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RESEARCH ARTICLE

On various parameters of $\mathbb{Z}_q$-simplex codes for an even integer q

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Abstract. In this paper, we defined the $\mathbb{Z}_q$-linear codes and discussed its various parameters. We constructed $\mathbb{Z}_q$-Simplex code and $\mathbb{Z}_q$-MacDonald code and found its parameters. We have given a lower and an upper bounds of its covering radius for q is an even integer.

1. Introduction

A code C is a subset of $\mathbb{Z}_q^n$, where $\mathbb{Z}_q$ is the set of integer modulo q and n is any positive integer. Let $x, y \in \mathbb{Z}_q^n$, then the distance between x and y is the number of coordinates in which they differ. It is denoted by $d(x, y)$. Clearly $d(x, y) = wt(x - y)$, the number of non-zero coordinates in $x - y$. wt(x) is called *weight of x*. The minimum distance d of C is defined by

$$d = \min\{d(x, y) \mid x, y \in C \text{ and } x \neq y\}.$$

The minimum weight of C is $\min\{wt(c) \mid c \in C \text{ and } c \neq 0\}$. A code of length n cardinality M with minimum distance d over $\mathbb{Z}_q$ is called $(n, M, d)q$-ary code. For basic results on coding theory, we refer [16].

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We know that $\mathbb{Z}_q$ is a group under addition modulo q. Then $\mathbb{Z}_q^n$ is a group under coordinatewise addition modulo q. A subset C of $\mathbb{Z}_q^n$ is said to be a *q-ary code.* If C is a subgroup of $\mathbb{Z}_q^n$, then C is called a $\mathbb{Z}_q$*-linear code.* Some authors are called this code as *modular code* because $\mathbb{Z}_q^n$ is a module over the ring $\mathbb{Z}_q$. In fact, it is a free $\mathbb{Z}_q$-module. Since $\mathbb{Z}_q^n$ is a free $\mathbb{Z}_q$-module, it has a basis. Therefore, every $\mathbb{Z}_q$-linear code has a basis. Since $\mathbb{Z}_q$ is finite, it is finite dimension.

Every k dimension $\mathbb{Z}_q$-linear code with length n and minimum distance d is called $[n, k, d]$ $\mathbb{Z}_q$*-linear code.* A matrix whose rows are a basis elements of the $\mathbb{Z}_q$-linear code is called a *generator matrix* of C. There are many researchers doing research on code over finite rings [4, 9–11, 13, 14, 18]. In the last decade, there are many researchers doing research on codes over $\mathbb{Z}_4$ [1–3, 8, 15].

In this correspondence, we concentrate on code over $\mathbb{Z}_q$ where q is even. We constructed some new codes and obtained its various parameters and its covering radius. In particular, we defined $\mathbb{Z}_q$-Simplex code, $\mathbb{Z}_q$-MacDonald code and studied its various parameters. Section 2 contains basic results for the $\mathbb{Z}_q$-linear codes and we constructed some $\mathbb{Z}_q$-linear code and given its parameters. $\mathbb{Z}_q$-Simplex code is given in section 3 and finally, section 4 we determined the covering radius of these codes and $\mathbb{Z}_q$-MacDonald code.

2. $\mathbb{Z}_q$-linear code

Let C be a $\mathbb{Z}_q$-linear code. If $x, y \in C$, then $x - y \in C$. Let us consider the minimum distance of C is $d = \min\{d(x, y) \mid x, y \in C \text{ and } x \neq y\}$. Then

$$d = \min\{wt(x - y) \mid x, y \in C \text{ and } x \neq y\}.$$

Since C is $\mathbb{Z}_q$-linear code and $x, y \in C$, $x - y \in C$. Since $x \neq y$,

$$\min\{wt(x - y) \mid x, y \in C \ \text{ and } x \neq y\} = \min\{wt(c) \mid c \in C \ \text{ and } c \neq 0\}.$$

Thus, we have

Lemma 1. *In a $\mathbb{Z}_q$-linear code, the minimum distance is the same as the minimum weight.*

Let q be an even integer and let $x, y \in \mathbb{Z}_q^n$ such that $x_i, y_i \in \{0, \frac{q}{2}\}$, then $x_i \pm y_i \in \{0, \frac{q}{2}\}$.

