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On Pseudo-valuation rings and their extensions

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Abstract. Let R be a commutative Noetherian $\mathbb{Q}$-algebra ($\mathbb{Q}$ is the field of rational numbers). Let σ be an automorphism of R and δ a σ-derivation of R. We define a δ-divided ring and prove the following:

(1) If R is a pseudo-valuation ring such that $x \notin P$ for any prime ideal P of $R[x;\sigma,\delta]$, and $P \cap R$ is a prime ideal of R with $\sigma(P \cap R) = P \cap R$ and $\delta(P \cap R) \subseteq P \cap R$, then $R[x;\sigma,\delta]$ is also a pseudo-valuation ring.

(2) If R is a δ-divided ring such that $x \notin P$ for any prime ideal P of $R[x;\sigma,\delta]$, and $P \cap R$ is a prime ideal of R with $\sigma(P \cap R) = P \cap R$ and $\delta(P \cap R) \subseteq P \cap R$, then $R[x;\sigma,\delta]$ is also a δ-divided ring.

1. Introduction

Throughout this paper, all rings are associative with identity $1 \neq 0$. Let now R be a ring. $N(R)$ denotes the set of all nilpotent elements of R. $Z(R)$ denotes the center of R. $\mathbb{Q}$ denotes the field of rational numbers and $\mathbb{Z}$ denotes the ring of integers unless otherwise stated. We recall that as in Hedstrom [12], an integral domain R with quotient field F, is called a pseudo-valuation domain (PVD) if each prime ideal P of R is strongly prime ($ab \in P$, $a \in F$, $b \in F$ implies that either $a \in P$ or $b \in P$). For example let F $= \mathbb{Q}(\sqrt{2})$. Set $V = F + xF[[x]] = F[[x]]$. Then V is a pseudo-valuation domain. In Badawi, Anderson and Dobbs [4], the study

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of pseudo-valuation domains was generalized to arbitrary rings in the following way:

A prime ideal P of R is said to be strongly prime if aP and bR are comparable (under inclusion; i.e. $aP \subseteq bR$ or $bR \subseteq aP$) for all $a, b \in R$. The sets of prime ideals of R and strongly prime ideals of R are denoted by $Spec(R)$ and $S.Spec(R)$ respectively.

A ring R is said to be a pseudo-valuation ring (PVR) if each prime ideal P of R is strongly prime. We note that a PVR is quasilocal by Lemma 1(b) of Badawi, Anderson and Dobbs [4].

An integral domain is a PVR if and only if it is a PVD by Proposition (3.1) of Anderson [1], Proposition (4.2) of Anderson [2] and Proposition (3) of Badawi [5]. We recall that a prime ideal P of R is said to be divided if it is comparable (under inclusion) to every ideal of R. A ring R is called a divided ring if every prime ideal of R is divided.

In Badawi [6], another generalization of PVDs is given in the following way:

For a ring R with total quotient ring Q such that $N(R)$ is a divided prime ideal of R, let $\phi : Q \to R_{N(R)}$ such that $\phi(a/b) = a/b$ for every $a \in R$ and every $b \in R \backslash Z(R)$. Then ϕ is a ring homomorphism from Q into $R_{N(R)}$, and ϕ restricted to R is also a ring homomorphism from R into $R_{N(R)}$ given by $\phi(r) = r/1$ for every $r \in R$. Denote $R_{N(R)}$ by T. A prime ideal P of $\phi(R)$ is called a T-strongly prime ideal if $xy \in P$, $x \in T$, $y \in T$ implies that either $x \in P$ or $y \in P$. $\phi(R)$ is said to be a T-pseudo-valuation ring (T-PVR) if each prime ideal of $\phi(R)$ is T-strongly prime. A prime ideal S of R is called ϕ-strongly prime ideal if $\phi(S)$ is a T-strongly prime ideal of $\phi(R)$. If each prime ideal of R is ϕ-strongly prime, then R is called a ϕ-pseudo-valuation ring ($\phi - PVR$).

This article is concerned with the study of skew polynomial rings over PVDs. Let R be a ring and σ be an automorphism of R and δ be a σ-derivation of R ($\delta : R \to R$ is an additive map with $\delta(ab) = \delta(a)\sigma(b) + a\delta(b)$, for all $a, b \in R$).

Example 1.1. Let σ be an automorphism of a ring R and $\delta : R \to R$ any map. Let $\phi : R \to M_2(R)$ defined by
$\phi(r) = \begin{pmatrix} \sigma(r) & 0 \\ \delta(r) & r \end{pmatrix}$, for all $r \in R$. Then δ is a σ-derivation of R if and only if ϕ is a homomorphism.

We denote the Ore extension $R[x; \sigma, \delta]$ by $O(R)$. If I is an ideal of R such that I is σ-stable; i.e. $\sigma(I) = I$ and I is δ-invariant; i.e. $\delta(I) \subseteq I$, then we denote $I[x; \sigma, \delta]$ by $O(I)$. We would like to mention that $R[x; \sigma, \delta]$ is the usual set of polynomials with coefficients in R, i.e. $\{\sum_{i=0}^{n} x^i a_i, a_i \in R\}$

in which multiplication is subject to the relation $ax = x\sigma(a) + \delta(a)$ for all $a \in R$.

