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On a semigroup of closed connected partial homeomorphisms of the unit interval with a fixed point

Ivan Chuchman

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Abstract. In this paper we study the semigroup $\mathfrak{IC}(I,[a])$ ($\mathfrak{IO}(I,[a])$) of closed (open) connected partial homeomorphisms of the unit interval I with a fixed point $a \in I$. We describe left and right ideals of $\mathfrak{IC}(I,[0])$ and the Green's relations on $\mathfrak{IC}(I,[0])$. We show that the semigroup $\mathfrak{IC}(I,[0])$ is bisimple and every non-trivial congruence on $\mathfrak{IC}(I,[0])$ is a group congruence. Also we prove that the semigroup $\mathfrak{IC}(I,[0])$ is isomorphic to the semigroup $\mathfrak{IO}(I,[0])$ and describe the structure of a semigroup $\mathfrak{II}(I,[0]) = \mathfrak{IC}(I,[0]) \sqcup \mathfrak{IO}(I,[0])$. As a corollary we get structures of semigroups $\mathfrak{IC}(I,[a])$ and $\mathfrak{IO}(I,[a])$ for an interior point $a \in I$.

1. Introduction and preliminaries

Furthermore we shall follow the terminology of [2] and [6]. For a semigroup S we denote the semigroup S with the adjoined unit by S^1 (see [2]).

A semigroup S is called *inverse* if for any element $x \in S$ there exists a unique element $x^{-1} \in S$ (called the *inverse* of x) such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. If S is an inverse semigroup, then the function $\mathrm{inv}\colon S \to S$ which assigns to every element x of S its inverse element x^{-1} is called *inversion*.

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If S is a semigroup, then we shall denote the subset of idempotents in S by $E(S)$. If S is an inverse semigroup, then $E(S)$ is closed under multiplication and we shall refer to $E(S)$ a *band* (or the *band of* S). If the band $E(S)$ is a non-empty subset of S, then the semigroup operation on S determines the following partial order $\leqslant$ on $E(S)$: $e \leqslant f$ if and only if $ef = fe = e$. This order is called the *natural partial order* on $E(S)$. A *semilattice* is a commutative semigroup of idempotents. A semilattice E is called *linearly ordered* or a *chain* if its natural order is a linear order. Let E be a semilattice and $e \in E$. We denote ${\downarrow}e = \{f \in E \mid f \leqslant e\}$ and ${\uparrow}e = \{f \in E \mid e \leqslant f\}$.

If S is a semigroup, then we shall denote by $\mathscr{R}$, $\mathscr{L}$, $\mathscr{J}$, $\mathscr{D}$ and $\mathscr{H}$ the Green relations on S (see [2]):

$$\begin{aligned}
&a\mathscr{R}b \text{ if and only if } aS^1 = bS^1;\\
&a\mathscr{L}b \text{ if and only if } S^1a = S^1b;\\
&a\mathscr{J}b \text{ if and only if } S^1aS^1 = S^1bS^1;\\
&\mathscr{D} = \mathscr{L} \circ \mathscr{R} = \mathscr{R} \circ \mathscr{L};\\
&\mathscr{H} = \mathscr{L} \cap \mathscr{R}.
\end{aligned}$$

A semigroup S is called *simple* if S does not contain proper two-sided ideals and *bisimple* if S has only one $\mathscr{D}$-class.

A congruence $\mathfrak{C}$ on a semigroup S is called *non-trivial* if $\mathfrak{C}$ is distinct from universal and identity congruence on S, and *group* if the quotient semigroup $S/\mathfrak{C}$ is a group.

The bicyclic semigroup $\mathscr{C}(p,q)$ is the semigroup with the identity 1 generated by elements p and q subject only to the condition $pq = 1$. The distinct elements of $\mathscr{C}(p,q)$ are exhibited in the following useful array:

$$\begin{array}{ccccc}
1 & p & p^2 & p^3 & \cdots\\
q & qp & qp^2 & qp^3 & \cdots\\
q^2 & q^2p & q^2p^2 & q^2p^3 & \cdots\\
q^3 & q^3p & q^3p^2 & q^3p^3 & \cdots\\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

The bicyclic semigroup is bisimple and every one of its congruences is either trivial or a group congruence. Moreover, every non-annihilating homomorphism h of the bicyclic semigroup is either an isomorphism or the image of $\mathscr{C}(p,q)$ under h is a cyclic group (see [2, Corollary 1.32]). The bicyclic semigroup plays an important role in algebraic theory of semigroups and in the theory of topological semigroups. For example the well-known Andersen's result [1] states that a (0–)simple semigroup is completely (0–)simple if and only if it does not contain the bicyclic semigroup.

Let $\mathscr{I}_X$ denote the set of all partial one-to-one transformations of an non-empty set X together with the following semigroup operation:

$$x(\alpha\beta) = (x\alpha)\beta \quad \text{if} \quad x \in \operatorname{dom}(\alpha\beta) = \{y \in \operatorname{dom}\alpha \mid y\alpha \in \operatorname{dom}\beta\},$$

for $\alpha, \beta \in \mathscr{I}_X$. The semigroup $\mathscr{I}_X$ is called the *symmetric inverse semigroup* over the set X (see [2, Section 1.9]). The symmetric inverse semigroup was introduced by Wagner [10] and it plays a major role in the theory of semigroups.

Let I be an interval $[0,1]$ with the usual topology. A partial map $\alpha\colon I \rightharpoonup I$ is called:

- *closed*, if $\operatorname{dom}\alpha$ and $\operatorname{ran}\alpha$ are closed subsets in I;
- *open*, if $\operatorname{dom}\alpha$ and $\operatorname{ran}\alpha$ are open subsets in I;
- *convex*, if $\operatorname{dom}\alpha$ and $\operatorname{ran}\alpha$ are convex non-singleton subsets in I;
- *monotone*, if $x_1 \leqslant x_2$ implies $(x_1)\alpha \leqslant (x_2)\alpha$, for all $x_1, x_2 \in \operatorname{dom}\alpha$;
- a *local homeomorphism*, if the restriction $\alpha|_{\operatorname{dom}\alpha}\colon \operatorname{dom}\alpha \to \operatorname{ran}\alpha$ is a homeomorphism.

We fix an arbitrary $a \in I$. Hereafter we shall denote by:

- $\mathfrak{IC}(I,[a])$ the semigroup of all closed connected partial homeomorphisms α such that $\operatorname{Int}_I(\operatorname{dom}\alpha) \neq \varnothing$ and $(a)\alpha = a$;
- $\mathfrak{IO}(I,[a])$ the semigroup of all open connected partial homeomorphisms α such that $(a)\alpha = a$;
- $\mathfrak{H}(I)$ the group of all homeomorphisms of I;
- $\mathfrak{H}^{\nearrow}(I)$ the group of all monotone homeomorphisms of I;
- $\mathbb{I}$ the identity map from I onto I.

Remark 1. We observe that for every $a \in I$ the semigroups $\mathfrak{IC}(I,[a])$ and $\mathfrak{IO}(I,[a])$ are inverse subsemigroups of the symmetric inverse semigroup $\mathscr{I}_I$ over the set I.

