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Quasi-duo Partial skew polynomial rings

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Abstract. In this paper we consider rings R with a partial action α of $\mathbb{Z}$ on R. We give necessary and sufficient conditions for partial skew polynomial rings and partial skew Laurent polynomial rings to be quasi-duo rings and in this case we describe the Jacobson radical. Moreover, we give some examples to show that our results are not an easy generalization of the global case.

Introduction

Partial actions of groups have been introduced in the theory of operator algebras giving powerful tools of their study (see [3] and the literature quoted therein). In [3], the authors introduced partial actions on rings in a pure algebraic context and studied partial skew group rings. In [2], the authors defined a partial action as follows: let R be a ring with an identity 1_R and let $\mathbb{Z}$ be the additive group of integers. A partial action α of $\mathbb{Z}$ on R is a collection of ideals S_i, $i \in \mathbb{Z}$, isomorphisms of rings $\alpha_i : S_{-i} \to S_i$ and the following conditions hold:

(i) $S_0 = R$ and α_0 is the identity map of R;

(ii) $S_{-(i+j)} \supseteq \alpha_i^{-1}(S_i \cap S_{-j})$,

(iii) $\alpha_j \circ \alpha_i(a) = \alpha_{j+i}(a)$, for any $a \in \alpha_i^{-1}(S_i \cap S_{-j})$.

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The above properties easily imply that $\alpha_j(S_{-j} \cap S_i) = S_j \cap S_{i+j}$, for all $i, j \in \mathbb{Z}$, and that $\alpha_{-i} = \alpha_i^{-1}$, for every $i \in \mathbb{Z}$.

Following [2], the *partial skew Laurent polynomial ring* $R\langle x; \alpha\rangle$ in an indeterminate x is the set of all finite formal sums $\sum_{i=-n}^{m} a_i x^i$, $a_i \in S_i$, where the addition is defined in the usual way and the multiplication is defined by $(a_i x^i)(a_j x^j) = \alpha_i(\alpha_{-i}(a_i)a_j)x^{i+j}$, for any $i, j \in \mathbb{Z}$. The *partial skew polynomial ring* $R[x; \alpha]$ is the subring of $R\langle x; \alpha\rangle$ whose elements are the polynomials $\sum_{i=0}^{n} a_i x^i$, $a_i \in S_i$.

Given a partial action α of $\mathbb{Z}$ on R, an enveloping action is a ring T containing R together with a global action $\beta = \{\sigma^i : i \in \mathbb{Z}\}$ on T, where σ is an automorphism of T such that the partial action α_i is given by the restriction of σ^i ([3], Definition 4.2). Note that T does not necessarily have an identity, since the group acting on R is infinite. It is shown in ([3], Theorem 4.5) that a partial action α has an enveloping action if and only if all the ideals S_i are generated by central idempotents of R.

When α has an enveloping action (T, σ), where σ is an automorphism of T, we may consider that R is an ideal of T and the following properties hold:

(i) $T = \sum_{i\in\mathbb{Z}} \sigma^i(R)$;

(ii) $S_i = R \cap \sigma^i(R)$, for every $i \in \mathbb{Z}$;

(iii) $\alpha_i(a) = \sigma^i(a)$, for all $i \in \mathbb{Z}$ and $a \in S_{-i}$.

In order to have associative rings and apply the results which are known for skew polynomial rings and skew Laurent polynomial rings, we assume throughout the paper that all ideals S_i are generated by central idempotents of R. The idempotent corresponding to S_i will be denoted by 1_i and the enveloping action of α by (T, σ), where σ is an automorphism of T. By condition (ii) above we have that $1_i = 1_R\sigma^i(1_R)$. This fact and conditions (i) and (iii) above will be used freely in the paper. Also the following remark will be used without further mention: if I is an ideal of R, then I is also an ideal of T. In fact, if $a \in I$ and $t \in T$ we have $ta = t1_Ra \in Ra \subseteq I$, and similarly $at \in I$.

The *skew Laurent polynomial ring* $T\langle x; \sigma\rangle$ is the set of formal finite sums $\sum_{i=p}^{q} a_i x^i$, $a_i \in T$, with usual sum and the multiplication is given by $xa = \sigma(a)x$, for all $a \in T$. The partial skew Laurent polynomial ring $R\langle x; \alpha\rangle$ is a subring of $T\langle x; \sigma\rangle$. Moreover, $R[x; \alpha]$ is a subring of the skew polynomial ring $T[x; \sigma]$.