Lemma 2. *Let q be an integer even. If $x, y \in \mathbb{Z}_q^n$ such that $x_i, y_i \in \{0, \frac{q}{2}\}$, then the coordinates of $x \pm y$ are either 0 or $\frac{q}{2}$.*

Now, we construct a new code and discuss its parameters. Let C be an [n, k, d] $\mathbb{Z}_q$-linear code. Define

$$D = \{(c0c\cdots c)+\alpha(\mathbf{0112\cdots q-1}) \mid \alpha\in\mathbb{Z}_q, c\in C \text{ and } \mathbf{i}=ii\cdots i\in\mathbb{Z}_q^n\}.$$

Then, $D=\{c0c\cdots c, c0c\cdots c+\mathbf{0112\cdots q-1}, c0c\cdots c+2(\mathbf{0112\cdots q-1}), \cdots, c0c\cdots c+(q-1)(\mathbf{0112\cdots q-1}) \mid c\in C \text{ and } \mathbf{i}\in\mathbb{Z}_{\mathbf{q}}^{\mathbf{n}}\}$. Since any $\mathbb{Z}_q$-linear combination of D is again an element in D, therefore the minimum distance of D is $d(D)=\min\{wt(c0c\cdots c), wt(c0c\cdots c+\mathbf{0112\cdots q-1}), wt(c0c\cdots c+2(\mathbf{0112\cdots q-1})), \cdots, wt(c0c\cdots c+(q-1)(\mathbf{0112\cdots q-1})) \mid c\in C \text{ and } \mathbf{i}\in\mathbb{Z}_{\mathbf{q}}^{\mathbf{n}}\}$.

Clearly $\min\{wt(c0c\cdots c)\mid c\in C \& c\neq 0\}\geqslant qd$.

Let $c\in C$. Let us take c has r_i $i's$ where $i=0,1,2,\cdots,q-1$. Then for $1\leqslant i\leqslant q-1$,

$$wt(c+\mathbf{i})=\sum_{j=0}^{q-1} r_j - r_{q-i}.$$

That is $wt(c+\mathbf{i})=n-r_{q-i}$. Therefore

$$\begin{aligned}
&wt(c0c\cdots c+\mathbf{0112\cdots q-1})\\
&\quad= wt(c+\mathbf{0})+1+wt(c+\mathbf{1})+wt(c+\mathbf{2})+\cdots+wt(c+\mathbf{q-1})\\
&\quad= n-r_0+1+n-r_{q-1}+n-r_{q-2}+\cdots+n-r_1\\
&\quad= (q-1)n+1.
\end{aligned}$$

Similarly, for every integer i which is relatively prime to q

$$wt((c0c\cdots c)+i(\mathbf{0112\cdots q-1})) \quad = \quad (q-1)n+1.$$

For other i's

$$\begin{aligned}
&\min_{i\in\mathbb{Z}_q}\{wt(c0c\cdots c+i(\mathbf{0112\cdots q-1}))\}\\
&\qquad= wt(c+\mathbf{0})+1+wt(c\cdots c+\frac{q}{2}(\mathbf{12\cdots q-1}))\\
&\qquad= wt(c+\mathbf{0})+1+wt(c\cdots c+(\mathbf{\frac{q}{2}0\frac{q}{2}0\cdots\frac{q}{2}0\frac{q}{2}}))\\
&\qquad= \frac{q}{2}wt(c+\mathbf{0})+1+\frac{q}{2}wt(c+\mathbf{\frac{q}{2}})\\
&\qquad= \frac{q}{2}(n-r_0)+1+\frac{q}{2}(n-r_{\frac{q}{2}})\\
&\qquad= \frac{q}{2}n+1+\frac{q}{2}(n-r_0-r_{\frac{q}{2}}).
\end{aligned}$$

