($n$ factors).

Given $P_n$ and $P_{n+1}$ ($n \in \mathbb{N}$) define the mapping $\delta_n : P_n \to P_{n+1}$. If $u$ is a tableau of type (5) then

$$\delta_n(u) = [0, a_1, a_2(x_2), \ldots, a_n(x_2, \ldots, x_n)] \in P_{n+1}.$$

By the direct calculations (or see [9], Lemma 4) it is easy to show that $\delta_n$ is a *strictly diagonal* (in the sense of the article [8]) embedding. Moreover, $\delta_n$ is, actually, the embedding of $P_n$ onto the *diagonal* of the wreath product $P_{n+1} = P_n \wr C_p$, where $C_p$ is the active group.

**Lemma 1.** [9] *The group $U_p^\omega$ is isomorphic to the direct limit of the direct system $\langle P_n, \delta_n\rangle_{n=1}^\infty$.*

## 2. Characteristic subgroups of $U_p^\omega$

This section is devoted to some necessary statements regarding description of characteristic subgroups of $U_p^\omega$. All results are taken from [10] and [11], where they are proved for the case of a generalized infinitely iterated wreath product of elementary abelian groups.

**Definition 1.** *The* weighted degree *of the monomial* $x_1^{k_1}x_2^{k_2}\dots x_n^{k_n}$ *is the positive rational number*

$$h=\sum_{i=1}^{n}k_ip^{-i}+1.$$

*The weighted degree of a polynomial is the maximum among the weighted degrees of its monomials. Let also* $h[0]=0$.

Thus, if $u=[a_i(\overline{x}_{i+1,k})]_{i=1}^{\infty}\in U_p^{\omega}$ then $h[a_i(\overline{x}_{i+1,k})]\in\{0\}\cup[1;1+p^{-i})$.

Given $u=[a_i(\overline{x}_{i+1,k})]_{i=1}^{\infty}\in U_p^{\omega}$ we denote the weighted degree of $a_i(\overline{x}_{i+1,k})$ by $|u|_i$. The sequence $|u|=(|u|_i)_{i=1}^{\infty}$ is called the *multidegree* of the tableau $u$. The set of all multidegrees can be partially ordered as follows: $|u|\preceq|v|$ if and only if $|u|_i\le|v|_i$ (with respect to the natural order on $\mathbb{Q}$) for all $i\in\mathbb{N}$.

**Definition 2.** *A subgroup $R$ of the group $U_p^{\omega}$ is called a* parallelotopic subgroup *if $u\in R$ and $|v|\preceq|u|$ yield $v\in R$.*

For every parallelotopic subgroup $R$ we put in correspondence the sequence $|R|=(\chi_i^{\varepsilon})_{i=1}^{\infty}$ such that

1) $\chi_i=\sup_{u\in R}|u|_i$;

2) if $R$ contains such a tableau $u$ that $|u|_i=\chi_i$, then $\varepsilon=$ "+";

3) otherwise, $\varepsilon=$ "−".

This sequence is called the *indicatrix* of $R$. If $\chi_i\neq 0$ for finitely many indices $i$ only then $d(R)=\max\{i\mid\chi_i\neq 0\}$ is called the *depth of the parallelotopic subgroup* $R$. Otherwise, we put $d(R)=+\infty$.

**Definition 3.** *A group $G$ is called* characteristically *(*fully invariantly *or* verbally*)* simple *if only its characteristic (fully invariant or verbal) subgroups are $E$ (the identity subgroup) or $G$.*

In [12] it was shown that $U_p^{\omega}$ is verbally complete (a group $G$ is called *verbally complete* if for an arbitrary $g\in G$ and for an arbitrary non-trivial word $w(x_1,x_2,\dots,x_n)$ there are $g_1,g_2,\dots,g_n\in G$ such that $w(g_1,g_2,\dots,g_n)=g$). Consequently, $U_p^{\omega}$ is verbally simple.