We recall some terminology from [2]. We say that an ideal I of R is an α-ideal (α-invariant ideal) if $\alpha_i(I \cap S_{-i}) \subseteq I \cap S_i$, for all $i \geq 0$ ($\alpha_i(I \cap S_{-i}) = I \cap S_i$, for all $i \in \mathbb{Z}$). Note that I is an α-ideal of R if and only if the set of all polynomials $\sum_{i\geq 0} a_i x^i$, where $a_i \in I \cap S_i$, is an ideal of $R[x;\alpha]$. A similar result holds in $R\langle x;\alpha\rangle$ if I is an α-invariant ideal of R.

A ring R is called *right (left) quasi-duo* if every maximal right (left) ideal of R is two-sided or, equivalently, every right (left) primitive homomorphic image of R is a division ring [7]. We refer [7] for further information on quasi-duo rings.

Let $J(R)$ be the Jacobson radical of R. Then from the definition we have that R is right (left) quasi-duo if and only if $R/J(R)$ is right (left) quasi-duo and in this case $R/J(R)$ is a reduced ring. We will use this property in the paper without further mention.

Next we recall some terminology and definitions on $\mathbb{Z}$-graded rings (see [8] for further details). A ring R is a *$\mathbb{Z}$-graded ring* if $R=\bigoplus_{n\in\mathbb{Z}} R_n$, where each R_n is an additive subgroup of R such that $R_nR_m\subseteq R_{n+m}$, for all $n,m\in\mathbb{Z}$. It is known that $1_R\in R_0$. An ideal I of a $\mathbb{Z}$-graded ring R is called homogeneous if $I=\bigoplus_{n\in\mathbb{Z}}(I\cap R_n)$. Note that the rings $R[x;\alpha]$ and $R\langle x;\alpha\rangle$ are naturally $\mathbb{Z}$-graded rings.

The main purpose of this paper is to study partial skew polynomial rings and partial skew Laurent polynomial rings which are quasi-duo. In Section 1 we give necessary and sufficient conditions for a partial skew polynomial rings to be quasi-duo and in this case we give an explicit description of the Jacobson radical.

In Section 2 we consider a partial action of finite type α of $\mathbb{Z}$ on R (we recall this definition in the beginning of the section). Then we give necessary and sufficient conditions for a partial skew Laurent polynomial ring to be quasi-duo. We also give an explicit description of the Jacobson radical in this case.

In Section 3 we give some examples to show that our results are not easy generalizations of the global case.

1. Quasi-duo partial skew polynomial rings

We begin with the following.

Proposition 1. *Let (R,α) be a partial action of $\mathbb{Z}$ on R. Then R is right quasi-duo if and only if T is right quasi-duo.*

Proof. If T is right quasi-duo, then R is right quasi-duo by ([5], Corollary 2), since the natural mapping $\varphi: T\to R$ defined by $\varphi(t)=t.1_R$, for all $t\in T$, is a surjective homomorphism.

Conversely, suppose that R is right quasi-duo and let M be a maximal right ideal of T. Then there exists $s\in\mathbb{Z}$ such that $M\cap\sigma^s(R)$ is a proper right ideal of $\sigma^s(R)$. Let L be a maximal right ideal of $\sigma^s(R)$ with

$M \cap \sigma^s(R) \subseteq L$. Put $X = \{t \in T : t\sigma^s(1_R) \in L\}$ and note that X is a right ideal of T. Also, since $\sigma^s(R) \simeq R$ is right quasi duo it follows that L is two-sided and hence X is also a two-sided ideal of T. Finally, if $x \in M$, then $x\sigma^s(1_R) \in M \cap \sigma^s(R) \subseteq L$ and so $M \subseteq X$. Therefore $M = X$ is a two-sided ideal of T. □

A skew polynomial ring $S[x;\sigma]$ of automorphism type is a commutative ring if and only if S is a commutative ring and $\sigma = id_S$.

For the partial case, we have the following result.

