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Diagonalizability theorem for matrices over certain domains

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Abstract. It is proved that R is a commutative adequate domain, then R is the domain of stable range 1 in localization in multiplicative closed set which corresponds s-torsion in the sense of Komarnitskii.

Introduction

A question of quasi-reduction of matrices over a commutative domain with so-called L_φ condition is considered by J. Szucs [1]. B. Zabavsky [2] proved that the L_φ condition for commutative Bezout domain is nothing else than of stable range condition. More over increase in the reduction in localization of given ring to reduction in basic one is shown.

In this work we continued this research and more precisely proved that an adequate domain is the domain of stable range 1 in localization in multiplicative closed set which corresponds s-torsion in the sense of Komarnitskii [3].

Let R be a commutative Bezout domain. An element $a \in R$ is called an adequate element if for any b from R the element a can be represented as a product $a = rs$, where $rR + bR = R$ and for any non invertible divisor s' of s we have obtain $s'R + bR \neq R$.

A ring R is called a Bezout ring if every finitely generated ideal is principal. A commutative Bezout ring in which any nonzero element is

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adequate is called an adequate ring [4]. An element $a \in R$ is called co-adequate if every non zero element $b \in R$ can be represented as a product $b = rs$ where $rR + aR = R$ and for any non invertible divisor s' of s we have obtain $s'R + bR \neq R$ [5].

A ring R is the ring of stable range 1 (in denotation $st.r.(R) = 1$), if the condition $aR + bR = R$ for every elements $a, b \in R$ implies that there exist element $t \in R$ such that $a + bt$ is an invertible element of the ring R [6].

An element $a \in R$ is called the element of almost stable range 1 if $st.r.(R/aR) = 1$ [7]. A ring where every non zero and non invertible element is almost stable range 1 is called the ring of almost stable range 1 [7,8]. The ring, where every finitely presented module is decomposed in the direct sum of cyclic modules, is called elementary divisors ring [4].

By $a|b$ we denote the fact that an element a of a ring R is a divisor of an element b of R. Let's denote as $J(R)$ a radical of Jacobson of the ring R and $U(R)$ – the group of units of the ring R.

1. Main result

Let R be a commutative Bezout domain, a – nonzero and non invertible element of domain R. Let's denote the set by

$$S_a = \{b | b \in R, aR + bR = R.\}$$

Proposition 1. *The set S_a is saturated and multiplicative closed.*

Proof. Let $c, b \in S_a$. According to the determination, there exist elements $u_1, u_2, v_1, v_2 \in R$ such, that

$$au_1 + bv_1 = 1,$$

$$au_2 + bv_2 = 1.$$

Multiplying these equalities, we get

$$aw_1 + cbw_2 = 1$$

for some elements $w_1, w_2 \in R$. Therefore $cb \in S_a$.

If $b = cd \in S_a$, then

$$au + c(dv) = 1$$

for some elements $u, v \in R$. So $c \in S_a$ and S_a – saturated and multiplicative closed set. □

Let a is nonzero and non invertible element of R. Let's denote

$$R_a = RS_a^{-1}.$$

Proposition 2. *If a – adequate element of domain R, then $st.r.(R_a) = 1$.*

Proof. Let

$$\frac{b}{s}R_a + \frac{c}{s}R_a = R_a.$$

Then

$$\frac{b}{s} \cdot \frac{u}{s_1} + \frac{c}{s} \cdot \frac{v}{s_2} = t,$$

where $s_1, s_2, t \in S_a$. Hence $bu' + cv' = ss_1s_2t \in S_a$ for some $u', v' \in R$. So $(bu + cv)R + aR = R$ and therefore

$$aR + bR + cR = R.$$

Since element a adequate, there exist element $r \in R$, such as $aR + (b + cr)R = R$, that is $u = b + cr \in S_a$. Another words, $(b + cr)R_a = R_a$. More,

$$\frac{b}{s} + \frac{c}{s} \cdot \frac{r}{1} = su \in R_a,$$

that is

$$(\frac{b}{s} + \frac{c}{s} \cdot \frac{r}{1})R_a = R_a.$$

And that mean the stable range of the ring R_a is equal 1. □

Obviously, from the proposition 2 we get next proposition.

Proposition 3. *Let R be an adequate domain. Then for any nonzero and non invertible element $a \in R$ the set R_a is a commutative Bezout domain of the stable range 1.*

The next question arose: what this commutative Bezout domain is, where for any element a localization is a commutative Bezout domain with the stable range 1?

It is worth to remarks that the stable range of the commutative Bezout domain doesn't exceed 2, that's why the stable range of R_a doesn't exceed 2 either [9].