Hence, $d(D)=\min\{qd,(q-1)n+1,\frac{q}{2}n+1+\frac{q}{2}(n-r_0-r_{\frac{q}{2}})\}$. Thus, we have

Theorem 1. *Let C be an $[n,k,d]$ $\mathbb{Z}_q$-linear code, then the*

$$D = \{c0c\cdots c + \alpha(\mathbf{0112\cdots q-1}) \mid \alpha \in \mathbb{Z}_q, c \in C \text{ and } \mathbf{i} = ii\cdots i \in \mathbb{Z}_q^n\}$$

is a $[qn+1, k+1, d(D)]$ $\mathbb{Z}_q$-linear code.

If there is a codeword $c \in C$ such that it has only 0 and $\frac{q}{2}$ as coordinates, then

$$\begin{aligned} wt(c0c\cdots c + \mathbf{0\frac{q}{2}\frac{q}{2}0\frac{q}{2}\cdots 0\frac{q}{2}}) \\ &= wt(c+0) + 1 + wt(c+\frac{q}{2}) + wt(c+0) + \cdots + w(c+\frac{q}{2}) \\ &= 1 + r_{\frac{q}{2}} + r_0 + r_{\frac{q}{2}} + \cdots + r_0 \\ &= \frac{q}{2}(r_0 + r_{\frac{q}{2}}) + 1 = \frac{q}{2}n + 1. \end{aligned}$$

Hence, $d(D) = \min\{qd, \frac{q}{2}n+1\}$. Thus, we have

Corollary 1. *If there is a codeword $c \in C$ such that $c_i = 0$ or $\frac{q}{2}$ and if $n \leqslant 2d-1$, then $d(D) = \frac{q}{2}n+1$.*

3. $\mathbb{Z}_q$-simplex codes

Let G be a matrix over $\mathbb{Z}_q$ whose columns are one non-zero element from each 1-dimensional submodule of $\mathbb{Z}_q^2$. Then this matrix is equivalent to

$$G_2 = \left[\begin{array}{c|c|cccc} 0 & 1 & 1 & 2 & \cdots & q-1 \\ 1 & 0 & 1 & 1 & \cdots & 1 \end{array}\right].$$

Clearly G_2 generates $[q+1, 2, \frac{q}{2}+1]$ code. Inductively, we define

$$G_{k+1} = \left[\begin{array}{c|c|c|c|c|c} 00\cdots 0 & 1 & 11\cdots 1 & 22\cdots 2 & \cdots & q-1q-1\cdots q-1 \\ \hline & 0 & & & & \\ G_k & \vdots & G_k & G_k & \cdots & G_k \\ & 0 & & & & \end{array}\right]$$

for $k \geqslant 2$. Clearly this G_{k+1} matrix generates $[n_{k+1} = \frac{q^{k+1}-1}{q-1},\ k+1,\ d]$ code. We call this code as $\mathbb{Z}_q$-*Simplex code.* This type of k-dimensional code is denoted by $S_k(q)$. For simplicity, we denote it by S_k.

Theorem 2. *$S_k(q)$ is an $[n_k = \frac{q^k-1}{q-1},\ k,\ \frac{q}{2}n_{k-1}+1]$ $\mathbb{Z}_q$-linear code.*