Characteristic subgroups of $U_p^{\omega}$ were investigated in [10] and [11]. Moreover, in these papers even more general case (the infinitely iterated wreath powers (we denote it by $U_{p,n}^{\infty}$) of the elementary abelian groups of rank $n$) was considered. If we put $n=1$ then $U_{p,1}^{\infty}\cong U_p^{\omega}$. Thereby, from [10] and [11] it is known that $U_p^{\omega}$ has non-trivial proper characteristic subgroups, i.e. $U_p^{\omega}$ is not characteristically simple.

**Theorem 1.** [11] *If $p \neq 2$ and $R$ is a characteristic (fully invariant or verbal) subgroup of the group $U_p^\omega$, then*
1) *$R$ is a parallelotopic subgroup of $U_p^\omega$;*
2) *$d(R) < +\infty$ ($R$ has finite depth).*

## 3. Fully invariant subgroups of $U_p^\omega$

Recall that a subgroup $H$ of a group $G$ is called *fully invariant* if it is invariant under all endomorphisms (homomorphisms of $G$ into $G$) of $G$.

**Lemma 2.** *Let $u = [a_i(\overline{x}_{i+1,k})]_{i=1}^{\infty} \in U_p^\omega$. Then the mapping*

$$\varphi : U_p^\omega \to U_p^\omega,$$

*which acts on the elements of $U_p^\omega$ by the rule:*

$$[\varphi(u)]_1 = 0, \quad [\varphi(u)]_{i+1} = a_i(x_{i+2}, \dots, x_{k+1}) \text{ fol all } i \in \mathbb{N}$$

*is an endomorphism of $U_p^\omega$.*

**Remark 1.** For any $u \in U_p^\omega$ the mapping $\varphi$ actually is a coordinate-wise translation to the right (all variables also must be shifted: $x_j \to x_{j+1}$).

*Proof of lemma 2.* Let

$$u = [a_i(\overline{x}_{i+1,k})]_{i=1}^{\infty} \in U_p^\omega \quad \text{and} \quad v = [b_i(\overline{x}_{i+1,k})]_{i=1}^{\infty} \in U_p^\omega$$

(without loss of generality we assume that $[u]_i = [v]_i = 0$ for all $i > n$, where $n < k$). Obviously, $[\varphi(uv)]_1 = [\varphi(u)\varphi(v)]_1 = 0$. Therefore, we consider $[\varphi(uv)]_i$ and $[\varphi(u)\varphi(v)]_i$, where $i \geq 2$. According to the formula (4) we have

$$\begin{aligned} [uv]_i &= a_i(\overline{x}_{i+1,k}) + b_i(\overline{x}^u_{i+1,k}) = \\ &= a_i(\dots, x_j, \dots) + b_i(\dots, x_j + a_j(x_{j+1}, \dots, x_k), \dots), \end{aligned}$$

where $j \in \{i+1, \dots, k\}$. Thus

$$[\varphi(uv)]_{i+1} = a_i(\dots, x_{j+1}, \dots) + b_i(\dots, x_{j+1} + a_j(x_{j+2}, \dots, x_{k+1}), \dots),$$

where $j \in \{i+1, \dots, k\}$.

On the other hand, since

$$[\varphi(u)]_{i+1} = a_i(x_{i+2}, \dots, x_{k+1}) \quad \text{and} \quad [\varphi(v)]_{i+1} = b_i(x_{i+2}, \dots, x_{k+1})$$

then

$$\begin{aligned} [\varphi(u)\varphi(v)]_{i+1} &= a_i(\dots, x_{j+1}, \dots) + b_i(\dots, x_{j+1} + [\varphi(u)]_{j+1}, \dots) = \\ &= a_i(\dots, x_{j+1}, \dots) + b_i(\dots, x_{j+1} + a_j(x_{j+2}, \dots, x_{k+1}), \dots), \end{aligned}$$

where $j \in \{i+1, \dots, k\}$.

Hence, $\varphi(uv) = \varphi(u)\varphi(v)$, i.e. $\varphi$ is an endomorphism of $U_p^\omega$. $\square$

Now we can prove the main result.