Let for any nonzero and non invertible element $x \in R$ the stable range of the ring R_x is equal 1 and b, d – nonzero elements of R such as $dR = aR + bR$, moreover d – non invertible element in R. Then elements $u, v, a_0, b_0 \in R$ such that $au + bv = d$, $a = da_0$, $b = db_0$ exist.

According to the restrictions imposed on R, the stable range of the ring R_d equals 1. Since R is domain, then $a_0u+b_0v=1$, that is $a_0R+b_0R=R$. Than $a_0R_d+b_dR=R_d$.

Once again st.r.(R_d)=1, so elements $q\in R$ i $u,p\in S_d$ such that

$$\frac{a_0}{1}\frac{q}{p}+\frac{b_0}{1}=u$$

exist. It follows that $a_0q+b_0p=up$. According to the proposition 1 $up\in S_d$, that is

$$(a_0p+b_0q)R+dR=R$$

and

$$pR+dR=R.$$

We shall notice that $a=da_0$, $b=db_0$. So for any nonzero and co-prime elements b,d there exist p,q such that

$$aq+bp=(a,b)=d$$

and $(p,d)=1$.

So, we proofed the next proposition.

Proposition 4. *Let R be such commutative Bezout domain for any nonzero element $a\in R$ st.r.$(R_a)=1$. Than for any nonzero and co-prime $a,b\in R$ elements $p,q\in R$ such that*

$$aR+bR=(ap+bq)R$$

and

$$qR+(ap+bq)R=R$$

exist.

As obviously corollary from this proposition we got the next result.

Proposition 5. *Let R be such commutative Bezout domain for any nonzero element $a\in R$ st.r.$(R_a)=1$. Than R is elementary divisors ring.*

Proof. To prove it is sufficient to show that for any $a,b,c\in R$ such that $(a,b,c)=1$ the matrix

$$A=\begin{pmatrix} c & a\\ 0 & b\end{pmatrix}$$

diagonalizes [4].

Let's consider possible case.

1) $c = 0$, another words the matrix A look as

$$A = \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}.$$

Let $aR+bR = dR$. Then the elements $a_0, b_0, u, v \in R$, such as $au+bv = d$, $a = da_0$ $b = db_0$ exist. Hence $a_0u + b_0v = 1$ and

$$\begin{pmatrix} u & v \\ -b_0 & a_0 \end{pmatrix}\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & d \\ 0 & 0 \end{pmatrix};$$

$$\begin{pmatrix} 0 & d \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix}.$$

Obviously, the matrices $\begin{pmatrix} u & v \\ -b_0 & a_0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are invertible.

2) $c \in U(R)$. Then

$$\begin{pmatrix} c & a \\ 0 & b \end{pmatrix}\begin{pmatrix} c^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix};$$

$$\begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix}\begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}.$$

Obviously, the matrices $\begin{pmatrix} c^{-1} & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix}$ are invertible.

3) Let $c \neq 0$, $c \in U(R)$. According to the condition and the proposition 4 the elements $p, q \in R$ such as

$$aR + bR = (ap + bq)R$$

and

$$qR + (ap + bq)R = R$$

exist. It follows that $(ap+bq, p) = 1$. Since $ap+bq = (a, b)$ and $(a, b, c) = 1$, than $(c, ap + bq) = 1$ and therefore $(cp, ap + bq) = 1$. We shall notice that $(p, q) = 1$ and

$$(p \quad q)\begin{pmatrix} c & a \\ 0 & b \end{pmatrix} = (cp \quad ap + bq).$$

Another words the matrix $\begin{pmatrix} c & a \\ 0 & b \end{pmatrix}$ diagonalizes. □

Proposition 6. *Let R a commutative Bezout domain, a – any nonzero element of R. Then $J(R_a) \neq 0$.*

Proof. Let M a maximal ideal of the ring R_a such as a doesn't belong to M. Than $M + aR_a = R_a$, that is the elements $m \in M$ and $\frac{r}{s} \in R_a$ such as

$$m + a\frac{r}{s} = 1.$$

Hence $ms + ar = s$.

Let's consider $(m, a) = n$. If n doesn't belong to $U(R)$, than

$$n(m_0 s + a_0 r) = s,$$

where $m = nm_0$, $a = na_0$.

From $\frac{n}{s} \in S$ follows that $(n, a) = 1$. It's impossible, because n doesn't belong to $U(R)$ and $n \mid a = 1$. So $(m, a) = 1$. That is $m \in U(R_a)$, but it's impossible, because $m \in M \in mspecR_a$.