Proof. We prove this theorem by induction on k. For $k = 2$, from the generator matrix G_2, it is clear that $d = \frac{q}{2}+1$ and the theorem is true. Since there is a codeword $c = 0\frac{q}{2}\frac{q}{2}0\frac{q}{2}\cdots 0\frac{q}{2}0\frac{q}{2} \in S_2$ and $n = q+1 \leqslant 2(\frac{q}{2}+1)-1 = 2d-1$, by Corollary 1 implies $d(S_3) = \frac{q}{2}n_2 + 1$ and hence the S_3 is $[n_3 = \frac{q^3-1}{q-1}, 3, \frac{q}{2}n_2+1]$ code. Since $c0c\cdots c+\frac{q}{2}(\mathbf{0112}\cdots\mathbf{q-1}) \in S_3$ whose coordinates are either 0 or $\frac{q}{2}$ and satisfies the conditions of the Corollary 1, therefore $d(S_4) = \frac{q}{2}n_3+1$ and hence the S_4 is $[n_4 = \frac{q^4-1}{q-1}, 4, \frac{q}{2}n_3+1]$ code. By induction we can assume that this theorem is true for all less than k. That is, there is a code $c \in S_{k-1}$ whose coordinates are either 0 or $\frac{q}{2}$ and $n_{k-1} \leqslant 2d_{k-1} - 1$. By Corollary 1, $d_k = \frac{q}{2}n_{k-1} + 1$. Therefore $S_k(q)$ is an $[\frac{q^k-1}{q-1},\ k,\ \frac{q}{2}n_{k-1} + 1]$ $\mathbb{Z}_q$-linear code. Thus we proved. □

Now, we are going to see the minimum distance of the dual code of this $\mathbb{Z}_q$-Simplex code. Since the matrix $G_k(q)$ has no zero columns, therefore, the minimum distance of its dual is greater than or equal to 2. Since in the first block of the matrix G_k, there are two columns whose transpose matrices are $(0,0,\cdots,0,1,1)$ and $(0,0,\cdots,0,a,1)$. Since addition and multiplications are modulo q and q is even, $\frac{q}{2}(0,0,\cdots,0,1,1) + \frac{q}{2}(0,0,\cdots,0,q-1,1) = 0$. That is, there are two linearly dependent columns. Therefore, the minimum distance of the dual code is less than or equal to 2. Hence the dual of S_k is $[n_k = \frac{q^k-1}{q-1}, n_k - k, 2]$ $\mathbb{Z}_q$-linear code.

4. Covering radius

The *covering radius* of a code C over $\mathbb{Z}_q$ with respect to the Hamming distance d is given by

$$R(C) = \max_{u\in\mathbb{Z}_q^n}\left\{\min_{c\in C}\{d(u,c)\}\right\}.$$

It is easy to see that $R(C)$ is the least positive integer r such that

$$\mathbb{Z}_q^n = \cup_{c\in C} S_r(c)$$

where

$$S_r(u) = \left\{v \in \mathbb{Z}_q^n\} \mid d(u,v) \leqslant r\right\}$$

for any $u \in \mathbb{Z}_q^n$.

Proposition 1 ([5]). *If appending(puncturing) r number of columns in a code C, then the covering radius of C is increased(decreased) by r.*

Proposition 2 ([17]). *If C_0 and C_1 are codes over $\mathbb{Z}_q^n$ generated by matrices G_0 and G_1 respectively and if C is the code generated by*

$$G = \left(\begin{array}{c|c} 0 & G_1 \\ \hline G_0 & A \end{array}\right),$$

then $r(C) \leqslant r(C_0) + r(C_1)$ and the covering radius of C satisfy the following

$$r(C) \geqslant r(C_0) + r(C_1).$$

Since the covering radius of C generated by

$$G = \left(\begin{array}{c|c} 0 & G_1 \\ \hline G_0 & A \end{array}\right),$$

is greater than or equal to $r(C_0) + r(C')$ where C_0 and C' are codes generated by $\left[\begin{array}{c} 0 \\ \hline G_0 \end{array}\right] = \left[\begin{array}{c} G_0 \end{array}\right]$ and $\left[\begin{array}{c} G_1 \\ \hline A \end{array}\right]$, respectively, this implies $r(C) \geqslant r(C_0) + r(C_1)$ because C_1 is a subcode of the code C'.

A q-ary repetition code C over a finite field $\mathbb{F}_q$ with q elements is an $[n, 1, n]$ linear code. The covering radius of C is $\left\lfloor \frac{n(q-1)}{q} \right\rfloor$ [12]. For basic results on covering radius, we refer to [5], [6]. Now, we consider the repetition code over $\mathbb{Z}_q$. There are two types of repetition codes.