In [3, 4] Gluskin studied the semigroup S of homeomorphic transformations of the unit interval. He described all ideals, homomorphisms and automorphisms of the semigroup S and congruence-free subsemigroups of S. This studies was continued in [7] by Shneperman. In [9] Shneperman described the structure of the semigroup of homeomorphisms of a simple arc. In the paper [8] he studied a semigroup $G(X)$ of all continuous transformations of a closed subset X of the real line.

In our paper we study the semigroup $\mathfrak{IC}(I,[a])$ ($\mathfrak{IO}(I,[a])$) of closed (open) connected partial homeomorphisms of the unit interval I with a fixed point $a \in I$. We describe left and right ideals of $\mathfrak{IC}(I,[0])$ and the Green's relations on $\mathfrak{IC}(I,[0])$. We show that the semigroup $\mathfrak{IC}(I,[0])$ is bisimple and every non-trivial congruence on $\mathfrak{IC}(I,[0])$ is a group congruence. Also we prove that the semigroup $\mathfrak{IC}(I,[0])$ is isomorphic to the semigroup $\mathfrak{IO}(I,[0])$ and describe the structure of a semigroup $\mathfrak{II}(I,[0]) = \mathfrak{IC}(I,[0]) \sqcup \mathfrak{IO}(I,[0])$. As a corollary we get structures of semigroups $\mathfrak{IC}(I,[a])$ and $\mathfrak{IO}(I,[a])$ for an interior point $a \in I$.

2. On the semigroup $\mathfrak{IC}(I,[0])$

Proposition 1. *The following conditions hold:*

(i) *every element of the semigroup* $\mathfrak{IC}(I,[0])$ ($\mathfrak{IO}(I,[1])$) *is a monotone partial map;*

(ii) *the semigroups* $\mathfrak{IC}(I,[0])$ *and* $\mathfrak{IC}(I,[1])$ *are isomorphic;*

(iii) $\max\{\operatorname{dom}\alpha\}$ *exists for every* $\alpha \in \mathfrak{IC}(I,[0])$*;*

(iv) $\sup\{\operatorname{dom}\alpha\}$ *exists for every* $\alpha \in \mathfrak{IO}(I,[0])$*;*

(v) $(0)\alpha = 0$ *and* $(1)\alpha = 1$ *for every* $\alpha \in \mathfrak{H}^{\nearrow}(I)$.

Proof. Statements (i), (iii), (iv) and (v) follow from elementary properties of real-valued continuous functions.

(ii) A homomorphism $\mathfrak{i}\colon \mathfrak{IC}(I,[0]) \to \mathfrak{IC}(I,[1])$ we define by the following way:

$$(\alpha)\mathfrak{i} = \beta, \quad \text{where } \begin{array}{l} \operatorname{dom}\beta = \{1-x \mid x \in \operatorname{dom}\alpha\}, \\ \operatorname{ran}\beta = \{1-x \mid x \in \operatorname{ran}\alpha\}, \text{ and} \\ (a)\beta = 1-(1-a)\alpha \text{ for all } a \in \operatorname{dom}\beta. \end{array}$$

Simple verifications show that such defined map $\mathfrak{i}$ is an isomorphism from the semigroup $\mathfrak{IC}(I,[0])$ onto the semigroup $\mathfrak{IO}(I,[1])$. □

Proposition 2. *The following statements hold:*

(i) *an element* α *of the semigroup* $\mathfrak{IC}(I,[0])$ *is an idempotent if and only if* $(x)\alpha = x$ *for every* $x \in \operatorname{dom}\alpha$*;*

(*ii*) *If* $\varepsilon, \iota \in E(\mathfrak{IC}(I,[0]))$*, then* $\varepsilon \leqslant \iota$ *if and only if* $\operatorname{dom}\varepsilon \subseteq \operatorname{dom}\iota$*;*

(*iii*) *The semilattice* $E(\mathfrak{IC}(I,[0]))$ *is isomorphic to the semilattice* $((0,1], \min)$ *under the mapping* $(\varepsilon)h = \max\{\operatorname{dom}\varepsilon\}$*;*

(*iv*) $\alpha\mathscr{R}\beta$ *in* $\mathfrak{IC}(I,[0])$ *if and only if* $\operatorname{dom}\alpha = \operatorname{dom}\beta$*;*

(*v*) $\alpha\mathscr{L}\beta$ *in* $\mathfrak{IC}(I,[0])$ *if and only if* $\operatorname{ran}\alpha = \operatorname{ran}\beta$*.*

(*vi*) $\alpha\mathscr{H}\beta$ *in* $\mathfrak{IC}(I,[0])$ *if and only if* $\operatorname{dom}\alpha = \operatorname{dom}\beta$ *and* $\operatorname{ran}\alpha = \operatorname{ran}\beta$*.*

(*vii*) *for every distinct idempotents* $\varepsilon, \iota \in \mathfrak{IC}(I,[0])$ *there exists an element* α *of the semigroup* $\mathfrak{IC}(I,[0])$ *such that* $\alpha\cdot\alpha^{-1} = \varepsilon$ *and* $\alpha^{-1}\cdot\alpha = \iota$*;*

(*viii*) $\alpha\mathscr{D}\beta$ *for all* $\alpha, \beta \in \mathfrak{IC}(I,[0])$*, and hence the semigroup* $\mathfrak{IC}(I,[0])$ *is bisimple;*

(*ix*) $\alpha\mathscr{J}\beta$ *for all* $\alpha, \beta \in \mathfrak{IC}(I,[0])$*, and hence the semigroup* $\mathfrak{IC}(I,[0])$ *is simple;*

(*x*) *a subset* $\mathscr{L}$ *is a left ideal of* $\mathfrak{IC}(I,[0])$ *if and only if there exists* $a \in (0,1]$ *such that either* $\mathscr{L} = \{\alpha \in \mathfrak{IC}(I,[0]) \mid \operatorname{ran}\alpha \subseteq [0,a)\}$ *or* $\mathscr{L} = \{\alpha \in \mathfrak{IC}(I,[0]) \mid \operatorname{ran}\alpha \subseteq [0,a]\}$*;*

(*xi*) *a subset* $\mathscr{R}$ *is a right ideal of* $\mathfrak{IC}(I,[0])$ *if and only if there exists a* $a \in (0,1]$ *such that either* $\mathscr{R} = \{\alpha \in \mathfrak{IC}(I,[0]) \mid \operatorname{dom}\alpha \subseteq [0,a)\}$ *or* $\mathscr{R} = \{\alpha \in \mathfrak{IC}(I,[0]) \mid \operatorname{dom}\alpha \subseteq [0,a]\}$.

Proof. Statements (*i*), (*ii*) and (*iii*) are trivial and they follow from the definition of the semigroup $\mathfrak{IC}(I,[0])$.

(*iv*) Let be $\alpha, \beta \in \mathfrak{IC}(I,[0])$ such that $\alpha\mathscr{R}\beta$. Since $\alpha\mathfrak{IC}(I,[0]) = \beta\mathfrak{IC}(I,[0])$ and $\mathfrak{IC}(I,[0])$ is an inverse semigroup, Theorem 1.17 [2] implies that

$$\alpha\mathfrak{IC}(I,[0]) = \alpha\alpha^{-1}\mathfrak{IC}(I,[0]) \qquad \text{and} \qquad \beta\mathfrak{IC}(I,[0]) = \beta\beta^{-1}\mathfrak{IC}(I,[0]),$$

and hence we have that $\alpha\alpha^{-1} = \beta\beta^{-1}$. Therefore we get that $\operatorname{dom}\alpha = \operatorname{dom}\beta$.