In case δ is the zero map, we denote the skew polynomial ring $R[x;\sigma]$ by $S(R)$ and for any ideal I of R with $\sigma(I) = I$, we denote $I[x;\sigma]$ by $S(I)$.
In case σ is the identity map, we denote the differential operator ring $R[x;\delta]$ by $D(R)$ and for any ideal J of R with $\delta(J) \subseteq J$, we denote $J[x;\delta]$ by $D(J)$.

Ore-extensions including skew-polynomial rings and differential operator rings have been of interest to many authors. See [3, 7, 8, 9, 10, 13].

Polynomial rings over Pseudo-valuation rings

Recall that in Bhat [8], a prime ideal P of a ring R is σ-divided (σ is an automorphism of R) if it is comparable (under inclusion) to every σ-stable I of R. A ring R is called a σ-divided ring if every prime ideal of R is σ-divided. In this direction in Theorem (2.8) of Bhat [8] it has been proved that if R is a σ-divided Noetherian ring such that $x \notin P$ for any $P \in Spec(S(R))$, then $S(R)$ is also σ-divided Noetherian. Also in Theorem (2.6) of Bhat [8] it has been proved that if R is a commutative PVR such that $x \notin P$ for any $P \in Spec(S(R))$, then $S(R)$ is also a PVR.

In this paper, we generalize these results for $O(R)$ and answer Question (1) of [8], but before that we have the following:

Let R be a ring. Let σ be an automorphism of R and δ a σ-derivation of R. We say that a prime ideal P of R is δ-divided if it is comparable (under inclusion) to every σ-stable and δ-invariant ideal I of R. A ring R is called a δ-divided ring if every prime ideal of R is δ-divided.

Let now R be a commutative Noetherian $\mathbb{Q}$-algebra. Let σ be an automorphism of R and δ a σ-derivation of R. Then we prove the following:

(1) Let R be a pseudo-valuation ring such that $x \notin P$ for any $P \in Spec(O(R))$ and $P \cap R$ is a σ-stable and δ-invariant prime ideal of R. Further assume that for any $U \in S.Spec(R)$ with $\sigma(U) = U$ and $\delta(U) \subseteq U$ we have $O(U) \in S.Spec(O(R))$. Then $R[x;\sigma,\delta]$ is also a pseudo-valuation ring.

(2) Let R be a δ-divided ring such that $x \notin P$ for any $P \in Spec(O(R))$ and $P \cap R$ be a σ-stable and δ-invariant prime ideal of R. Then $R[x;\sigma,\delta]$ is also a δ-divided ring.

These results are proved in Theorems (2.3) and (2.8) respectively.

2. Polynomial rings

We begin with the following known results:

Lemma 2.1. Let R be a commutative Noetherian $\mathbb{Q}$-algebra. Let σ be an automorphism of R and δ be a σ-derivation of R. Then U is a prime ideal of R such that $\sigma(U) = U$ and $\delta(U) \subseteq U$ implies that $O(U)$ is a prime ideal of $O(R)$ and $O(U) \cap R = U$.

Proof. The proof is on the same lines as in Theorem (2.22) of Goodearl and Warfield [11] and Lemma (10.6.4) of McConnell and Robson [14]. □

Theorem 2.2. (Hilbert Basis Theorem): Let R be a right/left Noetherian ring. Let σ be an automorphism of R and δ a σ-derivation of R. Then the Ore extension $O(R) = R[x; \sigma, \delta]$ is right/left Noetherian.

Proof. See Theorem (1.12) of Goodearl and Warfield [11]. □

Theorem 2.3. Let R be a commutative Noetherian pseudo-valuation $\mathbb{Q}$-algebra such that $x \notin P$ for any $P \in Spec(O(R))$ and $P \cap R$ be a σ-stable and δ-invariant prime ideal of R. Further let any $U \in S.Spec(R)$ with $\sigma(U) = U$ and $\delta(U) \subseteq U$ implies that $O(U) \in S.Spec(O(R))$. Then $O(R)$ is a Noetherian pseudo-valuation $\mathbb{Q}$-algebra.

Proof. $O(R)$ is Noetherian by Theorem (2.2). Let $J \in Spec(O(R))$. Then $J \cap R \in Spec(R)$ with $\sigma(J \cap R) = J \cap R$ and $\delta(J \cap R) \subseteq J \cap R$. Now R is a pseudo-valuation $\mathbb{Q}$-algebra, therefore $J \cap R \in S.Spec(R)$. Now by hypothesis $O(J \cap R) \in S.Spec(O(R))$. Now it can be seen that $O(J \cap R) = J$. Therefore $J \in S.Spec(O(R))$. Hence $O(R)$ is a pseudo-valuation $\mathbb{Q}$-algebra. □

Corollary 2.4. Let R be a PVR such that $x \notin P$ for any $P \in Spec(S(R))$. Then $S(R)$ is also a PVR.