Type I. Unit repetition code generated by $G_u = [\overbrace{uu\ldots u}^{n}]$ where u is an unit element of $\mathbb{Z}_q$. This matrix generates C_u is $[n, 1, n]$ $\mathbb{Z}_q$-linear code. That is, C_u is (n, q, n) q-ary repetition code. We call this as *unit repetition code.*

Type II. Zero divisor repetition code is generated by the matrix $G_v = [\overbrace{vv\ldots v}^{n}]$ where v is a zero divisor in $\mathbb{Z}_q$. That is, v is not a relatively prime to q. This is an $(n, \frac{q}{v}, n)$ code over $\mathbb{Z}_q$. This code is denoted by C_v. This code is called *zero divisor repetition code.*

With respect to the Hamming distance the covering radius of C_u is $\left\lfloor \frac{n(q-1)}{q} \right\rfloor$ [12] but clearly the covering radius of C_v is n because code symbols appear in this code are zero divisors only. Thus, we have

Theorem 3. *$R(C_v) = n$ and $R(C_u) = \left\lfloor \frac{(q-1)n}{q} \right\rfloor$.*

Let $\phi(q) = \#\{i \mid 1 \leqslant i < q \ \&\ (i, q) = 1\}$ be the Euler ϕ-function. Let $U = \{i \in \mathbb{Z} \mid 1 \leqslant i < q \ \&\ (i, q) = 1\}$ be the set of all units in $\mathbb{Z}_q$ and let

$O = \mathbb{Z}_q \setminus U$ be the set which contains all zero divisors and 0. Let C be a $\mathbb{Z}_q$-linear code generated by the matrix

$$[\overbrace{11\ldots1}^{n}\overbrace{22\ldots2}^{n}\cdots\overbrace{q-1q-1\ldots q-1}^{n}],$$

then this code is equivalent to a code whose generator matrix is

$$[u_1u_1\cdots u_1u_2u_2\cdots u_2\cdots u_{\phi(q)}u_{\phi(q)}\cdots u_{\phi(q)}o_1o_1\cdots o_1o_2o_2\cdots o_2\cdots o_ro_r\cdots o_r]$$

where $r = q - 1 - \phi(q)$. Let A be a code equivalent to the unit repetition code of length $\phi(q)n$ generated by $[u_1u_1\cdots u_1u_2u_2\cdots u_2\cdots u_{\phi(q)}u_{\phi(q)}\cdots u_{\phi(q)}]$, then by the above theorem, $R(A) = \left\lfloor \frac{(q-1)\phi(q)n}{q} \right\rfloor$. Let B be a code equivalent to the zero divisor repetition code of length $(q-1-\phi(q))n$ generated by $[o_1o_1\cdots o_1o_2o_2\cdots o_2\cdots o_ro_r\cdots o_r]$, then by the above theorem, $R(B) = (q-1-\phi(q))n$. By Proposition 2, $R(C) \geqslant \left\lfloor \frac{(q-1)\phi(q)n}{q} \right\rfloor + (q-1-\phi(q))n$.

Without loss of generality we can assume that the generator matrix of A as $[111\cdots1]$. Since $R(A) = \left\lfloor \frac{(q-1)\phi(q)n}{q} \right\rfloor$ and C is obtained by appending some $(q-1-\phi(q))n$ columns to A, by Proposition 1 the covering radius of C is increased by at most $(q-1-\phi(q))n$. Therefore, $R(C) \leqslant \left\lfloor \frac{(q-1)\phi(q)n}{q} \right\rfloor + (q-1-\phi(q))n$. Thus, we have

Theorem 4. *Let C be a $\mathbb{Z}_q$-linear code generated by the matrix*

$$[\overbrace{11\ldots1}^{n}\ \overbrace{22\ldots2}^{n}\cdots\overbrace{q-1q-1\ldots q-1}^{n}].$$

Then C is a $[(q-1)n, 1, \frac{q}{2}n]$ $\mathbb{Z}_q$-linear code with $R(C) = \left\lfloor \frac{(q-1)\phi(q)n}{q} \right\rfloor + (q-1-\phi(q))n$.