So a belongs to all maximal ideals of R_a. □

Proposition 7. *The element a is a co-adequate element of R_a.*

Proof. As in R_a only units are co-prime with a elements, any non invertible element b has the form $b = 1 \cdot b$, where $1R_a + aR_a = R_a$. For any element b' doesn't belong to $U(R_a)$ such as $b' \mid b$ execute

$$b'R_a + aR_a \neq R_a.$$

□

Proposition 8. *Let R is such commutative Bezout domain that $J(R_a) = aRa$. Than $st.r.(R_a) = 1$, that is to say R is elementary divisors ring.*

Proof. Let R such commutative Bezout domain that $J(R_a) = aR_a$. That is to say $a \in J(R_a) \neq 0$ and $J(R_a) = aR_a$. So let's consider the factor ring R_a/aR_a. Obviously, the Jacobson radical of this factor ring R_a/aR_a equals zero and any element a of R_a/aR_a is zero divisor or invertible. As the factor ring R_a/aR_a is reduced, it is possible only if R_a/aR_a is a zero-dimensional ring. So st.r.$(R_a/aR_a) = 1$, this implies R_a is a ring of the almost stable range 1, which has a nonzero Jacobson radical.

Let $b, c \in R_a$ such that $bR_a + cR_a = R_a$. Let's consider $a \in R_a$. Then $aR_a + bR_a + cR_a = R_a$. Since R_a is a ring of almost stable range 1, there an element $r \in R_a$ such that $aR_a + (b + cr)R_a = R_a$ exist. It follows that

$$au + (b + cr)v = 1$$

for any $u, v \in R_a$. Another words, $(b + cr)v = 1 - au$.

From $a \in J(R_a)$ follows $(b + cr)v \in U(R_a)$, that is R_a is a ring of stable range 1, and therefore R is an elementary divisor ring. □

Proposition 9. *Let R is such a commutative Bezout domain that for any nonzero and non invertible element $a \in R$ the localization R_a is an adequate ring. Then R is an elementary divisors ring.*

Proof. According to the proposition 6 R_a is adequate domain with non zero Jacobson radical. In [9] is proofed that st.r.$(R_a) = 1$. Then according to the proposition 5, R is elementary divisor ring.

Let us denote $K = R_a$ and consider $\overline{K} = K/rad(aK)$. Let's suppose that there are regular elements in $\overline{K}$. Let it be $\bar{b}$. Since $ba_0 = ab_0$, where $(a, b) = d$, $a = a_0 d$, $b = b_0 d$, so $\bar{b} \cdot \overline{a_0} = \bar{0}$.

As $\bar{b}$ is regular, there exists $n \in \mathbf{N}$ such that $a_0^n = at$. It is follows that $a_0^{n-1} = dt$. As $(a_0, b_0) = 1$, then $(a_0^{n-1}, b_0) = 1$. Since $d \mid a_0^{n-1}$, that $(d, b_0) = 1$. Since $(a_0, b_0) = 1$, that $(a, b_0 = 1.)$ That's b_0 is invertible in R. So divisors of a is regular elements in $\overline{K}$. Another words $a = bc$. □

Let $a = bc$. What happens to the image $\bar{b}$ by homomorphism $K \longrightarrow K/rad(aK)$?

Proposition 10. *Let $R = K_a = \{\frac{b}{c} \mid (c, a) = 1\}$. If a is an adequate element in R , then st.r.$(R) = 1$.*

Proof. Considering $J(R) \neq 0$, and st.r.$(R/aR) = 1$ we get st.r.$(R) = 1$. □

At the end we answer the question: is it possible that non adequate element a in R is adequate in R_a?

Proposition 11. *Let $R = K_a = \{\frac{b}{c} \mid (c, a) = 1\}$ and the element a is non adequate in R. then $rad(r/aR) \neq 0$.*

Proof. First of all, we mast remark if a is non adequate element in R then there at least one representation $a = bc$ exist, where $(b, c) \neq 1$. Really, if all representation look as $a = bc$ and $(b, c) = 1$, then − adequate.

Let $d \in R$. If $(a, d) = 1$, all right. But if $(a, d) = \delta$, where δ does not belong to $U(R)$, then $a = a_0\delta$ $d = d_0\delta$ and $(a_0, b_0) = 1$. Since $(a_0, \delta) = 1$, representation $a = rs$ where $r = a_0$ $s = \delta$ is desired.

So let $a = bc$, where $(b, c) = \delta$ and δ does not belong to $U(R)$. It follows that $b = b_0\delta$, $c = c_0\delta$ and

$$(bc_0)^2 = b_0\delta\delta b_0 c_0 c_0 = \delta b_0 \delta c_0 (b_0 c_0) = ab_0 c_0 \in aR.$$

Another words $bc_0 \in rad(R/aR)$. Remarks that $\overline{bc_0} \neq \bar{0}$. Really $bc_0 = ac_0\delta t$. Hence $\delta t = 1$, $\delta \in U(R)$. It is contradiction. So this presentation is proofed. □

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