Proposition 2. *Let (R, α) be a partial action of $\mathbb{Z}$ on R. Then the following conditions are equivalent:*

(i) *R is commutative and $\alpha_i = id_{S_i}$, for all $i \in \mathbb{Z}$.*

(ii) *$R[x;\alpha]$ is commutative.*

(iii) *$R\langle x;\alpha\rangle$ is commutative.*

Proof. $(ii) \Rightarrow (i)$. We clearly have that R is commutative. Take any $a \in S_{-i}$. We have $a1_ix^i = 1_ix^ia = \alpha_i(1_{-i}a)x^i = \alpha_i(a)x^i$ and so $\alpha_i(a) = 1_ia$. Thus $S_i \subseteq S_{-i}$, for any $i > 0$. Also, $1_ix^i = 1_ix^i1_i = \alpha_i(1_{-i}1_i)x^i$ and we have $1_i = \alpha_i(1_{-i}1_i)$. Applying α_{-i} to this relation we obtain $1_{-i} = 1_{-i}1_i$, hence $S_{-i} \subseteq S_i$ and now (i) follows easily.

$(i) \Rightarrow (ii)$. By assumption we easily have that $a_ix^i1_jx^j = 1_jx^ja_ix^i$ and $ra_ix^i = a_ix^ir$, for every $j \in \mathbb{Z}$ and $r \in R$. So, $R[x;\alpha]$ is commutative.

The proof $(i) \Leftrightarrow (iii)$ is similar with the proof of $(i) \Leftrightarrow (ii)$. □

Recall that if S is a ring and $\sigma : S \to S$ is an automorphism, an element $a \in S$ is said to be *σ-nilpotent* if for every $m \geq 1$ there exists $n \geq 1$ such that $a\sigma^m(a)\sigma^{2m}(a)...\sigma^{mn}(a) = 0$ (see [7] for more details). A subset B of S is *σ-nil* if every element of B is σ-nilpotent. Now we extend this notion to partial actions.

Definition 1. *Let (R, α) be a partial action of $\mathbb{Z}$ on R. An element $a \in R$ is said to be α-nilpotent if for every $m \geq 1$ there exists $n \geq 1$ such that $a\alpha_m(a1_{-m})\alpha_{2m}(a1_{-2m})...\alpha_{mn}(a1_{-mn}) = 0$. A subset I of R is called α-nil if every element of I is α-nilpotent.*

We write $N^i_\alpha(R) = \{a \in R : \exists n \geq 1, a\alpha_i(a1_{-i})...\alpha_{ni}(a1_{-ni}) = 0\}$ and $N_\alpha(R) = \cap_{i\geq 1}N^i_\alpha(R)$. Also $N^i(T) = \{a \in T : \exists n \geq 1, a\sigma^i(a)...\sigma^{ni}(a) = 0\}$, for any $i \geq 1$, and $N(T) = \cap_{i\geq 1}N^i(T)$.

Lemma 1. (i) *$N_\alpha(R)$ contains all α-nil subsets I of R.*

(ii) *For any $n > 0$ we have $N^n_\alpha(R) = N^n(T) \cap R$. In particular, $N_\alpha(R) = N(T) \cap R$.*

(iii) $N_\alpha(R)$ *is an* α*-invariant subset of* R.

Proof. (i) is clear. (iii) follows from (ii) since $N(T)$ is a σ-invariant subset of T. Thus we only proof (ii). Assume that $a \in R$. Since $1_{-i} = 1_R\sigma^{-i}(1_R)$, for any i, there exists $m > 0$ with $a\alpha_n(a1_{-n})...\alpha_{nm}(a1_{-nm}) = 0$ if and only if for such m we have $a\sigma^n(a)\sigma^n(1_R)1_R...\sigma^{nm}(a)\sigma^{nm}(1_R)1_R = 0$. This is equivalent to $a\sigma^n(a)...\sigma^{nm}(a) = 0$ and so $a \in N^n(T)$. Thus $a \in N^n_\alpha(R)$ if and only if $a \in N^n(T) \cap R$. □

The Jacobson radical of a skew polynomial ring and a skew Laurent polynomial ring are described in ([1], Theorem 3.1). Now we obtain similar results for partial skew polynomial rings and partial skew Laurent polynomial rings.

Proposition 3. *Let* (R, α) *be a partial action of* $\mathbb{Z}$ *on* R*. Then there exist* α*-nil* α*-invariant ideals* $K \subseteq J(R)$ *and* I *of* R *such that* $J(R\langle x;\alpha\rangle) = K\langle x;\alpha\rangle$ *and* $J(R[x;\alpha]) = J(R) \cap I + \sum_{i\geq 1}(S_i \cap I)x^i$.

Proof. By ([4], Proposition 6.1) we have $J(R\langle x;\alpha\rangle) = J(T\langle x;\sigma\rangle) \cap R\langle x;\alpha\rangle$.