Now, we see the covering radius of $\mathbb{Z}_q$-Simplex code. The covering radius of Simplex codes and MacDonald codes over finite field and finite rings were discussed in [12], [14].

Theorem 5. *For $k \geqslant 2$,*

$$R(S_{k+1}) \leqslant \frac{(k-1)(q-1)\phi(q) + (q^2-q-\phi(q))(q^{k+1}-q^2)}{q(q-1)^2} + R(S_2).$$

Proof. For $k \geqslant 2$, S_{k+1} is $[n_{k+1} = \frac{q^{k+1}-1}{q-1}, k+1, \frac{q}{2}n_k + 1]$ $\mathbb{Z}_q$-linear code. By Proposition 2 and Theorem 4 give

$$R(S_{k+1}) \leqslant (1 + \left\lfloor \frac{(q-1)\phi(q)n_k}{q} \right\rfloor + (q-1-\phi(q))n_k) + R(S_k)$$

$$\begin{aligned}&\leqslant (1+\frac{(q-1)\phi(q)n_k}{q}+(q-1-\phi(q))n_k)+R(S_k)\\&\leqslant \left(1+\frac{q^2-q-\phi(q)}{q}n_k\right)+R(S_k).\end{aligned}$$

This implies

$$R(S_k)\leqslant (1+\frac{q^2-q-\phi(q)}{q}n_{k-1})+R(S_{k-1}).$$

Combining these two, we get

$$R(S_{k+1})\leqslant (1+\frac{q^2-q-\phi(q)}{q}n_k)+(1+\frac{q^2-q-\phi(q)}{q}n_{k-1})+R(S_{k-1})$$

Similarly, if we continue, we get

$$\begin{aligned}R(S_{k+1})\leqslant{}& (1+\frac{q^2-q-\phi(q)}{q}n_k)+(1+\frac{q^2-q-\phi(q)}{q}n_{k-1})+\cdots\\&+(1+\frac{q^2-q-\phi(q)}{q}n_2)+R(S_2).\end{aligned}$$

Since $n_k=\frac{q^k-1}{q-1}$, for $k\geqslant 2$, therefore

$$\begin{aligned}R(S_{k+1})&\leqslant (k-1)+\frac{q^2-q-\phi(q)}{q}\left(\frac{q^k-1}{q-1}+\frac{q^{k-1}-1}{q-1}+\cdots+\frac{q^2-1}{q-1}\right)+R(S_2)\\&\leqslant (k-1)+\frac{q^2-q-\phi(q)}{q}\left(\frac{q^k+q^{k-1}+\cdots+q^2-(k-1)}{q-1}\right)+R(S_2)\\&\leqslant \frac{(k-1)\phi(q)+(q^2-q-\phi(q))((q^{k+1}-1)/(q-1)-(q+1))}{q(q-1)}+R(S_2)\\&\leqslant \frac{(k-1)(q-1)\phi(q)+(q^2-q-\phi(q))(q^{k+1}-q^2)}{q(q-1)^2}+R(S_2).\end{aligned}$$

Hence the proof is complete. □

In particular, for $q=4$, $R(S_{k+1})\leqslant\frac{5.4^{k+1}+3k-29}{18}$ for $k\geqslant 2$ because of simple calculation $R(S_2)=3$.

Now, we can define a new code which is similar to the $\mathbb{Z}_q$-MacDonald code. Let

$$G_{k,u}=\left(G_k\setminus\begin{pmatrix}0\\G_u\end{pmatrix}\right)$$

for $2 \leqslant u \leqslant k-1$ where 0 is a $(k-u) \times \frac{q^u-1}{q-1}$ zero matrix and $\left(A \setminus B\right)$ is a matrix obtained from the matrix A by removing the matrix B. The code generated by $G_{k,u}$ is called $\mathbb{Z}_q$-*MacDonald* code. It is denoted by $M_{k,u}$. The Quaternary MacDonald codes were discussed in [7].