Conversely, let be $\alpha, \beta \in \mathfrak{IC}(I,[0])$ such that $\operatorname{dom}\alpha = \operatorname{dom}\beta$. Then $\alpha\alpha^{-1} = \beta\beta^{-1}$. Since $\mathfrak{IC}(I,[0])$ is an inverse semigroup, Theorem 1.17 [2] implies that

$$\alpha\mathfrak{IC}(I,[0]) = \alpha\alpha^{-1}\mathfrak{IC}(I,[0]) = \beta\beta^{-1}\mathfrak{IC}(I,[0]) = \beta\mathfrak{IC}(I,[0]),$$

and hence $\alpha\mathfrak{IC}(I,[0]) = \beta\mathfrak{IC}(I,[0])$.

The proof of statement (*v*) is similar to (*iv*).

Statement (vi) follows from (iv) and (v).

(vii) We fix arbitrary distinct idempotents ε and ι in $\mathfrak{IC}(I,[0])$. If $d_\varepsilon = \max\{\operatorname{dom}\varepsilon\}$ and $d_\iota = \max\{\operatorname{dom}\iota\}$, then $d_\varepsilon \neq 0$, $d_\iota \neq 0$, and ε and ι are identity maps of intervals $[0,d_\varepsilon]$ and $[0,d_\iota]$, respectively. We define a partial map $\alpha\colon I \rightharpoonup I$ as follows:

$$\operatorname{dom}\alpha = [0,d_\varepsilon], \quad \operatorname{ran}\alpha = [0,d_\iota] \quad \text{and} \quad (x)\alpha = \frac{d_\iota}{d_\varepsilon}\cdot x, \text{ for all } x \in \operatorname{dom}\alpha.$$

Then we have that $\alpha \in \mathfrak{IC}(I,[0])$, $\alpha\cdot\alpha^{-1} = \varepsilon$ and $\alpha^{-1}\cdot\alpha = \iota$.

$(viii)$ Statement (vii) and Lemma 1.1 from [5] imply that $\mathfrak{IC}(I,[0])$ is a bisimple semigroup.

Since $\mathscr{D} \subseteq \mathscr{J}$, statement $(viii)$ implies assertion (ix).

(x) The semigroup operation on $\mathfrak{IC}(I,[0])$ implies that the sets $\{\alpha \in \mathfrak{IC}(I,[0]) \mid \operatorname{ran}\alpha \subseteq [0,a)\}$ and $\{\alpha \in \mathfrak{IC}(I,[0]) \mid \operatorname{ran}\alpha \subseteq [0,a]\}$ are left ideals in $\mathfrak{IC}(I,[0])$, for every $a \in (0,1]$.

Suppose that $\mathscr{L}$ is an arbitrary left ideal of the semigroup $\mathfrak{IC}(I,[0])$. We fix any $\alpha \in \mathscr{L}$. Then statements (i), (ii) and (v) imply that the left ideal $\mathscr{L}$ contains all $\beta \in \mathscr{L}$ such that $\operatorname{ran}\beta \subseteq \operatorname{ran}\alpha$. We put

$$A = \bigcup_{\alpha\in\mathscr{L}} \operatorname{ran}\alpha$$

and let $a = \sup A$. If there exists $\alpha \in \mathscr{L}$ such that $\sup\operatorname{ran}\alpha = a$ then statement (v) implies that $\mathscr{L} = \{\alpha \in \mathfrak{IC}(I,[0]) \mid \operatorname{ran}\alpha \subseteq [0,a]\}$. In other case we have that statement (v) implies that

$$\mathscr{L} = \{\alpha \in \mathfrak{IC}(I,[0]) \mid \operatorname{ran}\alpha \subseteq [0,a)\}.$$

The proof of statement (xi) is similar to statement (x). □

Definitions of the group $\mathfrak{H}^{\nearrow}(I)$ and the semigroup $\mathfrak{IC}(I,[0])$ imply the following:

Proposition 3. *The group of units of the semigroup $\mathfrak{IC}(I,[0])$ is isomorphic to (i.e., coincides with) the group $\mathfrak{H}^{\nearrow}(I)$.*

Proposition 2.20 of [2] states that every two subgroup which lie in some $\mathscr{D}$-class are isomorphic, and hence Proposition 3 implies the following:

Corollary 1. *Every maximal subgroup of the semigroup $\mathfrak{IC}(I,[0])$ is isomorphic to $\mathfrak{H}^{\nearrow}(I)$.*

Later we need the following two lemmas:

Lemma 1. *Let $\mathfrak{R}$ is an arbitrary congruence on a semilattice E and let a and b be elements of the semilattice E such that $a\mathfrak{R}b$. If $a \leqslant b$ then $a\mathfrak{R}c$ for all $c \in E$ such that $a \leqslant c \leqslant b$.*

The proof of the lemma follows from the definition of a congruence on a semilattice.

Lemma 2. *For arbitrary distinct idempotents α and β of the semigroup $\mathfrak{IC}(I,[0])$ there exists a subsemigroup $\mathscr{C}$ in $\mathfrak{IC}(I,[0])$ such that $\alpha,\beta \in \mathscr{C}$ and $\mathscr{C}$ is isomorphic to the bicyclic semigroup $\mathscr{C}(p,q)$.*

Proof. Without loss of generality we can assume that $\beta \leqslant \alpha$ in $E(\mathfrak{IC}(I,[0]))$. We define partial maps $\gamma,\delta\colon I \rightharpoonup I$ as follows:

$$\operatorname{dom}\gamma = [0,d_\alpha], \quad \operatorname{ran}\gamma = [0,d_\beta] \quad \text{and} \quad (x)\gamma = \frac{d_\beta}{d_\alpha}\cdot x, \text{ for all } x \in \operatorname{dom}\gamma,$$

and

$$\operatorname{dom}\delta = [0,d_\beta], \quad \operatorname{ran}\delta = [0,d_\alpha] \quad \text{and} \quad (x)\delta = \frac{d_\alpha}{d_\beta}\cdot x, \text{ for all } x \in \operatorname{dom}\delta,$$

where $d_\alpha = \max\{\operatorname{dom}\alpha\}$ and $d_\beta = \max\{\operatorname{dom}\beta\}$. Then we have that

$$\alpha\cdot\gamma = \gamma\cdot\alpha = \gamma, \quad \alpha\cdot\delta = \delta\cdot\alpha = \delta, \quad \gamma\cdot\delta = \alpha \quad \text{and} \quad \delta\cdot\gamma = \beta \neq \alpha.$$

Hence by Lemma 1.31 from [2] we get that a subsemigroup in $\mathfrak{IC}(I,[0])$ which is generated by elements γ and δ is isomorphic to the bicyclic semigroup $\mathscr{C}(p,q)$. □

Theorem 1. *Every non-trivial congruence on the semigroup $\mathfrak{IC}(I,[0])$ is a group congruence.*

Proof. Suppose that $\mathfrak{K}$ is a non-trivial congruence on the semigroup $\mathfrak{IC}(I,[0])$. Then there exist distinct elements α and β in $\mathfrak{IC}(I,[0])$ such that $\alpha\mathfrak{K}\beta$. We consider the following three cases:

(*i*) α and β are idempotents in $\mathfrak{IC}(I,[0])$;

(*ii*) α and β are not $\mathscr{H}$-equivalent in $\mathfrak{IC}(I,[0])$;

(*iii*) α and β are $\mathscr{H}$-equivalent in $\mathfrak{IC}(I,[0])$.