We note that Theorem (2.3) does not hold without the condition that $P \cap R$ is a σ-stable and δ-invariant prime ideal of R.

Example 2.5. Let $R = \mathbb{Q} \times \mathbb{Q}$. Let $\sigma : R \mapsto R$ be defined by $\sigma\big((a, b)\big) = (b, a)$, and $\delta = 0$. Then $P = 0$ is a prime ideal of $O(R)$ such that $x \notin P$, but $P \cap R$ is not a prime ideal of R.

Now let R and σ be as above, and $\delta = Id$ - σ. Then δ is a σ-derivation of R. Now it can be seen that $O(R)$ has the form $R[x - 1; \sigma]$. Now $P = (1, 0)R + (x - 1)O(R)$ is a prime ideal of $O(R)$ such that $x \notin P$, but $P \cap R = (1, 0)R$ is not σ-stable or δ-invariant.

We also note that in Theorem (2.3) the hypothesis that any $U \in S.Spec(R)$ with $\sigma(U) = U$ and $\delta(U) \subseteq U$ implies that $O(U) \in S.Spec(O(R))$ can not be deleted as an extension of a strongly prime ideal of R need not be a strongly prime ideal of $O(R)$.

Example 2.6. $R = \mathbb{Z}_{(p)}$. This is in fact a discrete valuation domain, and therefore, its maximal ideal $P = pR$ is strongly prime. But $pR[x]$ is not strongly prime in $R[x]$ because it is not comparable with $xR[x]$ (so the condition of being strongly prime in $R[x]$ fails for $a = 1$ and $b = x$).

Corollary 2.7. Let R be a commutative Noetherian pseudo-valuation $\mathbb{Q}$-algebra such that $x \notin P$ for any $P \in Spec(D(R))$. Then $D(R)$ is a Noetherian pseudo-valuation $\mathbb{Q}$-algebra.

We note that Corollary (2.7) does not hold without the condition that $x \notin P$ for any $P \in Spec(D(R))$. For example let $R = \mathbb{Q}[y]_{(y)}$ (the localization of the polynomial ring $\mathbb{Q}[y]$ at the maximal ideal (y)) and $\delta = y\frac{d}{dy}$. Then R is a commutative PVR. Now $P = yD(R) + xD(R)$ is a prime (maximal) ideal of $D(R)$, but xP is not comparable to $yD(R)$, therefore $D(R)$ is not a PVR.

Theorem 2.8. If R is a δ-divided commutative Noetherian $\mathbb{Q}$-algebra such that $x \notin P$ for any $P \in Spec(O(R))$ and $P \cap R$ is a σ-stable and δ-invariant prime ideal of R, then $O(R)$ is δ-divided Noetherian $\mathbb{Q}$-algebra.

Proof. $O(R)$ is Noetherian by Theorem (2.2). Let $J \in Spec(O(R))$ and $0 \neq K$ be a proper ideal of $O(R)$ such that $\sigma(K) = K$ and $\delta(K) \subseteq K$. Now we note that σ can be extended to an automorphism of $O(R)$ such that $\sigma(x) = x$ and δ can be extended to a σ-derivation of $O(R)$ such that $\delta(x) = 0$. Now $J \cap R \in Spec(R)$ with $\sigma(J \cap R) = J \cap R$ and $\delta(J \cap R) \subseteq J \cap R$. Also $K \cap R$ is an ideal of R with $\sigma(K \cap R) = K \cap R$ and $\delta(K \cap R) \subseteq K \cap R$. Now R is divided, therefore $J \cap R$ and $K \cap R$ are comparable under inclusion. Say $J \cap R \subseteq K \cap R$. Therefore $O(J \cap R) \subseteq O(K \cap R)$. Thus $J \subseteq K$. Hence $O(R)$ is δ-divided Noetherian. □

Corollary 2.9. Let R be a σ-divided Noetherian ring such that $x \notin P$ for any $P \in Spec(S(R))$. Then $S(R)$ is also σ-divided Noetherian.

Corollary 2.10. Let R be a divided commutative Noetherian $\mathbb{Q}$-algebra such that $x \notin P$ for any $P \in Spec(D(R))$. Then $D(R)$ is also divided Noetherian.

We note that Corollary (2.10) does not hold without the condition that $x \notin P$ for any $P \in Spec(D(R))$. For example let $R = \mathbb{Q}[y]_{(y)}$ (the localization of the polynomial ring $\mathbb{Q}[y]$ at the maximal ideal (y)) and

$\delta = y\frac{d}{dy}$. Then R is a commutative PVR, and so it is a divided ring. Now $P = yD(R)$ is a prime ideal of $D(R)$, but it is not comparable to the ideal $y^2D(R) + xD(R)$, and therefore $D(R)$ is not divided.

Question 2.11. (Question 1 of [8]): Let R be a PVR. Let σ be an automorphism of R and δ a σ-derivation of R. Is $O(R) = R[x; \sigma, \delta]$ a PVR (even if R is commutative Noetherian)?

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