By ([1], Theorem 3.1) $J(T\langle x;\sigma\rangle) = L\langle x;\sigma\rangle$, where $L \subseteq J(T)$ is σ-nil σ-invariant ideal of T. So $J(R\langle x;\alpha\rangle) = L\langle x;\sigma\rangle \cap R\langle x;\alpha\rangle = (L \cap R)\langle x;\alpha\rangle$, where $L \cap R \subseteq J(T) \cap R = J(R)$ is an α-nil α-invariant ideal of R. For $R[x;\alpha]$ the proof is similar. □

As in [8] we denote by $\mathcal{A}$ be the set of all maximal right ideals M of $R[x;\alpha]$ such that $S_ix^i \nsubseteq M$, for some $i \geq 1$, and by $\mathcal{B}$ the set of all remaining maximal right ideals of $R[x;\alpha]$. Since $R[x;\alpha]$ is naturally a $\mathbb{Z}$-graded ring, using ([8], Proposition 3) we have that

$$\mathcal{A}(R[x;\alpha]) = \bigcap_{M\in\mathcal{A}} M = \{f \in R[x;\alpha]; fS_ix^i \subseteq J(R[x;\alpha]),\ for\, all\, i \geq 1\}.$$

Also, we easily see that

$$B(R[x;\alpha]) = (\bigcap_{M\in\mathcal{B}} M \cap R) \oplus \sum_{i\geq 1} S_ix^i.$$

Note that, $J(R[x;\alpha]) = \mathcal{A}(R[x;\alpha]) \cap \mathcal{B}(R[x;\alpha])$.

The next result gives a characterization of $N_\alpha(R)$ when $R[x;\alpha]$ is right quasi-duo.

Lemma 2. *If* $R[x;\alpha]$ *is a right quasi-duo ring, then*

$$N_\alpha(R) = \mathcal{A}(R[x,\alpha]) \cap R = \{a \in R \mid a1_ix^i \in J(R[x;\alpha]),\ for\ all\ i \geq 1\}.$$

Moreover, $N_\alpha(R)$ *is an* α*-invariant ideal of* R.

Proof. Let $a \in N_\alpha(R)$. Then for all $i \geq 1$ there exists $n \geq 1$ such that $a\alpha_i(a1_{-i})\alpha_{2i}(a1_{-2i})...\alpha_{ni}(a1_{-ni}) = 0$. Consider $u = a1_ix^i + J(R[x;\alpha]) \in R[x,\alpha]/J(R[x,\alpha])$. By the above we have $u^n = 0$ and since $R[x,\alpha]/J(R[x,\alpha])$ is reduced we obtain $a1_ix^i \in J(R[x;\alpha])$. It follows that $a \in A(R[x,\alpha]) \cap R$.

On the other hand, let $a \in R$ be such that $a1_ix^i \in J(R[x;\alpha])$, for all $i \geq 1$. We fix such an i. Then by Proposition 1.5 there exists an α-nil ideal I of R such that $a1_i \in I$. Hence there exists m with

$$a1_i\alpha_i(a1_i1_{-i})\alpha_{2i}(a1_i1_{-2i})...\alpha_{mi}(a1_i1_{-mi}) = 0.$$

This easily gives

$$a1_R\sigma^i(1_R)\alpha_i(a1_{-i})\sigma^{2i}(1_R)\alpha_{2i}(a1_{-2i})...\sigma^{mi}(1_R)\alpha_{mi}(a1_{-mi})\sigma^{(m+1)i}(1_R) = 0$$

and it follows that

$$a\alpha_i(a1_{-i})\alpha_{2i}(a1_{-2i})...\alpha_{mi}(a1_{-mi})\alpha_{(m+1)i}(a1_{-(m+1)i}) = 0.$$

Hence $a \in N^i_\alpha(R)$, for all $i > 0$, i.e., $a \in N_\alpha(R)$. The rest is clear. □

As a consequence of Lemma 1.6 we have the following:

Corollary 1. *Suppose that $R[x;\alpha]$ is right quasi-duo. Then $N_\alpha(R)$ is an α-invariant ideal of R and $J(R[x;\alpha]) \subseteq N_\alpha(R)[x;\alpha] = \mathcal{A}(R[x;\alpha])$.*

Proof. The inclusion $N_\alpha(R)[x;\alpha] \subseteq \mathcal{A}(R[x;\alpha])$ is immediate from Lemma 1.6. Assume that $ax^i \in \mathcal{A}(R[x;\alpha])$, where $i > 0$. Then by Proposition 3 of [8] we have that $ax^i \in J(R[x;\alpha])$ and again by Lemma 1.6 we obtain $a \in N_\alpha(R)$. □

Next we will describe the Jacobson radical of $R[x;\alpha]$, when $R[x;\alpha]$ is quasi-duo.