Suppose case (*i*) holds and without loss of generality we assume that $\alpha \leqslant \beta$ in $E(\mathfrak{IC}(I,[0]))$. We define a partial map $\rho\colon I \rightharpoonup I$ as follows:

$$\operatorname{dom}\rho = \operatorname{dom}\beta, \quad \operatorname{ran}\rho = I \quad \text{and} \quad (x)\rho = \frac{1}{d_\beta}\cdot x, \text{ for all } x \in \operatorname{dom}\rho,$$

where $d_\beta = \max\{\operatorname{dom}\beta\}$. Then we have that $\rho^{-1}\cdot\beta\cdot\rho = \mathbb{I}$ and hence by Proposition 1(i) the element $\alpha_\beta = \rho^{-1}\cdot\alpha\cdot\rho$ is an idempotent of the semigroup $\mathfrak{IC}(I,[0])$. Obviously, $\alpha_\beta \leqslant \mathbb{I}$ in $E(\mathfrak{IC}(I,[0]))$, $\alpha_\beta \neq \mathbb{I}$ and $\alpha_\beta\mathfrak{K}\mathbb{I}$. Then by Lemma 2 there exist $\gamma,\delta\in\mathfrak{IC}(I,[0])$ such that

$$\mathbb{I}\cdot\gamma=\gamma\cdot\mathbb{I}=\gamma,\qquad \mathbb{I}\cdot\delta=\delta\cdot\mathbb{I}=\delta,\qquad \gamma\cdot\delta=\mathbb{I}\qquad\text{and}\qquad \delta\cdot\gamma=\alpha_\beta\neq\mathbb{I},$$

and a subsemigroup $\mathscr{C}\langle\gamma,\delta\rangle$ in $\mathfrak{IC}(I,[0])$ which is generated by elements γ and δ is isomorphic to the bicyclic semigroup $\mathscr{C}(p,q)$. Since by Corollary 1.32 from [2] every non-trivial congruence on the bicyclic semigroup $\mathscr{C}(p,q)$ is a group congruence on $\mathscr{C}(p,q)$ we get that all idempotents of the semigroup $\mathscr{C}\langle\gamma,\delta\rangle$ are $\mathfrak{K}$-equivalent. Also by Lemma 1.31 from [2] we get that every idempotent of the semigroup $\mathscr{C}\langle\gamma,\delta\rangle$ has a form

$$\delta^n\cdot\gamma^n = (\underbrace{\delta\cdot\ldots\cdot\delta}_{n-\text{times}})\cdot(\underbrace{\gamma\cdot\ldots\cdot\gamma}_{n-\text{times}}),\qquad\text{where}\quad n=0,1,2,3,\ldots,$$

and hence we get that $\operatorname{dom}\big(\delta^n\cdot\gamma^n\big)=[0,d^n]$, where $d=\max\{\operatorname{dom}\alpha_\beta\}$. This implies that for every idempotent $\varepsilon\in\mathfrak{IC}(I,[0])$ there exists a positive integer n such that $\delta^n\cdot\gamma^n\leqslant\varepsilon$, and hence by Lemma 1 we get that all idempotents of the semigroup $\mathfrak{IC}(I,[0])$ are $\mathfrak{K}$-equivalent. Then Lemma 7.34 and Theorem 7.36 from [2] imply that the quotient semigroup $\mathfrak{IC}(I,[0])/\mathfrak{K}$ is a group.

Suppose case (ii) holds: α and β are not $\mathscr{H}$-equivalent in $\mathfrak{IC}(I,[0])$. Since $\mathfrak{IC}(I,[0])$ is an inverse semigroup we get that either $\alpha\alpha^{-1}\neq\beta\beta^{-1}$ or $\alpha^{-1}\alpha\neq\beta^{-1}\beta$. Suppose inequality $\alpha\alpha^{-1}\neq\beta\beta^{-1}$ holds. Since $\alpha\mathfrak{K}\beta$ and $\mathfrak{IC}(I,[0])$ is an inverse semigroup, Lemma III.1.1 from [6] implies that $\big(\alpha\alpha^{-1}\big)\,\mathfrak{K}\,\big(\beta\beta^{-1}\big)$, and hence by case (i) we get that $\mathfrak{K}$ is a group congruence on the semigroup $\mathfrak{IC}(I,[0])$. In the case $\alpha^{-1}\alpha\neq\beta^{-1}\beta$ the proof is similar.

Suppose case (iii) holds: α and β are $\mathscr{H}$-equivalent in $\mathfrak{IC}(I,[0])$. Then Theorem 2.3 of [2] implies that without loss of generality we can assume that α and β are elements of the group of units $H(\mathbb{I})$ of the semigroup $\mathfrak{IC}(I,[0])$. Therefore we get that $\mathbb{I}=\alpha\cdot\alpha^{-1}$ and $\gamma=\beta\cdot\alpha^{-1}\in H(\mathbb{I})$ are $\mathscr{H}$-equivalent distinct elements in $\mathfrak{IC}(I,[0])$. Since $\mathbb{I}\neq\gamma$ we get that there exists $x_\gamma\in I$ such that $(x_\gamma)\gamma\neq x_\gamma$. We suppose $(x_\gamma)\gamma>x_\gamma$. We define a partial map $\delta\colon I\rightharpoonup I$ as follows:

$$\operatorname{dom}\delta=[0,(x_\gamma)\gamma],\qquad \operatorname{ran}\delta=[0,x_\gamma]\qquad\text{and}\qquad (x)\rho=\frac{x_\gamma}{(x_\gamma)\gamma}\cdot x,$$

for all $x\in\operatorname{dom}\delta$. Then we have that $\mathbb{I}\cdot\delta=\delta$ and hence we get that $(\gamma\cdot\delta)\mathfrak{K}\delta$. Since $\operatorname{dom}(\gamma\cdot\delta)=\operatorname{dom}\gamma\neq\operatorname{dom}\delta$, Proposition 1$(vi)$ implies that the elements $\gamma\cdot\delta$ and δ are not $\mathscr{H}$-equivalent. Therefore case (ii) holds, and hence $\mathfrak{K}$ is a group congruence on the semigroup $\mathfrak{IC}(I,[0])$.