Theorem 6. *For* $2 \leqslant u \leqslant r \leqslant k$,

$$R(M_{k+1,u}) \leqslant \frac{(k-r+1)(q-1)\phi(q)+(q^2-q-\phi(q))q^r(q^{k-r+1}-1)}{q(q-1)^2}+R(M_{r,u}).$$

Proof. By using, Proposition 2, we get

$$\begin{aligned} R(M_{k+1,u}) &\leqslant (1+\left\lfloor\frac{(q-1)\phi(q)n_k}{q}\right\rfloor + (q-1-\phi(q))n_k) + R(M_{k,u}) \\ &\leqslant (1+\frac{(q-1)\phi(q)n_k}{q} + (q-1-\phi(q))n_k) + R(M_{k,u}) \\ &\leqslant \left(1+\frac{q^2-q-\phi(q)}{q}n_k\right) + R(M_{k,u}). \end{aligned}$$

This implies $R(M_{k,u}) \leqslant (1+\frac{q^2-q-\phi(q)}{q}n_{k-1}) + R(M_{k-1,u})$. Combining these two, we get

$$R(M_{k+1,u}) \leqslant (1+\frac{q^2-q-\phi(q)}{q}n_k)+(1+\frac{q^2-q-\phi(q)}{q}n_{k-1})+R(M_{k-1,u}).$$

Similarly, if we continue, we get

$$\begin{aligned} R(M_{k+1,u}) \leqslant (1+\frac{q^2-q-\phi(q)}{q}n_k) + (1+\frac{q^2-q-\phi(q)}{q}n_{k-1}) \\ + \cdots + (1+\frac{q^2-q-\phi(q)}{q}n_r) + R(M_{r,u}). \end{aligned}$$

Since $n_k = \frac{q^k-1}{q-1}$, for $k \geqslant 2$, therefore

$$\begin{aligned} &R(M_{k+1,u}) \\ &\leqslant (k-r+1)+\frac{q^2-q-\phi(q)}{q}\left(\frac{q^k-1}{q-1}+\frac{q^{k-1}-1}{q-1}+\cdots+\frac{q^r-1}{q-1}\right)+R(M_{r,u}) \\ &\leqslant (k-r+1)+\frac{q^2-q-\phi(q)}{q}\left(\frac{q^k+q^{k-1}+\cdots+q^r-(k-r+1)}{q-1}\right)+R(M_{r,u}) \\ &\leqslant \frac{(k-r+1)\phi(q)+(q^2-q-\phi(q))q^r(q^{k-r}+q^{k-r-1}+\cdots+1)}{q(q-1)}+R(M_{r,u}) \end{aligned}$$

$$\leqslant \frac{(k-r+1)(q-1)\phi(q)+(q^2-q-\phi(q))q^r(q^{k-r+1}-1)}{q(q-1)^2}+R(M_{r,u}). \qquad \square$$

If $u = k$, then

$$R(M_{k+1,k}) \leqslant \left\lfloor \frac{(q-1)\phi(q)n_k}{q} \right\rfloor + (q-1-\phi(q))n_k + 1 \text{ for } k \geqslant 2.$$

In the above theorem, if we replace r by $u+1$, we get

$$R(M_{k+1,u}) \leqslant \frac{(k-u)(q-1)\phi(q) + (q^2-q-\phi(q))q^{u+1}(q^{k-u}-1)}{q(q-1)^2} + \frac{(q-1)\phi(q)n_u}{q} + (q-1-\phi(q))n_u + 1 \text{ for } u \geqslant 2.$$

Thus, we have

Corollary 2. *For $k \geqslant u \geqslant 2$,*

$$R(M_{k+1,u}) \leqslant \frac{(k-u)(q-1)\phi(q) + (q^2-q-\phi(q))q^{u+1}(q^{k-u}-1)}{q(q-1)^2} + \frac{(q-1)\phi(q)n_u}{q} + (q-1-\phi(q))n_u + 1.$$

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