Recall that a ring S is a subdirect product of the rings $\{S_i : i \in \Omega\}$ if for any $i \in \Omega$ there exists a surjective homomorphism $\varphi_i : S \to S_i$ such that $\bigcap_{i\in\Omega} ker\varphi_i = 0$.

Lemma 3. *Let U and V be ideals of R such that $U \subseteq V$ and V is α-invariant. Then $U + \sum_{i\geq 1}(V \cap S_i)x^i$ is a two-sided ideal of $R[x;\alpha]$ and $R[x;\alpha]/(U + \sum_{i\geq 1}(V \cap S_i)x^i)$ is a right quasi-duo ring if and only if R/U and $R[x;\alpha]/V[x;\alpha]$ are right quasi-duo rings.*

Proof. We clearly have that $U + \sum_{i\geq 1}(V \cap S_i)x^i$ is an ideal of $R[x;\alpha]$. Since V is α-invariant α induces a partial action $\overline{\alpha}$ of $\mathbb{Z}$ on R/V. Then there exists an isomorphism $(R/V)[x;\overline{\alpha}] \simeq R[x;\alpha]/V[x;\alpha]$ and note that

$$(U + \sum_{i\geq 1} S_ix^i) \cap V[x;\alpha] = U + \sum_{i\geq 1}(V \cap S_i)x^i.$$

We have an isomorphism

$$\varphi : R[x;\alpha]/(U + \sum_{i\geq 1}(V \cap S_i)x^i) \simeq R/U + \sum_{i\geq 1}\overline{S_i}x^i$$

defined by $\varphi(r + ax^i) = (r + U) + (a + V)x^i$, where $\overline{S_i} = (S_i + V)/V$, $i > 1$.

Consider the natural homomorphisms

$$\psi_1 : R[x;\alpha]/(U + \sum_{i\geq 1}(V \cap S_i)x^i) \to R/U \text{ and}$$
$$\psi_2 : R[x;\alpha]/(U + \sum_{i\geq 1}(V \cap S_i)x^i) \to (R/V)[x;\overline{\alpha}].$$

It is easy to see that $ker(\psi_1) \cap ker(\psi_2) = 0$ and so $R[x;\alpha]/(U + \sum_{i\geq 1}(V \cap S_i)x^i)$ is a subdirect product of R/U and $R/V[x;\overline{\alpha}]$. So the result follows from ([6], Corollary 3.6(2)). □

Theorem 1. *$R[x;\alpha]$ is right quasi-duo if and only if R is right quasi-duo, $J(R[x;\alpha]) = J(R) \cap N_\alpha(R) + \sum_{i\geq 1}(N_\alpha(R) \cap S_i)x^i$ and $(R/N_\alpha(R))[x;\overline{\alpha}]$ is commutative, where $\overline{\alpha}$ is the partial action induced by α on $R/N_\alpha(R)$.*

Proof. Suppose that $R[x;\alpha]$ is right quasi-duo. Then, by Corollary 1.7, we have that $N_\alpha(R)[x;\alpha] = \mathcal{A}(R[x;\alpha])$ and so

$$R[x;\alpha]/\mathcal{A}(R[x;\alpha]) \simeq (R/N_\alpha(R))[x;\overline{\alpha}].$$

Since the partial skew polynomial ring $R[x;\alpha]$ is $\mathbb{Z}$-graded, then by Theorem 5 of [8] we have that $(R/N_\alpha(R))[x;\overline{\alpha}]$ is commutative and R is right quasi-duo. Let $M \in \mathcal{B}$. Then we easily obtain that $M = M \cap R + \sum_{i\geq 1} S_i x^i$, where $M \cap R$ is a maximal ideal of R. Thus $\mathcal{B}(R[x;\alpha]) = J(R) + \sum_{i\geq 1} S_i x^i$. Hence,

$$J(R[x;\alpha]) = \mathcal{A}(R[x;\alpha]) \cap \mathcal{B}(R[x;\alpha]) = N_\alpha(R) \cap J(R) + \sum_{i\geq 1}(N_\alpha(R) \cap S_i)x^i.$$