In the case $(x_\gamma)\gamma<x_\gamma$ the proof that $\mathfrak{K}$ is a group congruence on the semigroup $\mathfrak{IC}(I,[0])$ is similar. This completes the proof of our theorem. □

Proposition 4. *The semigroups* $\mathfrak{IC}(I,[0])$ *and* $\mathfrak{IO}(I,[0])$ *are isomorphic.*

Proof. We define a map $\mathfrak{i}\colon \mathfrak{IC}(I,[0])\to\mathfrak{IO}(I,[0])$ by the following way: for arbitrary $\alpha\in\mathfrak{IC}(I,[0])$ we put $(\alpha)\mathfrak{i}$ is the restriction of α on the set $[0,a_\alpha]\setminus\{a_\alpha\}$, where $a_\alpha=\max\{\operatorname{dom}\alpha\}$, with $\operatorname{dom}((\alpha)\mathfrak{i})=\operatorname{dom}\alpha\setminus\{a_\alpha\}$ and $\operatorname{ran}((\alpha)\mathfrak{i})=\operatorname{ran}\alpha\setminus\{(a_\alpha)\alpha\}$. Simple verifications show that such defined map $\mathfrak{i}\colon \mathfrak{IC}(I,[0])\to\mathfrak{IO}(I,[0])$ is an isomorphism. □

3. On the semigroup $\mathfrak{II}(I,[0])$

We put $\mathfrak{II}(I,[0])=\mathfrak{IC}(I,[0])\sqcup\mathfrak{IO}(I,[0])$.

Later we shall denote elements of the semigroup $\mathfrak{IC}(I,[0])$ by $\overline{\alpha}$ and put $\mathring{\alpha}=(\overline{\alpha})\mathfrak{i}\in\mathfrak{IO}(I,[0])$, where $\mathfrak{i}\colon\mathfrak{IC}(I,[0])\to\mathfrak{IO}(I,[0])$ is the isomorphism which is defined in the proof of Proposition 4. Since the semigroups $\mathfrak{IC}(I,[0])$ and $\mathfrak{IO}(I,[0])$ are inverse subsemigroups of the symmetric inverse semigroup $\mathscr{I}_I$ over the set I and by Proposition 1 all elements of the semigroups $\mathfrak{IC}(I,[0])$ and $\mathfrak{IO}(I,[0])$ are monotone partial maps, the semigroup operation in $\mathscr{I}_I$ implies that for $\overline{\alpha}\in\mathfrak{IC}(I,[0])$ and $\mathring{\beta}\in\mathfrak{IO}(I,[0])$ we have that

$$\overline{\alpha}\cdot\mathring{\beta}=\begin{cases}\overline{\gamma}, & \text{if } \operatorname{ran}\overline{\alpha}\subset\operatorname{dom}\mathring{\beta};\\ \mathring{\gamma}, & \text{if } \operatorname{dom}\mathring{\beta}\subset\operatorname{ran}\overline{\alpha}\end{cases}\qquad\text{and}\qquad \mathring{\beta}\cdot\overline{\alpha}=\begin{cases}\mathring{\delta}, & \text{if } \operatorname{ran}\mathring{\beta}\subset\operatorname{dom}\overline{\alpha};\\ \overline{\delta}, & \text{if } \operatorname{dom}\overline{\alpha}\subset\operatorname{ran}\mathring{\beta},\end{cases}$$

where $\overline{\gamma}=\overline{\alpha}\cdot\overline{\beta}\in\mathfrak{IC}(I,[0])$ (i.e., $\mathring{\gamma}=\mathring{\alpha}\cdot\mathring{\beta}\in\mathfrak{IO}(I,[0])$) and $\overline{\delta}=\overline{\beta}\cdot\overline{\alpha}\in\mathfrak{IC}(I,[0])$ (i.e., $\mathring{\delta}=\mathring{\beta}\cdot\mathring{\alpha}\in\mathfrak{IO}(I,[0])$). Hence we get the following:

Proposition 5. $\mathfrak{II}(I,[0])$ *is an inverse semigroup.*

Given two partially ordered sets $(A,\leqslant_A)$ and $(B,\leqslant_B)$, the *lexicographical order* $\leqslant_{\mathbf{lex}}$ on the Cartesian product $A\times B$ is defined as follows:

$$(a,b)\leqslant_{\mathbf{lex}}(a',b')\quad\text{if and only if}\quad a<_A a'\quad\text{or}\quad(a=a'\text{ and }b\leqslant_B b').$$

In this case we shall say that the partially ordered set $(A\times B,\leqslant_{\mathbf{lex}})$ is the *lexicographic product* of partially ordered sets $(A,\leqslant_A)$ and $(B,\leqslant_B)$ and it is denoted by $A\times_{\mathbf{lex}}B$. We observe that a lexicographic order of two linearly ordered sets is a linearly ordered set.

Hereafter for every $\overline{\alpha} \in \mathfrak{IC}(I,[0])$ and $\mathring{\beta} \in \mathfrak{IO}(I,[0])$ we denote $d_\alpha = \max\{\operatorname{dom}\overline{\alpha}\}$, $r_\alpha = \max\{\operatorname{ran}\overline{\alpha}\}$, $d_\beta = \sup\{\operatorname{dom}\mathring{\beta}\}$ and $r_\beta = \sup\{\operatorname{ran}\mathring{\beta}\}$. Obviously we have that $d_\alpha = \sup\{\operatorname{dom}\mathring{\alpha}\}$ and $r_\alpha = \sup\{\operatorname{ran}\mathring{\alpha}\}$ for any $\mathring{\alpha} \in \mathfrak{IO}(I,[0])$.

Proposition 6. *The following conditions hold:*

(i) $E(\mathfrak{II}(I,[0])) = E(\mathfrak{IC}(I,[0])) \cup E(\mathfrak{IO}(I,[0]))$.

(ii) *If* $\overline{\alpha}, \mathring{\alpha}, \overline{\beta}, \mathring{\beta} \in E(\mathfrak{II}(I,[0]))$, *then*

- (a) $\mathring{\alpha} \leqslant \overline{\alpha}$;
- (b) $\overline{\alpha} \leqslant \overline{\beta}$ *if and only if* $d_\alpha \leqslant d_\beta$ $(r_\alpha \leqslant r_\beta)$;
- (c) $\mathring{\alpha} \leqslant \mathring{\beta}$ *if and only if* $d_\alpha \leqslant d_\beta$ $(r_\alpha \leqslant r_\beta)$;
- (d) $\overline{\alpha} \leqslant \mathring{\beta}$ *if and only if* $d_\alpha < d_\beta$ $(r_\alpha < r_\beta)$; *and*
- (e) $\mathring{\alpha} \leqslant \overline{\beta}$ *if and only if* $d_\alpha \leqslant d_\beta$ $(r_\alpha \leqslant r_\beta)$.

(iii) *The semilattice* $E(\mathfrak{II}(I,[0]))$ *is isomorphic to the lexicographic product* $(0;1] \times_{\boldsymbol{lex}} \{0;1\}$ *of the semilattices* $((0;1],\min)$ *and* $(\{0;1\},\min)$ *under the mapping* $(\overline{\alpha})\mathfrak{i} = (d_\alpha;1)$ *and* $(\mathring{\alpha})\mathfrak{i} = (d_\alpha;0)$, *and hence* $E(\mathfrak{II}(I,[0]))$ *is a linearly ordered semilattice.*

(iv) *The elements* α *and* β *of the semigroup* $\mathfrak{II}(I,[0])$ *are* $\mathscr{R}$*-equivalent in* $\mathfrak{II}(I,[0])$ *provides either* $\alpha,\beta \in \mathfrak{IC}(I,[0])$ *or* $\alpha,\beta \in \mathfrak{IO}(I,[0])$ *and moreover, we have that*

- (a) $\overline{\alpha}\mathscr{R}\overline{\beta}$ *in* $\mathfrak{II}(I,[0])$ *if and only if* $d_\alpha = d_\beta$; *and*
- (b) $\mathring{\alpha}\mathscr{R}\mathring{\beta}$ *in* $\mathfrak{II}(I,[0])$ *if and only if* $d_\alpha = d_\beta$.