**Theorem 2.** *If $p \neq 2$ then $U_p^\omega$ is fully invariantly simple.*

*Proof.* Let us assume that $R$ is a fully invariant subgroup of the group $U_p^\omega$, $R \neq E$ and $R \neq U_p^\omega$. Then $R$ is a parallelotopic subgroup of $U_p^\omega$ and $d(R) < +\infty$ (theorem 1). If $d(R) = r$ then $R$ contains the tableau

$$u = [\underbrace{0, \dots, 0}_{r-1}, 1, 0, \dots].$$

Given the endomorphism $\varphi$, defined in lemma 2, we have $\varphi(u) \notin R$ (since the depth of $\varphi(u)$ is equal to $r+1$). But, on the other hand, $\varphi(u) \in R$ (since $R$ is a fully invariant subgroup of $U_p^\omega$). This contradiction shows the falsity of the assumptions. $\square$

## 4. A note on the normalizer condition

A group $G$ is said to satisfy the *normalizer condition* if every proper subgroup $H$ is properly contained in its own normalizer, i.e. $H < N_G(H)$ for all $H < G$.

**Lemma 3.** [13] *If a group $G$ satisfies the normalizer condition then every subgroup $H$ of $G$ also satisfies normalizer condition.*

Let

$$\delta_{ij} = \begin{cases} 1, \text{ if } i = j; \\ 0, \text{ if } i \neq j. \end{cases}$$

Given a prime $p$ we consider the set $FM_p^\omega$ of all almost identity infinite matrices over $\mathbb{Z}_p$ ($\omega$ is the least infinite ordinal). In other words,

$$FM_p^\omega = \left\{ (a_{ij}) \,\middle|\, \begin{array}{l} a_{ij} \in \mathbb{Z}_p;\ a_{ij} = \delta_{ij} \text{ for all but} \\ \text{finitely many } (i,j) \in \mathbb{N} \times \mathbb{N} \end{array} \right\}.$$

Also, $FM_p^\omega$ is called the set of all *finitary* matrices. Finitary matrices can be multiplied by the usual rule:

$$(ab)_{ij} = \sum_k a_{ik} b_{kj},$$

since the sum on the right side contains only a finite number of nonzero terms. The set of all finitary invertible matrices with operation of matrix multiplication forms a group, which is called the *finitary linear group.*

The *finitary (upper) unitriangular group* is the group $UT_p^\omega$ of all finitary matrices over $\mathbb{Z}_p$ such that $a_{ij} = \delta_{ij}$ for all $i \geq j$ (see, for example, [14]). Similarly, we can consider the *lower* unitriangular group.

On the other hand, the group of all finitary invertible matrices can be considered as a permutation group on the set $X$ of all almost zero sequences over $\mathbb{Z}_p$ (see formula (2)) with natural action

$$t^A = t \cdot A = (\sum_{i=1}^{k} a_{i1}t_i, \sum_{i=1}^{k} a_{i2}t_i, \ldots, \sum_{i=1}^{k} a_{in}t_i, \ldots), \tag{6}$$

where $t = (t_i)_{i=1}^{\infty} \in X$ and $A = (a_{ij}) \in FM_p^{\omega}$, $i, j \in \mathbb{N}$.

In [14] (see theorem 1) it was shown that *the finitary unitriangular group $UT_p^{\omega}$ does not satisfy the normalizer condition.* And we get the following natural corollary.

**Theorem 3.** *The group $U_p^{\omega}$ does not satisfy the normalizer condition.*

*Proof.* According to lemma 3 it is sufficient to show that $U_p^{\omega}$ contains a group, which is isomorphic to the finitary (upper) unitriangular group.

Let $u \in U_p^{\omega}$ be a tableau with linear coordinates, i.e. $[u]_i$ is a homogeneous linear polynomial. By the direct computation it can be shown that the action of $u$ on $X$ (which is defined by the equation (3)) agrees with the action (6). Obviously, the subset of all tableaux with linear coordinates is a subgroup of $U_p^{\omega}$ and this subgroup is isomorphic to the finitary (lower) unitriangular matrix group.

Since the groups of upper and lower finitary unitriangular matrices are isomorphic (with isomorphism $x \to (x^{\mathsf{T}})^{-1}$, where $x \in UT_p^{\omega}$ and $x^{\mathsf{T}}$ is the transpose of $x$) the group $U_p^{\omega}$ does not satisfy the normalizer condition. $\square$

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Received by the editors: 15.04.2011
and in final form 19.12.2011.