Conversely, assume that R is right quasi-duo, $J(R[x;\alpha]) = J(R) \cap N_\alpha(R) + \sum_{i\geq 1}(N_\alpha(R) \cap S_i)x^i$ and $(R/N_\alpha(R))[x;\overline{\alpha}]$ is commutative. Then

$$R[x;\alpha]/J(R[x;\alpha]) = R[x;\alpha]/(J(R) \cap N_\alpha(R) + \sum_{i\geq 1}(N_\alpha(R) \cap S_i)x^i)).$$

Thus applying Lemma 1.8 with $U = J(R) \cap N_\alpha(R)$ and $V = N_\alpha(R)$ we easily conclude that $R[x;\alpha]/J(R[x;\alpha])$ is right quasi-duo and so $R[x;\alpha]$ is right quasi-duo. □

2. Quasi-duo partial skew Laurent polynomial rings

In this section we study quasi-duo partial skew Laurent polynomial rings. Let $\mathcal{A}$ be the set of all maximal right ideals M of $R\langle x;\alpha\rangle$ such that $1_n x^n \notin M$, for some $0 \neq n \in \mathbb{Z}$, and $\mathcal{B}$ the set of maximal right ideals M of $R\langle x;\alpha\rangle$ such that $1_i x^i \in M$, for all $0 \neq i \in \mathbb{Z}$. Then for any $M \in \mathcal{B}$ we easily have that $M = (M \cap R) \oplus \sum_{i\neq 0} S_i x^i$, with $S_i \subseteq (M \cap R)$ for all $i \neq 0$. Also we write $\mathcal{A}(R\langle x;\alpha\rangle) = \bigcap_{M\in\mathcal{A}} M$ and $\mathcal{B}(R\langle x;\alpha\rangle) = \cap_{M\in\mathcal{B}} M$.

We begin with the following easy remark.

Remark 1. Suppose that M is an ideal of $R < x;\alpha >$ such that $1_j x^j \notin M$, for some $0 \neq j \in \mathbb{Z}$. Then $1_{-j} x^{-j} \notin M$.

Lemma 4. *Suppose that $R < x;\alpha >$ is a right quasi-duo ring. Then $\mathcal{A}(R < x;\alpha >) = N_\alpha(R) < x;\alpha >$.*

Proof. First we show that $N_\alpha(R) = \mathcal{A}(R\langle x;\alpha\rangle) \cap R$. Suppose $r \in N_\alpha(R)$ and take any $i \geq 1$. Then there exists $n \geq 1$ such that $r\alpha_i(r1_{-i})...\alpha_{ni}(r1_{-ni}) = 0$ and hence $r1_i x^i \in R\langle x;\alpha\rangle$ is a nilpotent element. Since $R\langle x;\alpha\rangle/J(R\langle x;\alpha\rangle)$ is reduced it follows that $r1_i x^i \in J(R\langle x;\alpha\rangle)$. Hence $r1_i x^i \in M$, for all $M \in A$. Note that for each $M \in \mathcal{A}$ there exists $n_M \geq 1$ such that $1_{n_M} x^{n_M} \notin M$ and since $r1_i x^i \in M$, for all $i \geq 0$, then we have that $r \in M$, for all $M \in \mathcal{A}$. So $r \in \cap_{M\in A} M = \mathcal{A}(R\langle x;\alpha\rangle)$.

On the other hand, let $a \in \mathcal{A}(R\langle x;\alpha\rangle) \cap R$. Then we have that $a1_i x^i \in J(R\langle x;\alpha\rangle) = K\langle x;\alpha\rangle$, for all $i \geq 1$, where K is an α-nil ideal of R. Thus we easily obtain that $a \in N_\alpha(R)$.

From the first part we conclude that $N_\alpha(R)\langle x;\alpha\rangle \subseteq \mathcal{A}(R\langle x;\alpha\rangle)$.