(v) *The elements* α *and* β *of the semigroup* $\mathfrak{II}(I,[0])$ *are* $\mathscr{L}$*-equivalent in* $\mathfrak{II}(I,[0])$ *provides either* $\alpha,\beta \in \mathfrak{IC}(I,[0])$ *or* $\alpha,\beta \in \mathfrak{IO}(I,[0])$ *and moreover, we have that*

- (a) $\overline{\alpha}\mathscr{L}\overline{\beta}$ *in* $\mathfrak{II}(I,[0])$ *if and only if* $r_\alpha = r_\beta$; *and*
- (b) $\mathring{\alpha}\mathscr{L}\mathring{\beta}$ *in* $\mathfrak{II}(I,[0])$ *if and only if* $r_\alpha = r_\beta$.

(vi) *The elements* α *and* β *of the semigroup* $\mathfrak{II}(I,[0])$ *are* $\mathscr{H}$*-equivalent in* $\mathfrak{II}(I,[0])$ *provides either* $\alpha,\beta \in \mathfrak{IC}(I,[0])$ *or* $\alpha,\beta \in \mathfrak{IO}(I,[0])$ *and moreover, we have that*

- (a) $\overline{\alpha}\mathscr{H}\overline{\beta}$ *in* $\mathfrak{II}(I,[0])$ *if and only if* $d_\alpha = d_\beta$ *and* $r_\alpha = r_\beta$; *and*
- (b) $\mathring{\alpha}\mathscr{H}\mathring{\beta}$ *in* $\mathfrak{II}(I,[0])$ *if and only if* $d_\alpha = d_\beta$ *and* $r_\alpha = r_\beta$.

(*vii*) $\mathfrak{I}\mathfrak{I}(I,[0])$ *is a simple semigroup.*

(*viii*) *The semigroup* $\mathfrak{I}\mathfrak{I}(I,[0])$ *has only two distinct* $\mathscr{D}$*-classes: that are inverse subsemigroups* $\mathfrak{I}\mathfrak{C}(I,[0])$ *and* $\mathfrak{I}\mathfrak{O}(I,[0])$.

Proof. Statements (*i*), (*ii*) and (*iii*) follow from the definition of the semigroup $\mathfrak{I}\mathfrak{I}(I,[0])$ and Proposition 5.

Proofs of statements (*iv*), (*v*) and (*vi*) follow from Proposition 5 and Theorem 1.17 [2] and are similar to statements (*iv*), (*v*) and (*vi*) of Proposition 2.

(*vii*) We shall show that $\mathfrak{I}\mathfrak{I}(I,[0])\cdot\alpha\cdot\mathfrak{I}\mathfrak{I}(I,[0])=\mathfrak{I}\mathfrak{I}(I,[0])$ for every $\alpha\in\mathfrak{I}\mathfrak{I}(I,[0])$. We fix arbitrary $\alpha,\beta\in\mathfrak{I}\mathfrak{I}(I,[0])$ and show that there exist $\gamma,\delta\in\mathfrak{I}\mathfrak{I}(I,[0])$ such that $\gamma\cdot\alpha\cdot\delta=\beta$.

We consider the following four cases:

(1) $\alpha=\overline{\alpha}\in\mathfrak{I}\mathfrak{C}(I,[0])$ and $\beta=\overline{\beta}\in\mathfrak{I}\mathfrak{C}(I,[0])$;

(2) $\alpha=\overline{\alpha}\in\mathfrak{I}\mathfrak{C}(I,[0])$ and $\beta=\mathring{\beta}\in\mathfrak{I}\mathfrak{O}(I,[0])$;

(3) $\alpha=\mathring{\alpha}\in\mathfrak{I}\mathfrak{O}(I,[0])$ and $\beta=\overline{\beta}\in\mathfrak{I}\mathfrak{C}(I,[0])$;

(4) $\alpha=\mathring{\alpha}\in\mathfrak{I}\mathfrak{O}(I,[0])$ and $\beta=\mathring{\beta}\in\mathfrak{I}\mathfrak{O}(I,[0])$.

By Λ_a^b we denote a linear partial map from I into I with $\operatorname{dom}\Lambda_a^b=[0;a]$ and $\operatorname{ran}\Lambda_a^b=[0;b]$, and defined by the formula: $(x)\Lambda_a^b=\dfrac{b}{a}\cdot x$, for $x\in\operatorname{dom}\Lambda_a^b$.

We put:

$$\gamma=\Lambda_{d_\beta}^{d_\alpha} \text{ and } \delta=\alpha^{-1}\cdot\Lambda_{d_\alpha}^{d_\beta}\cdot\beta \text{ in case (1);}$$

$$\gamma=\Lambda_{d_\beta}^{d_\alpha} \text{ and } \delta=\alpha^{-1}\cdot\Lambda_{d_\alpha}^{d_\beta}\cdot\beta \text{ in case (2);}$$

$$\gamma=\Lambda_{d_\beta}^{a} \text{ and } \delta=\alpha^{-1}\cdot\Lambda_{a}^{d_\beta}\cdot\beta, \text{ where } 0<a<d_\alpha, \text{ in case (3);}$$

$$\gamma=\Lambda_{d_\beta}^{d_\alpha} \text{ and } \delta=\alpha^{-1}\cdot\Lambda_{d_\alpha}^{d_\beta}\cdot\beta \text{ in case (4).}$$

Elementary verifications show that $\gamma\cdot\alpha\cdot\delta=\beta$, and this completes the proof of assertion (*vii*).

Statement (*viii*) follows from statements (*iv*) and (*v*). □

On the semigroup $\mathfrak{I}\mathfrak{I}(I,[0])$ we determine a relation $\sim_{\mathfrak{i}\mathfrak{d}}$ by the following way. Let $\mathfrak{i}\colon\mathfrak{I}\mathfrak{C}(I,[0])\to\mathfrak{I}\mathfrak{O}(I,[0])$ be a map which is defined in the proof of Proposition 4. We put

$$\alpha\sim_{\mathfrak{i}\mathfrak{d}}\beta \quad \text{if and only if} \quad \alpha=\beta \;\text{ or }\; (\alpha)\mathfrak{i}\mathfrak{d}=\beta \;\text{ or }\; (\beta)\mathfrak{i}\mathfrak{d}=\alpha,$$

for $\alpha, \beta \in \mathfrak{I}\mathfrak{I}(I,[0])$. Simple verifications show that $\sim_{\mathfrak{id}}$ is an equivalence relation on the semigroup $\mathfrak{I}\mathfrak{I}(I,[0])$.