Conversely, let $f = \sum_{j=p}^{n} a_j x^j \in \mathcal{A}(R\langle x;\alpha\rangle)$. Then we have that $f1_i x^i \in J(R\langle x;\alpha\rangle)$, for all $0 \neq i \in \mathbb{Z}$. Fix any $i \neq 0$ with $p \leq i \leq n$. Then the coefficient of degree 0 in $f1_{-i}x^{-i}$ is a_i. Since $J(R\langle x;\alpha\rangle)$ is a homogeneous ideal it follows that $a_i \in J(R\langle x;\alpha\rangle) \subseteq \mathcal{A}(\langle x;\alpha\rangle)$, for any $p \leq i \leq n$. So $a_i x^i \in \mathcal{A}(R\langle x;\alpha\rangle)$, for $p \leq i \leq n$. Now arguing as before we have that $a_i \in N_\alpha(R)$ and we are done. □

The following definition was given in [4].

Definition 2. *Let R be a ring and α a partial action of $\mathbb{Z}$ on R. We say that α is of finite type if there exists $j_1, ..., j_n \in \mathbb{Z}$ such that for any $k \in \mathbb{Z}$ we have that $R = S_{-k+j_1} + ... + S_{-k+j_n}$.*

In the following lemma we show that when α is a partial action of finite type do not exist maximal right ideals of $R\langle x;\alpha\rangle$ in $\mathcal{B}$. So to compute the Jacobson radical of $R\langle x;\alpha\rangle$ it is enough to compute $\mathcal{A}(R\langle x;\alpha\rangle)$.

Lemma 5. *If α is a partial action of finite type of $\mathbb{Z}$ on R, then $\mathcal{B} = \emptyset$. In particular $J(R\langle x;\alpha\rangle) = \mathcal{A}(R\langle x;\alpha\rangle)$.*

Proof. Let I be a maximal right ideal in $\mathcal{B}$. Then $I = (I \cap R) \oplus \sum_{i\neq 0} S_i x^i$, where $I \cap R$ is a right ideal of R which contains S_i, for all $i \neq 0$. By the fact that α is of finite type we have that $R = \bigoplus_{i=1}^{n} S_{1+i} \subset I \cap R$, for some $n \geq 0$. It follows that $I = R < x;\alpha >$, which is a contradiction. □

From now on α is a partial action of finite type of $\mathbb{Z}$ on R and (T,σ) is the enveloping action of (R,α), where σ is an automorphism of T.

Now we give a precise description of the Jacobson radical of $R\langle x;\alpha\rangle$, when $R\langle x;\alpha\rangle$ is a quasi-duo ring.

Proposition 4. *If $R\langle x;\alpha\rangle$ is right quasi-duo, then*

$$J(R\langle x;\alpha\rangle) = N_\alpha(R)\langle x;\alpha\rangle.$$

In particular, $N_\alpha(R)$ is an α-invariant ideal of R.

Proof. The result follows from Lemmas 2.2 and 2.4. □

Finally we give the main result of this section which extends ([8], Corollary 10). The proof is an easy consequence of the previews results.

Theorem 2. *$R\langle x;\alpha\rangle$ is right quasi-duo if and only if $N_\alpha(R)$ is an α-invariant ideal of R,*

$$J(R\langle x;\alpha\rangle) = N_\alpha(R)\langle x;\alpha\rangle$$

and $(R/N_\alpha(R))\langle x;\overline{\alpha}\rangle$ is a commutative ring, where $\overline{\alpha}$ is the partial action induced by α on $R/N_\alpha(R)$.

3. Examples

In this section we give examples to answer some natural questions which can be risen after the results we obtained in the paper.

Example 1. Let K be a field, $T = Ke_1 \oplus Ke_2 \oplus Ke_3$, where $\{e_1, e_2, e_3\}$ are orthogonal central idempotents. We define an automorphism $\sigma : T \to T$ as follows: $\sigma(e_1) = e_2$, $\sigma(e_2) = e_3$, $\sigma(e_3) = e_1$ and $\sigma|_K = id_k$.

Now take $R = Ke_1 \oplus Ke_2$ and consider the partial action α of $\mathbb{Z}$ on R defined as the restriction of σ. This means that we take $S_i = Ke_1$, for all $i \equiv 2(mod 3)$, and $S_j = Ke_2$, for all $j \equiv 1(mod 3)$, $S_l = R$, for all $l \equiv 0(mod 3)$. Thus α is given by $\alpha_1(e_1) = e_2$, $\alpha_2(e_2) = e_1$, $\alpha_3 = id_R$, and so on. We clearly have that (T,σ) is the enveloping action of (R,α).

Note that $N^1_\alpha(R) = R$ because, for all $r = a_1e_1 + a_2e_2 \in R$, we have that $r\alpha_1(re_1)\alpha_2(re_2) = 0$. Since $\alpha_{3i} = id_R$, for all $i \in \mathbb{Z}$, we have that $N_\alpha(R) = \bigcap_{i\geq 1} N^i(R) = 0$. Moreover, we easily have that $0 = N_\sigma(T) \subsetneq N(T)$. Then, in this case,

$$N^1_\alpha(R) \supsetneq N_\alpha(R) = N_\sigma(T) \subsetneq N(T).$$

The next example shows that $R[x;\alpha]$ may be right quasi-duo even when $T[x;\sigma]$ is not right quasi-duo.

Example 2. Let K, T and σ be as in the last example. We consider $R = Ke_1$ and we have a natural partial action α of $\mathbb{Z}$ on R as follows: $S_i = R$ and $\alpha_i = id_R$, for all $i \equiv 0(mod 3)$; $S_j = 0$ and $\alpha_j = 0$ otherwise. Thus, $R[x;\alpha] = \bigoplus_{i\geq 0} Ke_1x^{3i}$ is right quasi-duo because is commutative.

We easily have that $e_i\sigma(e_i) = 0$, for $i = 1,2,3$. Thus $e_i \in N(T)$, for $i = 1,2,3$. Note that

$$(e_1 + e_2)\sigma(e_1 + e_2)\sigma^2(e_1 + e_2) = (e_1 + e_2)(e_2 + e_3)(e_3 + e_1) = 0$$

and we obtain that $e_1 + e_2 \in N(T)$ but $1 = e_1 + e_2 + e_3 \notin N(T)$. Hence $N(T)$ is not an ideal of T and by ([7], Proposition 2.3) $T[x;\sigma]$ is not right quasi-duo.

The next example shows that the Theorem 2.5 is does not hold when α is not of finite type.

Example 3. Let K be a field and $R = Ke_1 \oplus Ke_2$. We define a partial action of $\mathbb{Z}$ on R as follows: $S_0 = R$, $S_i = Ke_1$ for $i \neq 0$, $\alpha_0 = id_R$ and $\alpha_i = id_{S_i}$ for $i \neq 0$. Note that in this case $R < x;\alpha >$ is a right quasi-duo ring. We claim that $N_\alpha(R) = Ke_2$. In fact, $e_2\alpha_i(e_1e_2) = 0$, for any $i \neq 0$ and we obtain that $e_2 \in N_\alpha(R)$. Since $\alpha_i(e_1) = e_1$, for any $i \neq 0$, we have that $e_1 \notin N_\alpha(R)$. Thus $N_\alpha(R) = Ke_2$. It is not difficult to see that $M = Ke_1 + \sum_{i\neq 0} S_ix^i = Ke_1\langle x;\alpha\rangle$ is a maximal ideal of $R\langle x;\alpha\rangle$ and $\mathcal{B}(R\langle x;\alpha\rangle) = \{M\}$. Since $\mathcal{A}(R\langle x;\alpha\rangle) = N_\alpha(R)\langle x;\alpha\rangle$, then $J(R\langle x;\alpha\rangle = \mathcal{A}(R\langle x;\alpha\rangle) \cap \mathcal{B}(R\langle x;\alpha\rangle) = Ke_2\langle x;\alpha\rangle \cap Ke_1\langle x;\alpha\rangle = (0) \neq N_\alpha\langle x;\alpha\rangle$.

The next example shows that the converse of Lemma 2.4 is not true, in general.

Example 4. Let $R = \oplus^n_{i=-n,i\neq 0} Ke_i$ be a ring, where K is a field and $\{e_i : 1 \leq i \leq n, i \neq 0\}$ is a set of orthogonal idempotents. We define a partial action of $\mathbb{Z}$ of R as follows: the ideals are $S_i = 0$ for all $|i| > n$, $S_i = Ke_i$ for $i \neq 0$ and $-n \leq i \leq n$ and $S_0 = R$. The isomorphisms α_i are the zero application for all $|i| > n$, $\alpha_i(e_{-i}) = e_i$, for $i \neq 0$ and $-n \leq i \leq n$ and $\alpha_0 = id_R$. We easily have that α is not of finite type. We see that even in this case the set of maximal ideals in $\mathcal{B}$ is empty. In fact, if $M \in \mathcal{B}$, then $M \cap R$ contains S_i for all $i \neq 0$ and it follows that $R \subseteq M \cap R$. The result follows.

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