The following proposition immediately follows from Proposition 1(i) and the definition of the relation $\sim_{\mathfrak{id}}$ on the semigroup $\mathfrak{I}\mathfrak{I}(I,[0])$:

Proposition 7. *Let α and β are elements of the semigroup $\mathfrak{I}\mathfrak{I}(I,[0])$. Then $\alpha \sim_{\mathfrak{id}} \beta$ in $\mathfrak{I}\mathfrak{I}(I,[0])$ if and only if the following conditions hold:*

(i) $d_\alpha = d_\beta$;

(ii) $r_\alpha = r_\beta$;

(iii) $(x)\alpha = (x)\beta$ *for every* $x \in [0, d_\alpha)$;

(iv) $(y)\alpha = (y)\beta$ *for every* $y \in [0, d_\beta)$.

Proposition 8. *The relation $\sim_{\mathfrak{id}}$ is a congruence on the semigroup $\mathfrak{I}\mathfrak{I}(I,[0])$. Moreover, the quotient semigroup $\mathfrak{I}\mathfrak{I}(I,[0])/\sim_{\mathfrak{id}}$ is isomorphic to the semigroup $\mathfrak{I}\mathfrak{C}(I,[0])$.*

Proof. We fix arbitrary $\overline{\alpha}, \mathring{\alpha}, \overline{\beta}, \mathring{\gamma} \in \mathfrak{I}\mathfrak{I}(I,[0])$. It is complete to show that the following conditions hold:

(i) $\left(\overline{\alpha}\cdot\overline{\beta}\right) \sim_{\mathfrak{id}} \left(\mathring{\alpha}\cdot\overline{\beta}\right)$;

(ii) $\left(\overline{\beta}\cdot\overline{\alpha}\right) \sim_{\mathfrak{id}} \left(\overline{\beta}\cdot\mathring{\alpha}\right)$;

(iii) $\left(\overline{\alpha}\cdot\mathring{\gamma}\right) \sim_{\mathfrak{id}} \left(\mathring{\alpha}\cdot\mathring{\gamma}\right)$;

(iv) $\left(\mathring{\gamma}\cdot\overline{\alpha}\right) \sim_{\mathfrak{id}} \left(\mathring{\gamma}\cdot\mathring{\alpha}\right)$.

Suppose case (i) holds. If $d_\beta \leqslant r_\alpha$, then Proposition 1$(i)$ implies that $(x)\left(\overline{\alpha}\cdot\overline{\beta}\right) = (x)\left(\mathring{\alpha}\cdot\overline{\beta}\right)$ for all $x \in \left[0, (d_\beta)(\overline{\alpha})^{-1}\right)$, and hence by Proposition 7 we get that $\left(\overline{\alpha}\cdot\overline{\beta}\right) \sim_{\mathfrak{id}} \left(\mathring{\alpha}\cdot\overline{\beta}\right)$. If $d_\beta > r_\alpha$, then Proposition 1(i) implies that $(x)\left(\overline{\alpha}\cdot\overline{\beta}\right) = (x)\left(\mathring{\alpha}\cdot\overline{\beta}\right)$ for all $x \in [0, d_\alpha)$, and hence by Proposition 7 we get that $\left(\overline{\alpha}\cdot\overline{\beta}\right) \sim_{\mathfrak{id}} \left(\mathring{\alpha}\cdot\overline{\beta}\right)$.

In cases (ii), (iii) and (iv) the proofs are similar. Hence $\sim_{\mathfrak{id}}$ is a congruence on the semigroup $\mathfrak{I}\mathfrak{I}(I,[0])$.

Let $\Phi_{\sim_{\mathfrak{id}}} \colon \mathfrak{I}\mathfrak{I}(I,[0]) \to \mathfrak{I}\mathfrak{C}(I,[0])$ a natural homomorphism which is generated by the congruence $\sim_{\mathfrak{id}}$. Since the restriction $\Phi_{\sim_{\mathfrak{id}}}|_{\mathfrak{I}\mathfrak{C}(I,[0])} \colon \mathfrak{I}\mathfrak{C}(I,[0]) \to \mathfrak{I}\mathfrak{C}(I,[0])$ of the natural homomorphism $\Phi_{\sim_{\mathfrak{id}}} \colon \mathfrak{I}\mathfrak{I}(I,[0]) \to \mathfrak{I}\mathfrak{C}(I,[0])$ is an identity map we conclude that the semigroup $(\mathfrak{I}\mathfrak{I}(I,[0]))\Phi_{\sim_{\mathfrak{id}}}$ is isomorphic to the semigroup $\mathfrak{I}\mathfrak{C}(I,[0])$. □

Theorem 2. *Let $\mathfrak{K}$ be a non-trivial congruence on the semigroup $\mathfrak{I}\mathfrak{I}(I,[0])$. Then the quotient semigroup $\mathfrak{I}\mathfrak{I}(I,[0])/\mathfrak{K}$ is either a group or $\mathfrak{I}\mathfrak{I}(I,[0])/\mathfrak{K}$ is isomorphic to the semigroup $\mathfrak{I}\mathfrak{C}(I,[0])$.*

Proof. Since the subsemigroup of idempotents of the semigroup $\mathfrak{IJ}(I,[0])$ is linearly ordered we have that similar arguments as in the proof of Theorem 1 imply that there exist distinct idempotents ε and ι in $\mathfrak{IJ}(I,[0])$ such that $\varepsilon\mathfrak{K}\iota$ and $\varepsilon\leqslant\iota$. If the set $(\varepsilon,\iota)=\{\upsilon\in E(\mathfrak{IJ}(I,[0]))\mid\varepsilon<\upsilon<\iota\}$ is non-empty, then Lemma 1 and Theorem 1 imply that the quotient semigroup $\mathfrak{IJ}(I,[0])/\mathfrak{R}$ is inverse and it contains only one idempotent, and hence by Lemma II.1.10 from [6] we get that $\mathfrak{IJ}(I,[0])/\mathfrak{R}$ is a group. Otherwise Proposition 7(ii) implies that $\varepsilon=\mathring{\alpha}$ and $\iota=\overline{\alpha}$ for some idempotents $\mathring{\alpha}\in\mathfrak{IO}(I,[0])$ and $\overline{\alpha}\in\mathfrak{IC}(I,[0])$.

Since by Proposition 2(ix) the semigroup $\mathfrak{IC}(I,[0])$ is simple we get that for every $\overline{\beta}\in\mathfrak{IC}(I,[0])$ there exist $\overline{\gamma},\overline{\delta}\in\mathfrak{IC}(I,[0])$ such that $\overline{\beta}=\overline{\gamma}\cdot\overline{\alpha}\cdot\overline{\delta}$. Since $\mathfrak{IC}(I,[0])$ is an inverse semigroup and all elements of $\mathfrak{IC}(I,[0])$ are monotone partial maps of the unit interval I we conclude that that

$$\overline{\beta}=\overline{\beta}\cdot\overline{\beta}^{-1}\cdot\overline{\gamma}\cdot\overline{\alpha}\cdot\overline{\alpha}^{-1}\cdot\overline{\alpha}\cdot\overline{\alpha}^{-1}\cdot\overline{\alpha}\cdot\overline{\delta}\cdot\overline{\beta}^{-1}\cdot\overline{\beta},$$

and hence for elements

$$\overline{\gamma}_{\beta}=\overline{\beta}\cdot\overline{\beta}^{-1}\cdot\overline{\gamma}\cdot\overline{\alpha}\cdot\overline{\alpha}^{-1}\qquad\text{and}\qquad\overline{\delta}_{\beta}=\overline{\alpha}^{-1}\cdot\overline{\alpha}\cdot\overline{\delta}\cdot\overline{\beta}^{-1}\cdot\overline{\beta},$$

of the semigroup $\mathfrak{IC}(I,[0])$ the following conditions hold:

$$\overline{\beta}=\overline{\gamma}_{\beta}\cdot\overline{\alpha}\cdot\overline{\delta}_{\beta},\quad\operatorname{dom}\overline{\beta}=\operatorname{dom}\overline{\gamma}_{\beta},\quad\operatorname{ran}\overline{\gamma}_{\beta}=\operatorname{dom}\overline{\alpha},\quad\operatorname{ran}\overline{\alpha}=\operatorname{dom}\overline{\delta}_{\beta}\quad\text{and}\quad\operatorname{ran}\overline{\beta}=\operatorname{ran}\overline{\delta}_{\beta}.$$

Analogously, since all elements of the semigroups $\mathfrak{IC}(I,[0])$ and $\mathfrak{IO}(I,[0])$ are monotone partial maps of I we get that $\mathring{\beta}=\mathring{\gamma}_{\beta}\cdot\mathring{\alpha}\cdot\mathring{\delta}_{\beta}$ and hence $\overline{\beta}\mathfrak{R}\mathring{\beta}$. This implies that the congruence $\mathfrak{R}$ on the semigroup $\mathfrak{IJ}(I,[0])$ coincides with the congruence $\sim_{\mathfrak{io}}$ on $\mathfrak{IJ}(I,[0])$. Then Proposition 8 implies that the quotient semigroup $\mathfrak{IJ}(I,[0])/\mathfrak{K}$ is isomorphic to the semigroup $\mathfrak{IC}(I,[0])$. □

By S_2 we denote the cyclic group of order 2.

Theorem 3. *For arbitrary $a,b\in(0,1)$ the semigroups $\mathfrak{IC}(I,[a])$ and $\mathfrak{IC}(I,[b])$ are isomorphic. Moreover, for every $a\in(0,1)$ the semigroup $\mathfrak{IC}(I,[a])$ is isomorphic to the direct product*

$$S_2\times\mathfrak{IC}(I,[0])\times\mathfrak{IC}(I,[0]).$$

Proof. We fix an arbitrary $a\in(0,1)$. Obviously, the semigroup $\mathfrak{IC}(I,[a])$ is isomorphic to the direct product $S_2\times\mathfrak{IC}^{\nearrow}(I,[a])$, where $\mathfrak{IC}^{\nearrow}(I,[a])$ is a subsemigroup of $\mathfrak{IC}(I,[a])$ which consists of monotone partial maps of the unit interval I.

By $\mathfrak{IC}^{\nearrow}(I \sqcup I, [0])$ we denote the semigroup of all monotone convex closed partial local homeomorphisms α of the interval $[-1, 1]$ such that $(0)\alpha = 0$ and $0 \in \operatorname{Int}_{[-1,1]}(\operatorname{dom} \alpha)$. We define a map $\mathfrak{i}\colon \mathfrak{IC}^{\nearrow}(I, [a]) \to \mathfrak{IC}^{\nearrow}(I \sqcup I, [0])$ by the following way. For an arbitrary $\alpha \in \mathfrak{IC}^{\nearrow}(I, [a])$ we determine a partial map $\beta = (\alpha)\mathfrak{i} \in \mathfrak{IC}^{\nearrow}(I \sqcup I, [0])$ as follows:

(*i*) $\operatorname{dom} \beta = \left[\dfrac{d_m(\alpha) - a}{a}, \dfrac{d_M(\alpha) - a}{1 - a}\right]$, where $d_m(\alpha) = \min\{\operatorname{dom} \alpha\}$ and $d_M(\alpha) = \max\{\operatorname{dom} \alpha\}$;

(*ii*) $\operatorname{ran} \beta = \left[\dfrac{r_m(\alpha) - a}{a}, \dfrac{r_M(\alpha) - a}{1 - a}\right]$, where $r_m(\alpha) = \min\{\operatorname{ran} \alpha\}$ and $r_M(\alpha) = \max\{\operatorname{ran} \alpha\}$; and

(*iii*) $(x)\beta = \begin{cases} (ax + a)\alpha, & \text{if } x \leqslant 0 \\ ((1 - a)x + a)\alpha, & \text{if } x \geqslant 0 \end{cases}$, for all $x \in \operatorname{dom} \beta$.

Simple verifications show that such defined map $\mathfrak{i}\colon \mathfrak{IC}^{\nearrow}(I, [a]) \to \mathfrak{IC}^{\nearrow}(I \sqcup I, [0])$ is an isomorphism. This completes the first part of the proof of the theorem.

Next we define a map $\mathfrak{j}\colon \mathfrak{IC}^{\nearrow}(I \sqcup I, [0]) \to \mathfrak{IC}(I, [0]) \times \mathfrak{IC}(I, [0])$ by the following way. For an arbitrary $\alpha \in \mathfrak{IC}^{\nearrow}(I \sqcup I, [0])$ we determine a pair of partial maps $(\beta, \gamma) = (\alpha)\mathfrak{i} \in \mathfrak{IC}(I, [0]) \times \mathfrak{IC}(I, [0])$ as follows:

(*i*) $\operatorname{dom} \beta = \operatorname{dom} \alpha \cap [0, 1]$ and $\operatorname{ran} \beta = \operatorname{ran} \alpha \cap [0, 1]$;

(*ii*) $\operatorname{dom} \gamma = \{-x \mid x \in \operatorname{dom} \alpha \cap [0, 1]\}$ and $\operatorname{ran} \gamma = \{-x \mid x \in \operatorname{ran} \alpha \cap [0, 1]\}$;

(*iii*) $(x)\beta = (x)\alpha$ for $x \in \operatorname{dom} \beta$; and

(*iv*) $(x)\gamma = -(x)\alpha$ for $x \in \operatorname{dom} \gamma$.

Simple verifications show that such defined map $\mathfrak{j}\colon \mathfrak{IC}^{\nearrow}(I \sqcup I, [0]) \to \mathfrak{IC}(I, [0]) \times \mathfrak{IC}(I, [0])$ is an isomorphism. This completes the proof of the theorem. □

Theorem 3 implies the following:

Corollary 2. *For arbitrary* $a, b \in (0, 1)$ *the semigroups* $\mathfrak{II}(I, [a])$ *and* $\mathfrak{II}(I, [b])$ *are isomorphic. Moreover, for every* $a \in (0, 1)$ *the semigroup* $\mathfrak{II}(I, [a])$ *is isomorphic to the direct product*

$$S_2 \times \mathfrak{II}(I, [0]) \times \mathfrak{II}(I, [0]).$$

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Contact information

Ivan Chuchman — Department of Mechanics and Mathematics, Ivan Franko Lviv National University, Universytetska 1, Lviv, 79000, Ukraine
E-Mail: chuchman_i@mail.ru

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